A NOTE ON DERIVATIONS WITH POWER CENTRAL VALUES ON A LIE IDEAL

JEFFERY MARC BERGEN AND LUISA CARINI
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Let $R$ be a prime ring of characteristic $\neq 2$ with a derivation $d \neq 0$ and a non-central Lie ideal $U$ such that $d(u)^n$ is central, for all $u \in U$. We prove that $R$ must satisfy $s_4$, the standard identity in 4 variables; hence $R$ is either commutative or an order in a 4-dimensional simple algebra. This result extends a theorem of Herstein to Lie ideals.

In [2] Herstein shows that if $R$ is a prime ring with center $Z$ and if $d \neq 0$ is a derivation of $R$ such that $d(x)^n \in Z$, for all $x \in R$, then $R$ satisfies $s_4$, the standard identity in 4 variables. This theorem indicates that the global structure of a ring is often tightly connected to the behaviour of one of its derivations. The purpose of this note is to show that the same conclusion holds for prime rings of characteristic $\neq 2$ even if we assume only that $d(u)^n$ is central for those $u$ in some non-central Lie ideal.

We will proceed by first proving the result when $d$ is inner. We will then use Kharchenko's theorem on differential identities [5] to reduce to the case where $d$ is inner on the Martindale quotient ring of $R$. By using Kharchenko’s theorem, our proof will actually be somewhat simpler than the proof in [2].

In all that follows, unless stated otherwise, $R$ will be a prime ring of characteristic $\neq 2$, $U$ a non-central Lie ideal of $R$, $d \neq 0$ a derivation of $R$, and $n \geq 1$ a fixed integer such that $d(u)^n$ is central, for all $u \in U$. For any ring $S$, $Z = Z(S)$ will denote its center. For subsets $A, B \subset R$, $[A, B]$ will be the additive subgroup generated by all $[a, b] = ab - ba$; $a \in A$, $b \in B$. In addition, $s_4$ will denote the standard identity in 4 variables.

By a result of Herstein [3], $U \supset [I, R]$ for some $I \neq 0$, an ideal of $R$. Therefore, we will assume throughout that $U \supset [I, R]$.

We will also make frequent and important use of the following three results. We do not necessarily state them in their fullest generality.

1. (Carini-Giambruno [1].) If $U \not\subset Z(R)$ is a Lie ideal of a prime ring $R$ of characteristic $\neq 2$, and if $d$ is a derivation of $R$ such that $d(u)^n = 0$, for all $u \in U$, then $d = 0$. 

209
2. (Kharchenko [5].) If \( d \) is a derivation of a prime ring \( R \) and if there exist \( a_i, b_i, c_i, e_i \in R \) such that \( \Sigma a_i d(x) b_i = \Sigma c_i x e_i \), for all \( x \) in a non-zero ideal, then either \( d \) is inner in the Martindale quotient ring of \( R \) or \( \Sigma a_i x b_i = \Sigma c_i x e_i = 0 \) for all \( x \) in the ideal.

3. (Herstein-Procesi-Schacher [4].) If \( R \) is a prime ring satisfying a polynomial identity and if for all \( x, y \in R \) there exists an \( n = n(x, y) \geq 1 \) such that \( [x, y]^n \in Z(R) \), then \( R \) satisfies \( s_4 \).

We now begin the work necessary to prove our theorem with

**Lemma 1.** If \( J \neq 0 \) is an ideal of \( R \) then \( J \cap Z(R) \neq 0 \). Furthermore, \( R_Z \), the localization of \( R \) at \( Z(R) \), is simple with 1.

*Proof.* Let \( V = [I, J^2] \); it is easy to check that \( V \) is a non-central Lie ideal of \( R \) and \( V \subset U \). Since \( d(J^2) \subset Jd(J) + d(J)J \subset J \), we have

\[
d(V) = d([I, J^2]) \subset [I, d(J^2)] + [d(I), J^2] \subset J.
\]

By the result of Carini and Giambruno [1], there is some \( v \in V \), such that \( d(v)^n \neq 0 \).

Hence \( 0 \neq d(v)^n \in J \cap Z(R) \).

If \( K \neq 0 \) is an ideal of \( R_Z \), then \( K \cap R \) is a non-zero ideal of \( R \); hence \( (K \cap R) \cap Z(R) \neq 0 \). Therefore \( K \) contains invertible elements of \( R_Z \) and so, \( R_Z \) is simple with 1.

We can now prove the special case of our result when \( d \) is an inner derivation.

**Theorem 2.** Let \( R \) be a prime ring of characteristic \( \neq 2 \) and let \( a \in R, a \notin Z(R) \) be such that \((au - ua)^n \in Z(R)\), for all \( u \in U \), where \( U \notin Z(R) \) is a Lie ideal of \( R \). Then \( R \) satisfies \( s_4 \).

*Proof.* By Lemma 1 we may localize \( R \) at \( Z(R) \), and it follows that \((au - ua)^n \in Z(R_Z)\), for all \( u \in [R_Z, R_Z] \). Therefore in order to prove that \( R \) satisfies \( s_4 \), we may assume that \( R \) is simple with 1 and \( U \supset [R, R] \). Since \( a \notin Z \), \([a(xy - yx) - (xy - yx)a)^n, z] \) is a non-trivial generalized polynomial identity for \( R \). Thus, by a result of Martindale [6], \( R \) is primitive with minimal right ideal, whose commuting ring \( D \) is a division ring finite dimensional over \( Z(R) \). However, since \( R \) is simple with 1, \( R \) must be artinian; thus \( R = D_k \) for some \( k \geq 1 \).

If \( Z(R) \) is infinite, let \( F \) be a maximal subfield of \( D \) and consider \( \overline{R} = R \otimes_{Z(R)} F = F_m \), for some \( m \geq 1 \).
A Vandermonde determinant argument shows that in \( \overline{R} \), 
\[
((a(xy - yx) - (xy - yx)a)^n, z)
\] is still an identity. If \( Z(R) \) is finite then \( D \) is a finite division ring; hence \( D = Z(R) \). In either case, to prove that \( R \) satisfies \( s_4 \) it is enough to consider the case where \( R = Z(R)_m \), for some \( m \geq 1 \), and \( (a(xy - yx) - (xy - yx)a)^n \in Z(R) \) for all \( x, y \in R \).

\( R \) is now the ring of linear transformations over \( Z(R) \) of an \( m \)-dimensional vector space \( V \). Since \( a \not\in Z(R) \), we can find some \( v \in V \) such that \( v \) and \( va \) are linearly independent.

It suffices to show that \( m \leq 2 \), so we suppose not and let \( t_1, t_2, \ldots, t_{m-2} \in V \) be such that \( T = \{ v, va, t_1, \ldots, t_{m-2} \} \) is a basis for \( V \) over \( Z(R) \).

Let \( x \in R \) such that \( vax = t_1 \) and \( tx = 0 \), for all other \( t \in T \). In addition, let \( y \in R \) be such that \( t_1y = v \) and \( ty = 0 \), for all other \( t \in T \). Therefore
\[
v(a(xy - yx) - (xy - yx)a) = v \quad \text{and so,} \\
v(a(xy - yx) - (xy - yx)a)^n = v.
\]

Hence \( (a(xy - yx) - (xy - yx)a)^n \) is a non-zero element of \( Z(R) \). Thus \( m \) equals the rank of \( (a(xy - yx) - (xy - yx)a)^n \) as a linear transformation. However, \( xy - yx \) has rank 1; therefore \( a(xy - yx) - (xy - yx)a \) has rank \( \leq 2 \) and so, \( (a(xy - yx) - (xy - yx)a)^n \) also has rank \( \leq 2 \).

Therefore \( m \leq 2 \) and we have proved that \( R \) must satisfy \( s_4 \).

At this point we would like to reduce the general case down to the case where \( d \) is inner. To do this, in addition to using Kharchenko's theorem [5], we need the following lemma which is similar to Lemma 4 of [2].

**Lemma 3.** If \( \text{char } R = p > 0 \) and if \( d(Z) \neq 0 \), then \( R \) satisfies \( s_4 \).

**Proof.** Let \( \gamma \in Z \) be such that \( d(\gamma) \neq 0 \) and let \( K = \{ \alpha \in Z | d(\alpha) = 0 \} \). Since \( \text{char } R = p > 0 \), for every \( \beta \in Z \) we have \( \beta^p \in K \). Now, if \( K \) were finite, then \( K \) would be a field and the quotient field of \( Z \) would be algebraic over \( K \). Therefore there would exist an integer \( m \geq 1 \) such that \( \gamma^{p^m} = \gamma \), resulting in the contradiction \( 0 = d(\gamma^{p^m}) = d(\gamma) \). Hence \( K \) is infinite.

In \( R_Z \), let \( T = \{ d(\gamma)/(\alpha + \gamma) | \alpha \in K \} \); by the preceding argument, \( T \) is an infinite subset of \( Z(R_Z) \). If \( u \in [I, R] \) we note that \( (\alpha + \gamma)u \in [I, R] \), for all \( \alpha \in K \). Hence
\[
d((\alpha + \gamma)u)^n = (d(\gamma)u + (\alpha + \gamma)d(u))^n \in Z(R)
\]
for all \( u \in [I, R] \) and \( \alpha \in K \). Dividing by \( (\alpha + \gamma)^n \) results in

\[
(\lambda u + d(u))^n \in Z(R_Z),
\]

for all \( \lambda \in T \) and \( u \in [I, R] \). We again use a Vandermonde determinant argument, since \( T \) is infinite, to see that \( u^n \in Z(R_Z) \cap R = Z(R) \).

Thus \( [i, r]^n \in Z(R) \), for all \( i \in I, \ r \in R \); which implies that \( R \) satisfies a polynomial identity and, after localizing at \( Z(R) \), we have \( [x, y]^n \in Z(R_Z) \), for all \( x, y \in R_Z \).

By the result of Herstein-Procesi-Schacher [4], \( R_Z \) satisfies \( s_4 \), hence \( R \) satisfies \( s_4 \).

We can now prove the main result of this note.

**Theorem 4.** Let \( R \) be a prime ring of characteristic \( \neq 2 \) with a derivation \( d \neq 0 \) and a Lie ideal \( U \subset Z(R) \) such that \( d(u)^n \in Z(R) \), for all \( u \in U \). Then \( R \) satisfies \( s_4 \); hence \( R \) is commutative or an order in a 4 dimensional simple algebra.

**Proof.** Suppose \( d \) is inner in the Martindale quotient ring of \( R \) and is induced by some element \( q \). However, by Lemma 1, the Martindale quotient ring of \( R \) is \( R_Z \); hence \( q = a/\alpha \), for some \( a \in R, \ \alpha \in Z(R) \). Therefore the inner derivation of \( R \) induced by \( a \) satisfies all the hypotheses of Theorem 2, thus in this case, \( R \) satisfies \( s_4 \).

As a result, it now suffices to show that \( d \) is inner in the Martindale quotient ring of \( R \). Let \( x_1, \ldots, x_n \in I \) and \( y_1, \ldots, y_n \in R \) and let \( z_i = [x_i, y_i] \).

By hypothesis \( d(z_1 + \cdots + z_n)^n \in Z(R) \) and, by linearization, we obtain

\[
\sum_{\pi \in S_n} d(z_{\pi(1)}) \cdots d(z_{\pi(n)}) \in Z(R),
\]

where \( S_n \) denotes the symmetric group in \( n \) letters.

Expanding each \( d(z_i) \) we obtain

\[
(*) \sum_{\pi \in S_n} \left( [d(x_{\pi(1)}), y_{\pi(1)}] + [x_{\pi(1)}, d(y_{\pi(1)})] \right) \cdots \left( [d(x_{\pi(n)}), y_{\pi(n)}] + [x_{\pi(n)}, d(y_{\pi(n)})] \right) \in Z(R).
\]

If we multiply out the terms of (*) we will obtain a sum of terms all of whom mention \( y_1 \) or \( d(y_1) \) exactly once. Therefore, after commuting (*) with any \( r \in R \), we obtain an expression of the form

\[
\sum a_i y_i b_i + \sum c_i d(y_1) e_i = 0
\]

where \( a_i, b_i, c_i, e_i \) are products obtained from \( r \) and the \( x_i, d(x_i), y_i, \) and \( d(y_i) \), but not \( y_1 \) or \( d(y_1) \). By Kharchenko's result [5], if \( d \) is not
inner in the Martindale quotient ring then
\[ \sum a_i y_i b_i + \sum c_i y_i e_i = 0, \quad \text{for all } y_i \in R. \]

Thus in (*) we may replace all occurrences of \( d(y_1) \) by \( y_1 \). Similarly, we may sequentially replace each \( d(y_i) \) by \( y_i \) and then each \( d(x_i) \) by \( x_i \) to finally obtain
\[ (**) \quad 2^n \sum_{\pi \in S_n} [x_{\pi(1)}, y_{\pi(1)}] \cdots [x_{\pi(n)}, y_{\pi(n)}] \in Z(R). \]

Since \( \text{char} R \neq 2 \), \( R \) satisfies a polynomial identity. However, by Lemma 3, if \( \text{char} R = p > 0 \) and if \( R \) fails to satisfy \( s_4 \), then \( d(Z) = 0 \).

But now, by the Skolem-Noether theorem, \( d \) is inner on \( R_Z \) since \( d(Z(R_Z)) = 0 \). Thus in this case the proof is complete.

Finally, if \( \text{char} R = 0 \) then let \( x_1 = x_2 = \cdots = x_n \) and \( y_1 = y_2 = \cdots = y_n \); therefore (**) becomes \( 2^n n! [x_1, y_1]^n \in Z(R) \) and so, \( [x_1, y_1]^n \in Z(R) \). As in the proof of Lemma 3, by [4], \( R \) satisfies \( s_4 \), thereby completing the proof.

REFERENCES


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<table>
<thead>
<tr>
<th>Author(s)</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jeffery Marc Bergen and Luisa Carini</td>
<td>A note on derivations with power central values on a Lie ideal</td>
<td>209</td>
</tr>
<tr>
<td>Alfonso Castro and Sumalee Unsurangsie</td>
<td>A semilinear wave equation with nonmonotone nonlinearity</td>
<td>215</td>
</tr>
<tr>
<td>Marius Dadarlat</td>
<td>On homomorphisms of matrix algebras of continuous functions</td>
<td>227</td>
</tr>
<tr>
<td>A. Didierjean</td>
<td>Quelques classes de cobordisme non orienté refusant de se fibrer sur des sphères</td>
<td>233</td>
</tr>
<tr>
<td>Edward George Effros and Zhong-Jin Ruan</td>
<td>On matricially normed spaces</td>
<td>243</td>
</tr>
<tr>
<td>Sherif El-Helaly and Taqdir Husain</td>
<td>Orthogonal bases are Schauder bases and a characterization of ( \Phi )-algebras</td>
<td>265</td>
</tr>
<tr>
<td>Edward Richard Fadell and Peter N-S Wong</td>
<td>On deforming ( G )-maps to be fixed point free</td>
<td>277</td>
</tr>
<tr>
<td>Jean-Jacques Gervais</td>
<td>Stability of unfoldings in the context of equivariant contact-equivalence</td>
<td>283</td>
</tr>
<tr>
<td>Douglas Martin Grenier</td>
<td>Fundamental domains for the general linear group</td>
<td>293</td>
</tr>
<tr>
<td>Ronald Scott Irving and Brad Shelton</td>
<td>Loewy series and simple projective modules in the category ( \mathcal{O}_S )</td>
<td>319</td>
</tr>
<tr>
<td>Russell Allan Johnson</td>
<td>On the Sato-Segal-Wilson solutions of the K-dV equation</td>
<td>343</td>
</tr>
<tr>
<td>Thomas Alan Keagy and William F. Ford</td>
<td>Acceleration by subsequence transformations</td>
<td>357</td>
</tr>
<tr>
<td>Min Ho Lee</td>
<td>Mixed cusp forms and holomorphic forms on elliptic varieties</td>
<td>363</td>
</tr>
<tr>
<td>Charles Livingston</td>
<td>Indecomposable surfaces in 4-space</td>
<td>371</td>
</tr>
<tr>
<td>Geoffrey Lynn Price</td>
<td>Shifts of integer index on the hyperfinite ( \text{II}_1 ) factor</td>
<td>379</td>
</tr>
<tr>
<td>Andrzej Sładek</td>
<td>Witt rings of complete skew fields</td>
<td>391</td>
</tr>
</tbody>
</table>