REPRESENTING HOMOLOGY CLASSES OF $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

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In this paper we determine the set of all second homology classes in $CP^2 \# \overline{CP}^2$ which can be represented by smoothly embedded two-spheres in $CP^2 \# \overline{CP}^2$.

We say a class $u \in H_2(M^4, \mathbb{Z})$ can be represented by $S^2$ if it can be represented by a smoothly embedded 2-sphere in $M^4$. The purpose of this note is to prove the following.

**Theorem.** Let $\eta$, $\xi$ be canonical generators of $H_2(CP^2 \# \overline{CP}^2, \mathbb{Z})$. Then $\gamma = a\eta + b\xi$, $a, b \in \mathbb{Z}$, can be represented by $S^2$ if and only if $a, b$ satisfy one of the following conditions.

(i) $|a - b| \leq 1$, or
(ii) $(a, b) = (\pm 2, 0)$ or $(0, \pm 2)$.

**Remark 1.** The "if" part of the theorem is known (see Wall [7], Mandelbaum [5, the proof of Theorem 6.6]).

**Remark 2.** If $p \in \mathbb{Z}$, then $p\eta$ (or $p\xi$) is represented by $S^2$ if and only if $|p| \leq 2$ (see Rohlin [6]).

**Remark 3.** If $a, b$ are relatively prime integers, then $\gamma = a\eta + b\xi$ is realized by a topologically embedded locally flat 2-sphere by Freedman [2]. Hence smoothness condition in the theorem is essential.

By Remarks 1 and 2, the Theorem follows from the following.

**Proposition.** Let $a$ and $b$ be two integers satisfying

\[
\begin{cases}
(i) & ab \neq 0, \text{ and} \\
(ii) & |a| - |b| \geq 2.
\end{cases}
\]

Then $a\eta + b\xi$ is not represented by $S^2$.

**Proof.** Suppose conversely that $a\eta + b\xi$ is represented by $S^2$. By reversing orientation if necessary, we may assume $n = b^2 - a^2 > 0$. Let $M^4 = CP^2 \# \overline{CP}^2 \# (n - 1)CP^2$ with $\xi_i$'s the generators of
$H_2(M^4,\mathbb{Z})$ with respect to the additional $CP^2$'s. Then the homology class $\gamma = a\eta + b\xi + \sum_{i=1}^{n-1} \xi_i$ can be represented by a smoothly embedded $2$-sphere $S$ in $M^4$. The self-intersection number of $S$ is $S \cdot S = a^2 - b^2 + n - 1 = -1$. Hence the tubular neighborhood $N$ of $S$ in $M^4$ is the $(-1)$-Hopf bundle over $S$ and $\partial N$ is diffeomorphic to $S^3$. Set $W^4 = (M^4 - \hat{N})U_0D^4$. It is known that $W^4$ is a closed, simply connected smooth $4$-manifold with a positive definite intersection form (see Kuga [4, claim 1]). By Donaldson's result (see Donaldson [1]), the intersection form of $W^4$ is standard. On the other hand, $M^4 = W^4 \# \hat{N}^4$ where $\hat{N}^4 = N^4U_0D^4$. So, $(H_2(W^4,\mathbb{Z}), \langle , \rangle_{W^4})$ is isomorphic to $(\gamma^*, \langle , \rangle_{M^4})$. Hence there exist exactly $2n \alpha \in H_2(M^4,\mathbb{Z})$ such that $a\gamma = 0$ and $a\alpha = 1$. Writing out the conditions in terms of the base $(\eta, \xi, \xi_1, \xi_2, \ldots, \xi_{n-1})$ by letting $\alpha = x\eta + y\xi + \sum_{i=1}^{n-1} z_i\xi_i$, we obtain $2n$ ($\geq 16$) solutions of the system of Diophantine equations

$$
\begin{align*}
(1) \quad \langle ax - by + \sum_{i=1}^{n-1} z_i \rangle = 0, \\
(2) \quad \langle x^2 - y^2 + \sum_{i=1}^{n-1} z_i^2 \rangle = 1.
\end{align*}
$$

Claim. If $a$, $b$ satisfy (*), the above equations have at most four solutions.

Proof. We have $y^2 - x^2 = \sum_{i=1}^{n-1} z_i^2 - 1 \geq -1$. If $y^2 - x^2 = -1$, then $y = 0$, $x = \pm 1$, and $z_i = 0$ for all $i$. By (1), this implies $a = 0$; if $y^2 - x^2 = 0$, then only one of $z_i$'s is $\pm 1$, all others are zero. By (1), this implies that $||a| - |b|| \leq 1$; If $y^2 - x^2 = 1$, then $y = \pm 1$, $x = 0$, and only two of $z_i$'s are $\pm 1$, all others are zero. So (1) implies $|a| \leq 2$, but $|a| \leq |b|$ by assumption. Therefore, in all cases, $a$, $b$ fail to satisfy (*). Hence we have $y^2 - x^2 \geq 3$.

Assume $n'$ of the $z_i$'s are nonzero, say $z_{ij}$, $j = 1, 2, \ldots, n'$. Then we have

$$
\begin{align*}
(3) \quad (ax - by)^2 &= \left( \sum_{j=1}^{n'} z_{ij} \right)^2 \\
&\leq n' \cdot \left( \sum_{j=1}^{n'} z_{ij}^2 \right) \\
&= n'(1 + y^2 - x^2) = n' + n'(y^2 - x^2) \\
&\leq n' + (n - 1)(y^2 - x^2) = n' + (b^2 - a^2 - 1)(y^2 - x^2) \\
&= n' + b^2 y^2 - b^2 x^2 + a^2 x^2 - a^2 y^2 - (y^2 - x^2) \\
&= n' + a^2 x^2 + b^2 y^2 - b^2 x^2 - a^2 y^2 - \sum_{j=1}^{n'} z_{ij}^2 + 1,
\end{align*}
$$

where (3) follows from Cauchy-Schwarz inequality.
Expanding and re-arranging this implies
\[(5) \quad (bx - ay)^2 \leq \left( n' - \sum_{j=1}^{n'} z_j^2 \right) + 1.\]

Since each \(z_i \neq 0\), (5) implies all these \(z_i\)'s are ±1, and \((bx - ay)^2 \leq 1\).

There are now only two cases that might happen.

**Case 1.** \(bx - ay = \pm 1\).

Then equalities in (3) and (4) hold. So \(z_1 = \cdots = z_{n-1} = \pm 1\), and (1), (2) reduce to
\[(6) \quad ax - by = \pm (n - 1),\]
\[x^2 - y^2 + (n - 1) = 1.\]

The equation (6) and \(bx - ay = \pm 1\) give at most four solutions to the Diophantine equations (1), (2) according to the choice of plus or minus signs.

**Case 2.** \(bx - ay = 0\).

Then the equality in (3) must hold because if inequality holds, the left hand side of (3) will reduce at least -4 which contradicts (5) where the right hand side exceeds the left hand side by +1. By the same argument, the equality in (4) must hold since we have shown that \(y^2 - x^2 \geq 3\). Therefore, the equality in (5) holds which is again a contradiction. Hence this case gives no solution.

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**References**


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