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**POSITIVE ANALYTIC CAPACITY BUT ZERO BUFFON  
NEEDLE PROBABILITY**

PETER WILCOX JONES AND TAKAFUMI MURAI

## POSITIVE ANALYTIC CAPACITY BUT ZERO BUFFON NEEDLE PROBABILITY

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**There exists a compact set of positive analytic capacity but zero Buffon needle probability.**

**1. Introduction.** For a compact set  $E$  in the complex plane  $\mathbf{C}$ ,  $H^\infty(E^c)$  denotes the Banach space of bounded analytic functions outside  $E$  with supremum norm  $\|\cdot\|_{H^\infty(E^c)}$ . The analytic capacity of  $E$  is defined by

$$\gamma(E) = \sup\{|f'(\infty)|; f \in H^\infty(E^c), \|f\|_{H^\infty(E^c)} \leq 1\},$$

where  $f'(\infty) = \lim_{z \rightarrow \infty} z(f(z) - f(\infty))$  [1, p. 6]. Let  $\mathcal{L}(r, \theta)$  ( $r > 0$ ,  $-\pi < \theta \leq \pi$ ) denote the straight line defined by the equation  $x \cos \theta + y \sin \theta = r$ . The Buffon length of  $E$  is defined by

$$Bu(E) = \iint_{\{(r, \theta); \mathcal{L}(r, \theta) \cap E \neq \emptyset\}} dr d\theta.$$

Vitushkin [7] asked whether two classes of null-sets concerning  $\gamma(\cdot)$  and  $Bu(\cdot)$  are same or not (cf. [2], [3]). Mattila [4] showed that these two classes are different. (He showed that the class of null-sets concerning  $Bu(\cdot)$  is not conformal invariant. Hence his method does not give the information about the implication between these two classes.) The second author [5] showed that, for any  $0 < \varepsilon < 1$ , there exists a compact set  $E_\varepsilon$  such that  $\gamma(E_\varepsilon) = 1$ ,  $Bu(E_\varepsilon) \leq \varepsilon$ . The purpose of this note is to show

**THEOREM.** *There exists a compact set  $E_0$  such that  $\gamma(E_0) = 1$ ,  $Bu(E_0) = 0$ .*

**2. Cranks.** To construct  $E_0$ , we begin by defining cranks. The 1-dimension Lebesgue measure is denoted by  $|\cdot|$ . For a finite union  $E$  of segments in  $\mathbf{C}$ , its length is also denoted by  $|E|$ . For  $\rho > 0$ ,  $z \in \mathbf{C}$  and a set  $E \subset \mathbf{C}$ , we write  $[\rho E + z] = \{\rho\zeta + z; \zeta \in E\}$ . With  $0 \leq \varphi < 1$  and a segment  $J \subset \mathbf{C}$  parallel to the  $x$ -axis, we associate the closed segment  $J(\varphi)$  of the same midpoint as  $J$ , parallel to the  $x$ -axis and of

length  $(1 + \varphi)|J|$ . With a positive integer  $q, 0 \leq \varphi < 1$  and a segment  $J$  parallel to the  $x$ -axis, we associate

$$J(q, \varphi) = \bigcup_{k=1}^{2^{q-1}} [J_{2k-1}(\varphi) + i2^{-q}|J|] \cup \bigcup_{k=1}^{2^{q-1}} J_{2k}(\varphi),$$

where  $\{J_k\}_{k=1}^{2^q}$  are mutually non-overlapping segments on  $J$  of length  $2^{-q}|J|$ ; they are ordered from left to right. The set  $J(q, \varphi)$  is a union of  $2^q$  closed segments of length  $2^{-q}(1 + \varphi)|J|$ . The segment  $\Gamma_0 = \{x; 0 \leq x \leq 1\} \subset \mathbf{C}$  is called a crank of type 0. For a finite sequence  $\{\varphi_j\}_{j=0}^n, \varphi_0 = 0$  ( $n \geq 1$ ) of non-negative numbers less than 1, a finite union  $\Gamma$  of closed segments is called a crank of type  $\{\varphi_j\}_{j=0}^n$  if there exists a crank  $\Gamma' = \bigcup_{k=1}^l J_k$  ( $\{J_k\}_{k=1}^l$  are components of  $\Gamma'$ ) of type  $\{\varphi_j\}_{j=0}^{n-1}$  such that

$$\Gamma = \bigcup_{k=1}^l J_k(q_k, \varphi_n)$$

for some  $l$ -tuple  $(q_1, \dots, q_l)$  of positive integers larger than or equal to  $q_0 = 100$ . We write  $\Gamma' [\varphi_n] \Gamma$ . For a sequence  $\{\varphi_j\}_{j=0}^\infty, \varphi_0 = 0$  of non-negative numbers less than 1, a set  $\Gamma$  is called a crank of type  $\{\varphi_j\}_{j=0}^\infty$ , if there exists a sequence  $\{\Gamma_n\}_{n=0}^\infty$  of cranks such that

$$(1) \quad \Gamma_n \text{ is of type } \{\varphi_j\}_{j=0}^n,$$

$$(2) \quad \Gamma_0 [\varphi_1] \Gamma_1 [\varphi_2] \cdots,$$

$$(3) \quad \Gamma = \bigcap_{n=0}^{\infty} \overline{\bigcup_{j=n}^{\infty} \Gamma_j}.$$

We write by  $\mathbf{O}_n$  the finite sequence of  $n$  zeros ( $n \geq 1$ ). For a finite union  $\Gamma$  of segments,  $L^p(\Gamma)$  ( $1 \leq p \leq \infty$ ) denotes the  $L^p$  space on  $\Gamma$  with respect to the length element  $|dz|$ . We define an operator  $\mathcal{H}_\Gamma$  on  $L^p(\Gamma)$  by

$$\begin{aligned} \mathcal{H}_\Gamma f(z) &= \frac{1}{2\pi i} \text{p.v.} \int_\Gamma \frac{f(\zeta)}{\zeta - z} |d\zeta| \\ &= \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0} \int_{|\zeta - z| > \varepsilon, \zeta \in \Gamma} \frac{f(\zeta)}{\zeta - z} |d\zeta|. \end{aligned}$$

The following fact is already known.

LEMMA 1 ([5]). *For any positive integer  $m$ , there exist a crank  $\Gamma_m^*$  of type  $\mathbf{O}_{m+1}$  and a non-negative function  $g_m^*$  on  $\Gamma_m^*$  such that  $g_m^*$  is a constant on each component of  $\Gamma_m^*$ ,*

$$\|g_m^*\|_{L^1(\Gamma_m^*)} = 1, \quad \|g_m^*\|_{L^\infty(\Gamma_m^*)} \leq C_1, \quad \|\operatorname{Re} \mathcal{Z}_{\Gamma_m^*} g_m^*\|_{L^\infty(\Gamma_m^*)} \leq C_1 \sqrt{m},$$

$$Bu(\Gamma_m^*) \leq C_1/m^{9/10},$$

where  $\operatorname{Re} \zeta$  is the real part of  $\zeta$  and  $C_1$  is an absolute constant.

Our method is as follows. We define a sequence  $\{n(k)\}_{k=0}^\infty$  of non-negative integers with large gaps. Choosing  $\{\varphi_j\}_{j=0}^{10n(1)}$  suitably, we define a crank  $\Gamma_{10n(1)}$  of type  $\{\varphi_j\}_{j=0}^{10n(1)}$ . Then  $|\Gamma_{10n(1)}| = \prod_{\mu=1}^{10n(1)} (1 + \varphi_\mu)$ . Replacing each component of  $\Gamma_{10n(1)}$  by a crank similar to  $\Gamma_{n(2)-10n(1)}^*$  in Lemma 1, we construct a crank  $\Gamma_{n(2)}$  of type  $\{\varphi_j\}_{j=0}^{n(2)}$ , where  $\varphi_j = 0$  ( $10n(1) + 1 \leq j \leq n(2)$ ). Then we see that

$$1/\gamma(\Gamma_{n(2)}) \leq 1/\gamma(\Gamma_{10n(1)}) + \operatorname{Const}(n(2) - 10n(1))^{1/2} / \prod_{j=1}^{10n(1)} (1 + \varphi_j),$$

$$Bu(\Gamma_{n(2)}) \leq C_1 \prod_{j=1}^{10n(1)} (1 + \varphi_j)(n(2) - 10n(1))^{-9/10}.$$

Our sequence  $\{\varphi_j\}_{j=0}^{10n(1)}$  is chosen so that

$$n(2) - 10n(1) = \left\{ \prod_{j=1}^{10n(1)} (1 + \varphi_j) \right\}^{4/3}.$$

Hence

$$1/\gamma(\Gamma_{n(2)}) \leq 1/\gamma(\Gamma_{10n(1)}) + \operatorname{Const}(n(2) - 10n(1))^{-1/4},$$

$$Bu(\Gamma_{n(2)}) \leq C_1(n(2) - 10n(1))^{-3/20}.$$

Replacing each component of  $\Gamma_{n(2)}$  by a suitable crank, we construct a crank  $\Gamma_{10n(2)}$  of type  $\{\varphi_j\}_{j=0}^{10n(2)}$ . Replacing each component of  $\Gamma_{10n(2)}$  by a crank similar to  $\Gamma_{n(3)-10n(2)}^*$ , we construct a crank  $\Gamma_{n(3)}$  of type  $\{\varphi_j\}_{j=0}^{n(3)}$ , where  $\varphi_j = 0$  ( $10n(2) + 1 \leq j \leq n(3)$ ). The sequence  $\{\varphi_j\}_{j=n(2)+1}^{10n(2)}$  is chosen so that  $|(n(3) - 10n(2)) - (\prod_{j=1}^{10n(2)} (1 + \varphi_j))^{4/3}|$  is small. We see that

$$1/\gamma(\Gamma_{n(3)}) \leq 1/\gamma(\Gamma_{10n(1)}) + \operatorname{Const}(n(2) - 10n(1))^{-1/4}$$

$$+ \operatorname{Const}(n(3) - 10n(2))^{-1/4} + (\text{negligible quantity}),$$

$$Bu(\Gamma_{n(3)}) \leq C_1(n(3) - 10n(2))^{-3/20}.$$

Repeating this argument, we define a sequence  $\{\Gamma_{n(k)}\}_{k=2}^{\infty}$  of cranks such that

$$\limsup_{k \rightarrow \infty} 1/\gamma(\Gamma_{n(k)}) < \infty, \quad \lim_{k \rightarrow \infty} Bu(\Gamma_{n(k)}) = 0.$$

Then the analytic capacity of the limit crank is positive and its Buffon length is zero.

### 3. Lemmas.

**LEMMA 2.** *Let  $\Gamma_n$  be a crank of type  $\{\varphi_j\}_{j=0}^n$ ,  $g_n$  be a non-negative function on  $\Gamma_n$  such that  $g_n$  is a constant on each component of  $\Gamma_n$ , and let  $\{\varphi_j\}_{j=n+1}^{n+m}$  be non-negative numbers less than 1. Then there exist a crank  $\Gamma_{n+m}$  of type  $\{\varphi_j\}_{j=0}^{n+m}$  and a non-negative function  $g_{n+m}$  on  $\Gamma_{n+m}$  such that*

$$(4) \quad g_{n+m} \text{ is a constant on each component of } \Gamma_{n+m},$$

$$(5) \quad \|g_{n+m}\|_{L^1(\Gamma_{n+m})} = \|g_n\|_{L^1(\Gamma_n)},$$

$$(6) \quad \|g_{n+m}\|_{L^\infty(\Gamma_{n+m})} \leq \|g_n\|_{L^\infty(\Gamma_n)} \prod_{\mu=n+1}^{n+m} (1 + \varphi_\mu),$$

$$(7) \quad \|\operatorname{Re} \mathcal{R}_{\Gamma_{n+m}} g_{n+m}\|_{L^\infty(\Gamma_{n+m})} \\ \leq \|\operatorname{Re} \mathcal{R}_{\Gamma_n} g_n\|_{L^\infty(\Gamma_n)} + \|g_n\|_{L^\infty(\Gamma_n)} \sum_{j=n+1}^{n+m} \left\{ 1 / \prod_{\mu=n+1}^j (1 + \varphi_\mu) \right\}.$$

We can write  $\Gamma_n = \bigcup_{k=1}^{l_n} J_k^{(n)}$  with its components  $\{J_k^{(n)}\}_{k=1}^{l_n}$ . We put

$$\Gamma_{n+1} = \bigcup_{k=1}^{l_n} J_k^{(n)}(q_{n+1}, \varphi_{n+1}),$$

where  $q_{n+1}$  ( $\geq q_0 = 100$ ) is determined later. Suppose that  $\{\Gamma_\mu\}_{\mu=n+1}^j$  have been defined. We can write  $\Gamma_j = \bigcup_{k=1}^{l_j} J_k^{(j)}$  with its components  $\{J_k^{(j)}\}_{k=1}^{l_j}$ . We put

$$(8) \quad \Gamma_{j+1} = \bigcup_{k=1}^{l_j} J_k^{(j)}(q_{j+1}, \varphi_{j+1}).$$

Thus  $\{\Gamma_j\}_{j=n+1}^{n+m}$  are defined;  $\{q_j\}_{j=n+1}^{n+m}$  are determined later. Let  $n+1 \leq j \leq n+m$ . We define a non-negative function  $g_j$  on  $\Gamma_j$  as follows. Each component  $J_k^{(n)}$  of  $\Gamma_n$  generates  $2^{q_{n+1}+\dots+q_j}$  components of  $\Gamma_j$ . On these components, we put

$$g_j(z) = \left\{ \frac{1}{|J_k^{(n)}|} \int_{J_k^{(n)}} g_n(\zeta) |d\zeta| \right\} / \prod_{\mu=n+1}^j (1 + \varphi_\mu).$$

Since the total length of these  $2^{q_{n+1}+\dots+q_j}$  components is

$$|J_k^{(n)}| \prod_{\mu=n+1}^j (1 + \varphi_\mu),$$

the integration of  $g_j$  over these components is equal to  $\int_{J_k^{(n)}} g_n(\zeta) |d\zeta|$ . Hence  $\|g_j\|_{L^1(\Gamma_j)} = \|g_n\|_{L^1(\Gamma_n)}$ . Evidently,  $g_j$  is a constant on each component of  $\Gamma_j$ . We have

$$\|g_j\|_{L^\infty(\Gamma_j)} \leq \|g_n\|_{L^\infty(\Gamma_n)} / \prod_{\mu=n+1}^j (1 + \varphi_\mu).$$

In particular, (4)–(6) hold. To prove (7), we estimate

$$\|\operatorname{Re} \mathcal{R}_{\Gamma_{j+1}} g_{j+1}\|_{L^\infty(\Gamma_{j+1})}.$$

Recall (8). We have

$$\begin{aligned} J_k^{(j)}(q_{j+1}, \varphi_{j+1}) &= \bigcup_{\mu=1}^{\sigma_{j+1}} [J_{k,2\mu}^{(j)}(\varphi_{j+1}) + i2^{-q_{j+1}} |J_k^{(j)}|] \\ &\cup \bigcup_{\mu=1}^{\sigma_{j+1}} J_{k,2\mu-1}^{(j)}(\varphi_{j+1}) \quad (\sigma_{j+1} = 2^{q_{j+1}-1}, 1 \leq k \leq l_j), \end{aligned}$$

where  $\{J_{k,\mu}^{(j)}\}_{\mu=1}^{2\sigma_{j+1}}$  are mutually non-overlapping segments on  $J_k^{(j)}$  of

length  $2^{-q_{j+1}}|J_k^{(j)}|$ ; they are ordered from left to right. Let

$$z_0 \in \bigcup_{\mu=1}^{\sigma_{j+1}} [J_{k_0, 2\mu}^{(j)}(\varphi_{j+1}) + i2^{-q_{j+1}}|J_{k_0}^{(j)}|]$$

and let  $z_0^*$  be the nearest point on  $J_{k_0}^{(j)}$  to  $z_0$ . Then

$$\begin{aligned} L_1 &= \left| \operatorname{Re} \frac{1}{2\pi i} \text{p.v.} \int_{J_{k_0}^{(j)}(\varphi_{j+1}, \varphi_{j+1})} \frac{g_{j+1}(\zeta)}{\zeta - z_0} |d\zeta| \right. \\ &\quad \left. - \operatorname{Re} \frac{1}{2\pi i} \text{p.v.} \int_{J_{k_0}^{(j)}} \frac{g_j(\zeta)}{\zeta - z_0^*} |d\zeta| \right| \\ &= \left| \operatorname{Re} \frac{1}{2\pi i} \sum_{\mu=1}^{\sigma_{j+1}} \text{p.v.} \int_{J_{k_0, 2\mu-1}^{(j)}(\varphi_{j+1})} \frac{g_{j+1}(\zeta)}{\zeta - z_0} |d\zeta| \right| \\ &\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2^{-q_{j+1}}|J_{k_0}^{(j)}|}{(x - \operatorname{Re} z_0)^2 + (2^{-q_{j+1}}|J_{k_0}^{(j)}|)^2} \|g_{j+1}\|_{L^\infty(\Gamma_{j+1})} dx \\ &\leq \|g_n\|_{L^\infty(\Gamma_n)} / \left\{ 2 \prod_{\mu=n+1}^{j+1} (1 + \varphi_\mu) \right\}. \end{aligned}$$

Let

$$\rho_j = \min_{1 \leq k \leq l_j} \operatorname{dis}(J_k^{(j)}, \Gamma_j - J_k^{(j)}), \quad \tau(q_{j+1}) = 2^{-q_{j+1}} \max_{1 \leq k \leq l_j} |J_k^{(j)}|,$$

where  $\operatorname{dis}(\cdot, \cdot)$  is the distance. We choose, for a while,  $q_{j+1} (\geq q_0)$  so that  $\tau(q_{j+1}) \leq \rho_j/10$ . Since

$$\begin{aligned} \int_{[J_{k, 2\mu}^{(j)}(\varphi_{j+1}) + i2^{-q_{j+1}}|J_k^{(j)}|]} g_{j+1}(\zeta) |d\zeta| &= \int_{J_{k, 2\mu}^{(j)}} g_j(\zeta) |d\zeta|, \\ \int_{J_{k, 2\mu-1}^{(j)}(\varphi_{j+1})} g_{j+1}(\zeta) |d\zeta| &= \int_{J_{k, 2\mu-1}^{(j)}} g_j(\zeta) |d\zeta| \\ &\quad (1 \leq k \leq l_j, 1 \leq \mu \leq 2^{q_{j+1}-1} (= \sigma_{j+1})), \end{aligned}$$

we have

$$\begin{aligned}
L_2 &= \left| \operatorname{Re} \frac{1}{2\pi i} \int_{\Gamma_{j+1} - J_{k_0}^{(j)}(q_{j+1}, \varphi_{j+1})} \frac{g_{j+1}(\zeta)}{\zeta - z_0} |d\zeta| \right. \\
&\quad \left. - \operatorname{Re} \frac{1}{2\pi i} \int_{\Gamma_j - J_{k_0}^{(j)}} \frac{g_j(\zeta)}{\zeta - z_0^*} |d\zeta| \right| \\
&\leq \frac{1}{2\pi} \sum_{k \neq k_0} \left\{ \sum_{\mu=1}^{\sigma_{j+1}} \left| \int_{[J_{k,2\mu}^{(j)}(\varphi_{j+1}) + i2^{-q_{j+1}} |J_k^{(j)}|]} \frac{g_{j+1}(\zeta)}{\zeta - z_0} |d\zeta| \right. \right. \\
&\quad \left. \left. - \int_{J_{k,2\mu}^{(j)}} \frac{g_j(\zeta)}{\zeta - z_0^*} |d\zeta| \right| \right. \\
&\quad \left. + \sum_{\mu=1}^{\sigma_{j+1}} \left| \int_{J_{k,2\mu-1}^{(j)}(\varphi_{j+1})} \frac{g_{j+1}(\zeta)}{\zeta - z_0} |d\zeta| - \int_{J_{k,2\mu-1}^{(j)}} \frac{g_j(\zeta)}{\zeta - z_0^*} |d\zeta| \right| \right\} \\
&\leq \operatorname{Const} \tau(q_{j+1}) \rho_j^{-2} \sum_{k \neq k_0} \sum_{\mu=1}^{2^{q_{j+1}}} \int_{J_{k,\mu}^{(j)}} g_j(\zeta) |d\zeta| \\
&\leq \operatorname{Const} \tau(q_{j+1}) \rho_j^{-2} \|g_j\|_{L^1(\Gamma_j)} \\
&= \operatorname{Const} \tau(q_{j+1}) \rho_j^{-2} \|g_n\|_{L^1(\Gamma_n)}.
\end{aligned}$$

Thus

$$\begin{aligned}
(9) \quad |\operatorname{Re} \mathcal{H}_{\Gamma_{j+1}} g_{j+1}(z_0)| &\leq |\operatorname{Re} \mathcal{H}_{\Gamma} g_j(z_0^*)| + L_1 + L_2 \\
&\leq \|\operatorname{Re} \mathcal{H}_{\Gamma} g_j\|_{L^\infty(\Gamma_j)} + \|g_n\|_{L^\infty(\Gamma_n)} / \left\{ 2 \prod_{\mu=n+1}^{j+1} (1 + \varphi_\mu) \right\} \\
&\quad + \operatorname{Const} \tau(q_{j+1}) \rho_j^{-2} \|g_n\|_{L^1(\Gamma_n)}.
\end{aligned}$$

In the same manner, we have (9) for any point  $z_0$  in

$$\bigcup_{\mu=1}^{\sigma_{j+1}} J_{k_0, 2\mu-1}^{(j)}(\varphi_{j+1}).$$

Since  $k_0$  ( $1 \leq k_0 \leq l_j$ ) is arbitrary,  $\|\operatorname{Re} \mathcal{R}_{\Gamma_{j+1}} g_{j+1}\|_{L^\infty(\Gamma_{j+1})}$  is dominated by the summation of the last three quantities in (9). Consequently,

$$\begin{aligned}
(10) \quad & \|\operatorname{Re} \mathcal{R}_{\Gamma_{n+m}} g_{n+m}\|_{L^\infty(\Gamma_{n+m})} \\
& \leq \|\operatorname{Re} \mathcal{R}_{\Gamma_{n+m-1}} g_{n+m-1}\|_{L^\infty(\Gamma_{n+m-1})} \\
& \quad + \|g_n\|_{L^\infty(\Gamma_n)} / \left\{ 2 \prod_{\mu=n+1}^{n+m} (1 + \varphi_\mu) \right\} \\
& \quad + \operatorname{Const} \tau(q_{n+m}) \rho_{n+m-1}^{-2} \|g_n\|_{L^1(\Gamma_n)} \leq \cdots \leq \|\operatorname{Re} \mathcal{R}_{\Gamma_n} g_n\|_{L^\infty(\Gamma_n)} \\
& \quad + \|g_n\|_{L^\infty(\Gamma_n)} \sum_{j=n+1}^{n+m} 1 / \left\{ 2 \prod_{\mu=n+1}^j (1 + \varphi_\mu) \right\} \\
& \quad + \operatorname{Const} \|g_n\|_{L^1(\Gamma_n)} \sum_{j=n+1}^{n+m} \tau(q_j) \rho_{j-1}^{-2}.
\end{aligned}$$

Since  $\lim_{q \rightarrow \infty} \tau(q) = 0$ , we can inductively define  $\{q_j\}_{j=n+1}^{n+m}$  so that (7) holds. This completes the proof of Lemma 2.

**LEMMA 3.** *Let  $\Gamma_n$  be a crank of type  $\{\varphi_j\}_{j=0}^n$ ,  $g_n$  be a non-negative function on  $\Gamma_n$  such that  $g_n$  is a constant on each component of  $\Gamma_n$ , and let  $m$  be a positive integer. Then there exist a crank  $\Gamma_{n+m}$  of type  $\{\varphi_j\}_{j=0}^{n+m}$  with  $\varphi_j = 0$  ( $n+1 \leq j \leq n+m$ ) and a non-negative function  $g_{n+m}$  on  $\Gamma_{n+m}$  such that*

$$(11) \quad g_{n+m} \text{ is a constant on each component of } \Gamma_{n+m},$$

$$(12) \quad \|g_{n+m}\|_{L^1(\Gamma_{n+m})} = \|g_n\|_{L^1(\Gamma_n)},$$

$$(13) \quad \|g_{n+m}\|_{L^\infty(\Gamma_{n+m})} \leq C_1 \|g_n\|_{L^\infty(\Gamma_n)},$$

$$\begin{aligned}
(14) \quad & \|\operatorname{Re} \mathcal{R}_{\Gamma_{n+m}} g_{n+m}\|_{L^\infty(\Gamma_{n+m})} \\
& \leq \|\operatorname{Re} \mathcal{R}_{\Gamma_n} g_n\|_{L^\infty(\Gamma_n)} + C_2 \sqrt{m} \|g_n\|_{L^\infty(\Gamma_n)},
\end{aligned}$$

$$(15) \quad Bu(\Gamma_{n+m}) \leq C_1 |\Gamma_n| / m^{9/10},$$

where  $C_1$  is the constant in Lemma 1 and  $C_2$  is an absolute constant.

We can write  $\Gamma_n = \bigcup_{k=1}^l J_k$  with its components  $\{J_k\}_{k=1}^l$ . Let  $z_k$  be the left endpoint of  $J_k$  ( $1 \leq k \leq l$ ). We put

$$\Gamma_{n+m} = \bigcup_{k=1}^l \Lambda_k, \quad \Lambda_k = [ |J_k| \Gamma_m^* + z_k ],$$

$$g_{n+m}(z) = g_m^*((z - z_k)/|J_k|)g_n(z_k) \quad (z \in \Lambda_k, 1 \leq k \leq l),$$

where  $\Gamma_m^*$ ,  $g_m^*$  are the crank and the function in Lemma 1, respectively. Then  $\Gamma_{n+m}$  is a crank of type  $\{\varphi_j\}_{j=0}^{n+m}$ . Evidently, (11) and (12) hold. Lemma 1 immediately yields (13) and (15). Let  $z_0 \in \Lambda_{k_0}$  and let  $z_0^*$  be the projection of  $z_0$  to  $J_{k_0}$ . Then Lemma 1 shows that

$$\begin{aligned} & |\operatorname{Re} \mathcal{H}_{\Gamma_{n+m}} g_{n+m}(z_0) - \operatorname{Re} \mathcal{H}_{\Gamma_n} g_n(z_0^*)| \\ & \leq \left| \operatorname{Re} \frac{1}{2\pi i} \int_{\Lambda_{k_0}} \frac{g_{n+m}(\zeta)}{\zeta - z_0} |d\zeta| - \operatorname{Re} \frac{1}{2\pi i} \int_{J_{k_0}} \frac{g_n(\zeta)}{\zeta - z_0^*} |d\zeta| \right| + \frac{1}{2\pi} L^0 \\ & = \left| \operatorname{Re} \frac{1}{2\pi i} \int_{\Lambda_{k_0}} \frac{g_{n+m}(\zeta)}{\zeta - z_0} |d\zeta| \right| + \frac{1}{2\pi} L^0 \\ & = \left| \operatorname{Re} (\mathcal{H}_{\Gamma_m^*} g_m^*) \left( \frac{z_0 - z_k}{|J_k|} \right) \right| g_n(z_k) + \frac{1}{2\pi} L^0 \\ & \leq C_1 \sqrt{m} \|g_n\|_{L^\infty(\Gamma_n)} + \frac{1}{2\pi} L^0, \end{aligned}$$

where

$$L^0 = \sum_{k \neq k_0} \left| \int_{\Lambda_k} \frac{g_{n+m}(\zeta)}{\zeta - z_0} |d\zeta| - \int_{J_k} \frac{g_n(\zeta)}{\zeta - z_0^*} |d\zeta| \right|.$$

Let  $\{\Gamma_j\}_{j=0}^n$  be cranks such that

$$\Gamma_0 [\varphi_1 \Gamma_1 [\varphi_2 \cdots [\varphi_n \Gamma_n.$$

For  $1 \leq k \leq l$ ,  $0 \leq j \leq n$ ,  $\gamma_k(j)$  denotes the component of  $\Gamma_j$  generating  $J_k$ . In particular,  $\gamma_k(n) = J_k$  ( $1 \leq k \leq l$ ). We put

$$L_j^0 = \sum_{k \in \mathcal{F}_j} \left| \int_{\Lambda_k} \frac{g_{n+m}(\zeta)}{\zeta - z_0} |d\zeta| - \int_{J_k} \frac{g_n(\zeta)}{\zeta - z_0^*} |d\zeta| \right| \quad (1 \leq j \leq n),$$

where

$$\mathcal{F}_j = \{1 \leq k \leq l; k \neq k_0, \gamma_k(j-1) = \gamma_{k_0}(j-1), \gamma_k(j) \neq \gamma_{k_0}(j)\}.$$

Then

$$L^0 = \sum_{j=1}^n L_j^0.$$

Since  $\Gamma_m^*$  is a crank of type  $\mathbf{O}_{m+1}$ , a geometric observation shows that, for any  $z \in \Lambda_k$  ( $1 \leq k \leq l$ ),

$$\operatorname{dis}(z, J_k) \leq 2|J_k| \{2^{-q_0} + 2^{-2q_0} + \cdots + 2^{-mq_0}\} \leq \frac{1}{100} |J_k|.$$

Hence  $\Lambda_k$  is contained in the square  $Q_k = \{z + is; z \in J_k, 0 \leq s \leq |J_k|/100\}$  ( $1 \leq k \leq l$ ). Since  $|\gamma_k(n)| = |\gamma_{k_0}(n)|$  ( $k \in \mathcal{F}_n$ ), we have, for  $k \in \mathcal{F}_n$ ,

$$\begin{aligned} \text{dis}(Q_k, Q_{k_0}) &\geq \text{dis}(\gamma_k(n), \gamma_{k_0}(n)) - \frac{1}{100} \{|\gamma_k(n)| + |\gamma_{k_0}(n)|\} \\ &= \text{dis}(\gamma_k(n), \gamma_{k_0}(n)) - \frac{1}{50} |\gamma_{k_0}(n)|. \end{aligned}$$

For any  $1 \leq j \leq n-1, z \in Q_k$ ,

$$\begin{aligned} \text{dis}(z, \gamma_k(j)) &\leq \sum_{\mu=j+1}^n \left\{ \frac{|\gamma_k(\mu)|}{(1 + \varphi_\mu)} + |\gamma_k(\mu)| \right\} + \frac{1}{100} |J_k| \\ &\leq 2|\gamma_k(j)| \sum_{\mu=j+1}^n |\gamma_k(\mu)|/|\gamma_k(j)| + \frac{1}{100} |\gamma_k(j)| \\ &\leq 2|\gamma_k(j)| \{2^{-q_0}(1 + \varphi_{j+1}) + 2^{-2q_0}(1 + \varphi_{j+1})(1 + \varphi_{j+2}) \\ &\quad + \dots + 2^{-(n-j)q_0}(1 + \varphi_{j+1}) \dots (1 + \varphi_n)\} + \frac{1}{100} |\gamma_k(j)| \\ &\leq 2|\gamma_k(j)| \{2^{-(q_0-1)} + 2^{-2(q_0-1)} + \dots\} + \frac{1}{100} |\gamma_k(j)| \leq \frac{1}{50} |\gamma_k(j)|. \end{aligned}$$

Since  $|\gamma_k(j)| = |\gamma_{k_0}(j)|$  ( $k \in \mathcal{F}_j$ ), we have, for  $k \in \mathcal{F}_j, 1 \leq j \leq n-1$ ,

$$\begin{aligned} (16) \quad \text{dis}(Q_k, Q_{k_0}) &\geq \text{dis}(\gamma_k(j), \gamma_{k_0}(j)) - \frac{1}{50} \{|\gamma_k(j)| + |\gamma_{k_0}(j)|\} \\ &= \text{dis}(\gamma_k(j), \gamma_{k_0}(j)) - \frac{1}{25} |\gamma_{k_0}(j)|. \end{aligned}$$

Thus (16) holds for any  $k \in \mathcal{F}_j, 1 \leq j \leq n$ . Let  $1 \leq j \leq n$ . Since

$$\int_{\Lambda_k} g_{n+m}(\zeta) |d\zeta| = \int_{J_k} g_n(\zeta) |d\zeta| \quad (1 \leq k \leq l),$$

we have

$$\begin{aligned} (17) \quad L_j^0 &= \sum_{k \in \mathcal{F}_j} \left| \int_{\Lambda_k} \left\{ \frac{1}{\zeta - z_0} - \frac{1}{z_k - z_0^*} \right\} g_{n+m}(\zeta) |d\zeta| \right. \\ &\quad \left. + \int_{J_k} \left\{ \frac{1}{z_k - z_0^*} - \frac{1}{\zeta - z_0^*} \right\} g_n(\zeta) |d\zeta| \right| \\ &\leq \text{Const} \sum_{k \in \mathcal{F}_j} (|J_k| + |J_{k_0}|) \text{dis}(Q_k, Q_{k_0})^{-2} \int_{J_k} g_n(\zeta) |d\zeta| \\ &\leq \text{Const} \|g_n\|_{L^\infty(\Gamma_n)} \sum_{k \in \mathcal{F}_j} (|J_k| + |J_{k_0}|) |J_k| \text{dis}(Q_k, Q_{k_0})^{-2}. \end{aligned}$$

The segment  $\gamma_{k_0}(j-1)$  generates  $2^{q_j}$  components  $\{\lambda_\nu\}_{\nu=1}^{2^{q_j}}$  of  $\Gamma_j$  of length  $|\gamma_{k_0}(j)|$ , where  $q_j = \log\{(1+\varphi_j)|\gamma_{k_0}(j-1)|/|\gamma_{k_0}(j)|\}/\log 2$  ( $\geq q_0$ ). We may assume that  $\lambda_1 = \gamma_{k_0}(j)$ . Let

$$\mathcal{F}_{j,\nu} = \{k \in \mathcal{F}_j; \lambda_\nu = \gamma_k(j)\} \quad (2 \leq \nu \leq 2^{q_j}).$$

Then  $\mathcal{F}_j = \bigcup_{\nu=2}^{2^{q_j}} \mathcal{F}_{j,\nu}$ . We have, for  $2 \leq \nu \leq 2^{q_j}$ ,

$$\begin{aligned} & \sum_{k \in \mathcal{F}_{j,\nu}} (|J_k| + |J_{k_0}|) |J_k| \\ & \leq |\lambda_1| 2^{-q_0(n-j)} \prod_{j < \mu \leq n} (1 + \varphi_\mu) \sum_{k \in \mathcal{F}_{j,\nu}} |J_k| \\ & = |\lambda_1| 2^{-q_0(n-j)} \left\{ \prod_{j < \mu \leq n} (1 + \varphi_\mu) \right\}^2 \leq |\lambda_1|^2 2^{-(q_0-2)(n-j)}, \end{aligned}$$

where  $\prod_{j < \mu \leq n} (1 + \varphi_\mu)$  denotes 1 if  $j = n$ .

Hence a geometric observation and (16) show that the last quantity in (17) is dominated by

$$\begin{aligned} & \text{Const} \|g_n\|_{L^\infty(\Gamma_n)} \sum_{\nu=2}^{2^{q_j}} \sum_{k \in \mathcal{F}_{j,\nu}} (|J_k| + |J_{k_0}|) |J_k| \text{dis}(\mathcal{Q}_k, \mathcal{Q}_{k_0})^{-2} \\ & \leq \text{Const} \|g_n\|_{L^\infty(\Gamma_n)} \sum_{\nu=2}^{2^{q_j}} \text{dis}(\lambda_\nu, \lambda_1)^{-2} \sum_{k \in \mathcal{F}_{j,\nu}} (|J_k| + |J_{k_0}|) |J_k| \\ & \leq \text{Const} \|g_n\|_{L^\infty(\Gamma_n)} |\lambda_1|^2 2^{-(q_0-2)(n-j)} \sum_{\nu=2}^{2^{q_j}} \text{dis}(\lambda_\nu, \lambda_1)^{-2} \\ & \leq \text{Const} \|g_n\|_{L^\infty(\Gamma_n)} |\lambda_1|^2 2^{-(q_0-2)(n-j)} \sum_{\mu=1}^{\infty} (|\lambda_1| \mu)^{-2} \\ & \leq \text{Const} \|g_n\|_{L^\infty(\Gamma_n)} 2^{-(q_0-2)(n-j)}. \end{aligned}$$

Thus

$$\begin{aligned} |\text{Re } \mathcal{H}_{\Gamma_{n+m}} g_{n+m}(z_0)| & \leq |\text{Re } \mathcal{H}_{\Gamma_n} g_n(z_0^*)| \\ & \quad + C_1 \sqrt{m} \|g_n\|_{L^\infty(\Gamma_n)} + \frac{1}{2\pi} \sum_{j=1}^n L_j^0 \\ & \leq \|\text{Re } \mathcal{H}_{\Gamma_n} g_n\|_{L^\infty(\Gamma_n)} + C_1 \sqrt{m} \|g_n\|_{L^\infty(\Gamma_n)} \\ & \quad + \text{Const} \|g_n\|_{L^\infty(\Gamma_n)} \sum_{j=1}^n 2^{-(q_0-2)(n-j)}, \end{aligned}$$

which shows that

$$|\operatorname{Re} \mathcal{H}_{\Gamma_{n+m}} g_{n+m}(z_0)| \leq \|\operatorname{Re} \mathcal{H}_{\Gamma_n} g_n\|_{L^\infty(\Gamma_n)} + C_2 \sqrt{m} \|g_n\|_{L^\infty(\Gamma_n)}$$

for some absolute constant  $C_2$ . Since  $z_0 \in \Gamma_{n+m}$  is arbitrary, this gives (14). This completes the proof of Lemma 3.

**LEMMA 4.** *Let  $\Gamma$  be a crank of type  $\{\varphi_j\}_{j=0}^\infty$ , and let  $\{\Gamma_n\}_{n=0}^\infty$  be a sequence of cranks satisfying (1)–(3). If  $\liminf_{n \rightarrow \infty} \operatorname{Bu}(\Gamma_n) = 0$ , then  $\operatorname{Bu}(\Gamma) = 0$ .*

Let  $\mathcal{P}^\theta$  ( $-\pi/2 < \theta \leq \pi/2$ ) denote the straight line defined by the equation  $x \sin \theta - y \cos \theta = 0$ . For a set  $E \subset \mathbb{C}$ ,  $\operatorname{proj}_\theta(E)$  denotes the projection of  $E$  to  $\mathcal{P}^\theta$ . We have

$$\operatorname{Bu}(E) = \int_{-\pi/2}^{\pi/2} |\operatorname{proj}_\theta(E)| d\theta.$$

We can write  $\Gamma_n = \bigcup_{k=1}^{l_n} J_k^{(n)}$  with its components  $\{J_k^{(n)}\}_{k=1}^{l_n}$ . In the same manner as in the proof of (14), we have

$$\Gamma \subset \bigcup_{k=1}^{l_n} \{z; \operatorname{dis}(z, J_k^{(n)}) \leq |J_k^{(n)}|\} \left( = \bigcup_{k=1}^{l_n} R_k^{(n)}, \text{ say} \right).$$

Hence, for any  $-\pi/2 < \theta \leq \pi/2$ ,

$$|\operatorname{proj}_\theta(\Gamma)| \leq \left| \operatorname{proj}_\theta \left( \bigcup_{k=1}^{l_n} R_k^{(n)} \right) \right|.$$

We can decompose  $\{k; 1 \leq k \leq l_n\}$  into a finite number of mutually disjoint sets  $\{\mathcal{E}_\mu^\theta\}_{\mu=1}^{\nu_\theta}$  so that  $\operatorname{proj}_\theta(\bigcup_{k \in \mathcal{E}_\mu^\theta} J_k^{(n)})$  is connected. Then a geometric observation shows that

$$\begin{aligned} \left| \operatorname{proj}_\theta \left( \bigcup_{k \in \mathcal{E}_\mu^\theta} R_k^{(n)} \right) \right| &\leq \left| \operatorname{proj}_\theta \left( \bigcup_{k \in \mathcal{E}_\mu^\theta} J_k^{(n)} \right) \right| \\ &\quad + \operatorname{Const} \left( \frac{\pi}{2} - |\theta| \right)^{-1} \max_{k \in \mathcal{E}_\mu^\theta} |\operatorname{proj}_\theta(J_k^{(n)})| \\ &\leq \operatorname{Const} \left( \frac{\pi}{2} - |\theta| \right)^{-1} \left| \operatorname{proj}_\theta \left( \bigcup_{k \in \mathcal{E}_\mu^\theta} J_k^{(n)} \right) \right| \quad (1 \leq \mu \leq \nu_\theta), \end{aligned}$$

and hence

$$\begin{aligned} |\text{proj}_\theta(\Gamma)| &\leq \text{Const} \left( \frac{\pi}{2} - |\theta| \right)^{-1} \sum_{\mu=1}^{\nu_\theta} \left| \text{proj}_\theta \left( \bigcup_{k \in \mathcal{E}_\mu^\theta} J_k^{(n)} \right) \right| \\ &= \text{Const} \left( \frac{\pi}{2} - |\theta| \right)^{-1} |\text{proj}_\theta(\Gamma_n)|. \end{aligned}$$

We have, for any  $0 < \varepsilon < \pi/2$ ,

$$\begin{aligned} \int_{-(\pi/2)+\varepsilon}^{(\pi/2)-\varepsilon} |\text{proj}_\theta(\Gamma)| d\theta &\leq \text{Const} \int_{-(\pi/2)+\varepsilon}^{(\pi/2)-\varepsilon} \left( \frac{\pi}{2} - |\theta| \right)^{-1} |\text{proj}_\theta(\Gamma_n)| d\theta \\ &\leq \text{Const} \varepsilon^{-1} Bu(\Gamma_n). \end{aligned}$$

Since  $\liminf_{n \rightarrow \infty} Bu(\Gamma_n) = 0$ , this shows that the first quantity equals zero. Since  $0 < \varepsilon < \pi/2$  is arbitrary,  $Bu(\Gamma) = 0$ . This completes the proof of Lemma 4.

**4. Construction of  $E_0$ .** Let  $p_n$  be the integral part of  $(3/2)^{4n/3}$  ( $n \geq 1$ ). We define a sequence  $\{n(k)\}_{k=1}^\infty$  of positive integers by  $n(1) = 10$ ,

$$n(k+1) = 10n(k) + p_{10n(k)} \quad (k \geq 1).$$

We define a sequence  $\{\varphi_j\}_{j=0}^\infty$  of non-negative numbers by  $\varphi_0 = 0$ ,

$$\begin{aligned} \varphi_j &= \frac{1}{2} & (1 \leq j \leq n(1)), \\ \varphi_j &= \frac{1}{2} & (n(k) < j \leq 10n(k), \quad k \geq 1), \\ \varphi_j &= 0 & (10n(k) < j \leq n(k+1), \quad k \geq 1). \end{aligned}$$

We use Lemma 2 with  $\Gamma_0, g_0 = 1$  and  $\{\varphi_j\}_{j=0}^{10n(1)}$ . There exist a crank  $\Gamma_{10n(1)}$  of type  $\{\varphi_j\}_{j=0}^{10n(1)}$  and a non-negative function  $g_{10n(1)}$  on  $\Gamma_{10n(1)}$  such that  $g_{10n(1)}$  is a constant on each component of  $\Gamma_{10n(1)}$ ,

$$\|g_{10n(1)}\|_{L^1(\Gamma_{10n(1)})} = 1, \quad \|g_{10n(1)}\|_{L^\infty(\Gamma_{10n(1)})} \leq 1 / \prod_{\mu=1}^{10n(1)} (1 + \varphi_\mu),$$

$$\begin{aligned} &\|\text{Re } \mathcal{H}_{\Gamma_{10n(1)}} g_{10n(1)}\|_{L^\infty(\Gamma_{10n(1)})} \\ &\leq \|\text{Re } \mathcal{H}_{\Gamma_0} g_0\|_{L^\infty(\Gamma_0)} + \sum_{j=1}^{10n(1)} 1 / \prod_{\mu=1}^j (1 + \varphi_\mu) \\ &= \left\{ \sum_{j=1}^{n(1)} 1 / \prod_{\mu=1}^j (1 + \varphi_\mu) \right\} + \sum_{j=n(1)+1}^{10n(1)} 1 / \prod_{\mu=1}^j (1 + \varphi_\mu). \end{aligned}$$

Using Lemma 3 with  $n = 10n(1)$ ,  $m = p_{10n(1)}$ , we obtain a crank  $\Gamma_{n(2)}$  of type  $\{\varphi_j\}_{j=0}^{n(2)}$  and a non-negative function  $g_{n(2)}$  on  $\Gamma_{n(2)}$  such that  $g_{n(2)}$  is a constant on each component of  $\Gamma_{n(2)}$ ,

$$\|g_{n(2)}\|_{L^1(\Gamma_{n(2)})} = \|g_{10n(1)}\|_{L^1(\Gamma_{10n(1)})} = 1,$$

$$\|g_{n(2)}\|_{L^\infty(\Gamma_{n(2)})} \leq C_0 \|g_{10n(1)}\|_{L^\infty(\Gamma_{10n(1)})} \leq C_0 \Big/ \prod_{\mu=1}^{10n(1)} (1 + \varphi_\mu),$$

$$\begin{aligned} & \|\operatorname{Re} \mathcal{H}_{\Gamma_{n(2)}} g_{n(2)}\|_{L^\infty(\Gamma_{n(2)})} \\ & \leq \|\operatorname{Re} \mathcal{H}_{\Gamma_{10n(1)}} g_{10n(1)}\|_{L^\infty(\Gamma_{10n(1)})} + C_0 \sqrt{p_{10n(1)}} \|g_{10n(1)}\|_{L^\infty(\Gamma_{10n(1)})} \\ & \leq \left\{ \sum_{j=1}^{n(1)} 1 \Big/ \prod_{\mu=1}^j (1 + \varphi_\mu) \right\} + \sum_{j=n(1)+1}^{10n(1)} 1 \Big/ \prod_{\mu=1}^j (1 + \varphi_\mu) \\ & \quad + C_0 \sqrt{p_{10n(1)}} \Big/ \prod_{\mu=1}^{10n(1)} (1 + \varphi_\mu), \end{aligned}$$

$$Bu(\Gamma_{n(2)}) \leq C_0 |\Gamma_{10n(1)}| / p_{10n(1)}^{9/10} = C_0 \prod_{\mu=1}^{10n(1)} (1 + \varphi_\mu) / p_{10n(1)}^{9/10},$$

where  $C_0 = \max\{C_1, C_2\}$ . Using Lemma 2 with  $n = n(2)$ ,  $m = 9n(2)$ , we obtain a crank  $\Gamma_{10n(2)}$  and a non-negative function  $g_{10n(2)}$ . Using Lemma 3 with  $n = 10n(1)$ ,  $m = p_{10n(2)}$ , we obtain a crank  $\Gamma_{n(3)}$  and a non-negative function  $g_{n(3)}$ . Repeating this argument, we obtain a crank  $\Gamma_{n(k)}$  ( $k \geq 2$ ) of type  $\{\varphi_j\}_{j=0}^{n(k)}$  and a non-negative function  $g_{n(k)}$  on  $\Gamma_{n(k)}$  such that  $g_{n(k)}$  is a constant on each component of  $\Gamma_{n(k)}$ ,

$$\|g_{n(k)}\|_{L^1(\Gamma_{n(k)})} = 1,$$

$$\|g_{n(k)}\|_{L^\infty(\Gamma_{n(k)})} \leq C_0^{k-1} \Big/ \prod_{\mu=1}^{10n(k-1)} (1 + \varphi_\mu),$$

$$\begin{aligned} & \|\operatorname{Re} \mathcal{H}_{\Gamma_{n(k)}} g_{n(k)}\|_{L^\infty(\Gamma_{n(k)})} \\ & \leq \left\{ \sum_{j=1}^{n(1)} 1 \Big/ \prod_{\mu=1}^j (1 + \varphi_\mu) \right\} + \sum_{\nu=1}^{k-1} \sum_{j=n(\nu)+1}^{10n(\nu)} \left\{ C_0^{\nu-1} \Big/ \prod_{\mu=1}^j (1 + \varphi_\mu) \right\} \\ & \quad + \sum_{\nu=1}^{k-1} \left\{ C_0^\nu \sqrt{p_{10n(\nu)}} \Big/ \prod_{\mu=1}^{10n(\nu)} (1 + \varphi_\mu) \right\}, \end{aligned}$$

$$Bu(\Gamma_{n(k)}) \leq C_0 \prod_{\mu=1}^{10n(k-1)} (1 + \varphi_\mu) / p_{10n(k-1)}^{9/10}.$$

Let  $\Gamma = \bigcap_{j=1}^{\infty} \overline{\bigcup_{k=2}^{\infty} \Gamma_{n(k)}}$ . Then  $\Gamma$  is a crank of type  $\{\varphi_j\}_{j=0}^{\infty}$ . We have

$$\begin{aligned} Bu(\Gamma_{n(k)}) &\leq C_0 \prod_{\mu=1}^{10n(k-1)} (1 + \varphi_\mu) p_{10n(k-1)}^{-9/10} \\ &\leq \text{Const} \left(\frac{3}{2}\right)^{10n(k-1)} \left(\frac{3}{2}\right)^{-(4/3)(9/10)10n(k-1)} \\ &= \text{Const} \left(\frac{3}{2}\right)^{-2n(k-1)}, \end{aligned}$$

which shows that  $\lim_{k \rightarrow \infty} Bu(\Gamma_{n(k)}) = 0$ . Hence Lemma 4 gives that  $Bu(\Gamma) = 0$ .

We now show that  $\gamma(\Gamma) > 0$ . Let  $k \geq 1$ . Then

$$\int_{\Gamma_{n(k)}} g_{n(k)}(\zeta) |d\zeta| = 1.$$

Since  $n(\nu) \geq 10n(\nu - 1)$  ( $\nu \geq 2$ ),  $n(1) = 10$ , we have  $n(\nu) \geq 10^\nu$  ( $\nu \geq 1$ ), and hence

$$\|g_{n(k)}\|_{L^\infty(\Gamma_{n(k)})} \leq C_0^{k-1} \left(\frac{3}{2}\right)^{-9n(k-1)} \leq \text{Const}.$$

Since

$$\begin{aligned} \sqrt{p_{10n(\nu)}} \left\{ \prod_{\mu=1}^{10n(\nu)} (1 + \varphi_\mu) \right\}^{-1} &\leq \sqrt{p_{10n(\nu)}} \left(\frac{3}{2}\right)^{-9n(\nu)} \\ &\leq \text{Const} \left(\frac{3}{2}\right)^{(4/3)(1/2)10n(\nu)} \left(\frac{3}{2}\right)^{-9n(\nu)} \\ &= \text{Const} \left(\frac{3}{2}\right)^{-(7/3)n(\nu)} \quad (\nu \geq 1), \end{aligned}$$

we have

$$\|\text{Re } \mathcal{R}_{\Gamma_{n(k)}} g_{n(k)}\|_{L^\infty(\Gamma_{n(k)})} \leq \text{Const}.$$

Hence we can define a non-negative function  $h_k$  on  $\Gamma_{n(k)}$  so that

$$\begin{aligned} \int_{\Gamma_{n(k)}} h_k(\zeta) |d\zeta| &= \eta_0, \quad \|h_k\|_{L^\infty(\Gamma_{n(k)})} \leq 1/2, \\ \|\text{Re } \mathcal{R}_{\Gamma_{n(k)}} h_k\|_{L^\infty(\Gamma_{n(k)})} &\leq 1/2, \\ h_k(\zeta) &= 0 \quad \text{at endpoints of each component of } \Gamma_{n(k)}, \\ h_k &\text{ is differentiable along } \Gamma_{n(k)}, \end{aligned}$$

where  $\eta_0$  is an absolute constant. Let

$$\hat{h}_k(z) = \frac{1}{2\pi i} \int_{\Gamma_{n(k)}} \frac{h_k(\zeta)}{\zeta - z} |d\zeta|,$$

$$u_k(z) = \operatorname{Re} \hat{h}_k(z), \quad v_k(z) = (\text{the imaginary part of } \hat{h}_k(z)),$$

$$f_k(z) = \{1 - \exp(i\hat{h}_k(z))\} / \{1 + \exp(i\hat{h}_k(z))\} \quad (z \notin \Gamma_{n(k)})$$

(cf. [1, p. 30]). We see easily that  $f_k$  is analytic outside  $\Gamma_{n(k)}$  and

$$f'_k(\infty) = \frac{1}{4\pi} \int_{\Gamma_{n(k)}} h_k(\zeta) |d\zeta| = \eta_0/4\pi.$$

The non-tangential limit of  $|u_k(z)|$  to each point on  $\Gamma_{n(k)}$  is dominated by

$$\|h_k\|_{L^\infty(\Gamma_{n(k)})} + \|\operatorname{Re} \mathcal{N}_{\Gamma_{n(k)}} h_k\|_{L^\infty(\Gamma_{n(k)})} \leq 1.$$

Since  $|u_k|$  is sub-harmonic in  $\Gamma_{n(k)}^c$  and continuous in  $\mathbf{C} \cup \{\infty\}$ , we have  $\sup_{z \in \Gamma_{n(k)}^c} |u_k(z)| \leq 1$ . Hence, for any  $z \notin \Gamma_{n(k)}$ ,

$$|f_k(z)|^2 = \frac{1 + \exp(-2v_k(z)) - 2 \exp(-v_k(z)) \cos(u_k(z))}{1 + \exp(-2v_k(z)) + 2 \exp(-v_k(z)) \cos(u_k(z))} \leq 1,$$

which shows that  $\|f_k\|_{H^\infty(\Gamma_{n(k)}^c)} \leq 1$ . Since  $k \geq 1$  is arbitrary, using an argument of normal families, we obtain  $f \in H^\infty(\Gamma^c)$  satisfying  $f'(\infty) = \eta_0/4\pi$ ,  $\|f\|_{H^\infty(\Gamma^c)} \leq 1$ . This shows that  $\gamma(\Gamma) \geq \eta_0/4\pi$ . Normalizing  $\Gamma$ , we obtain the required set  $E_0$ .

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