THE MAZUR PROPERTY FOR COMPACT SETS

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We give a “convex” characterization to the following smoothness property, denoted by (CI): every compact convex set is the intersection of balls containing it. This characterization is used to give a transfer theorem for property (CI). As an application we prove that the family of spaces which have an equivalent norm with property (CI) is stable under $c_0$ and $l_p$ sums for $1 \leq p < \infty$. We also prove that if $X$ has a transfinite Schauder basis, and $Y$ has an equivalent norm with property (CI) then the space $X \hat{\otimes}_p Y$ has an equivalent norm with property (CI), for every tensor norm $p$.

Similar results are obtained for the usual Mazur property (I), that is, the family of spaces which have an equivalent norm with property (I) is stable under $c_0$ and $l_p$ sums for $1 < p < \infty$.

Introduction. Mazur [6] was the first who considered the following separation property, denoted by (I):

Every bounded closed convex set is the intersection of balls containing it.

Later, Phelps [7] proved that property (I) is weaker than the Fréchet differentiability of the norm, and gave a dual characterization for (I) in the finite dimensional case.

Phelps’ theorem was extended to the infinite dimensional case in [3], where the property (I) was dually characterized.

Here we will give another extension of Phelps’ theorem by characterizing the following property, denoted by (CI):

Every compact convex set is an intersection of balls.

This property was recently introduced by Whitfield and Zizler [9].

We use this characterization to give a “transfer theorem” for property (CI), which is analogous to the one given for property (I) [2].

We also prove a stability result for property (CI), which is of the same nature as the one given by Zizler for l.u.c. renormings [10]. Our proof can be modified to give a similar stability result for property (I).

Some renorming results of Whitfield-Zizler [9], and Deville [2] are particular cases of these stability results.
Notation. Our notation is standard. A point \( x \in X \) is said to be extremal if \( x = 0 \) or \( x/\|x\| \) is an extreme point of the unit ball of \( X \). Similar conventions will be used for \( w^* \)-exposed points, \( w^* \)-denting points, and \( w^* \)-strongly exposed points.

The unit ball and the unit sphere of a Banach space \( X \) will be denoted by \( B(X) \) and \( S(X) \) respectively. We also denote by \( B(z, r) \) [resp. \( S(z, r) \)] the ball [resp. the sphere] centered at \( z \) and of radius \( r \) (the underlying Banach space is understood).

For a subset \( C \) of a Banach space \( X \) we denote by \( cv(C) \) [resp. \( \overline{cv}(C) \)] the convex [resp. closed convex] hull of \( C \).

1. Dual characterization for property \((CI)\). The following theorem is analogous to the one given for property \((I)\) [3]. Techniques used in the proof can be found in Phelps’ paper [7].

Theorem 1. Let \( X \) be a Banach space. The following properties are equivalent:

(i) Every compact convex set is the intersection of balls containing it.

(ii) The cone of extreme points of \( X^* \) is dense in \( X^* \) for the topology \( \mathcal{T} \) of uniform convergence on compact sets of \( X \).

Proof. (i) \( \Rightarrow \) (ii). Let \( f \in S(X^*) \), \( K \) a compact subset of \( B(X) \), and \( \varepsilon > 0 \). We want to find \( g \in \text{Ext}(B(X^*)) \), and \( \lambda > 0 \), such that

\[
\|f - \lambda g\|_K = \sup_{K} |f - \lambda g| \leq \varepsilon.
\]

Without loss of generality we can suppose that \( K \) is absolutely convex and \( \|f\|_K \geq 1 - \varepsilon/2 \).

(Indeed, let \( x \in B(X) \) such that \( f(x) > 1 - \varepsilon/2 \), and let \( L \) be the closed convex symmetric hull of \( K \cup \{x\} \). The above mentioned reduction is then possible since \( \|\cdot\|_L \geq \|\cdot\|_K \).) Let \( u \in K \) be such that \( f(u) = 1 - \varepsilon/2 \), and put \( u' = (\varepsilon/4)u \), and \( D = K \cap f^{-1}(0) \). By (i), there exists \( z \in X, r > 0 \), such that \( u' \notin B(z, r) \), and \( D \subset B(z, r) \).

Let \( w \) be the unique element of \( [S(z, r) \cap cv(u', z)] \). Put \( x = (w - z)/r \), and let \( g \in \text{Ext}(B(X^*)) \) such that \( \|x\| = g(x) = 1 \). Then it is easy to see that:

\[
0 \leq g(w) = \sup_{B(z, r)} g < g(u'), \quad \text{so } \|g\|_K > 0.
\]

Let \( \lambda > 0 \) be such that \( \|\lambda g\|_K = 1 \). Then for every \( k \in D \) we have:

\[
\lambda g(k) \leq \lambda g(u') = \varepsilon \lambda g(u')/4 \leq \varepsilon/4,
\]

and by symmetry of \( D \), we have \( \|g\|_D \leq \varepsilon/4 \).
Phelps' lemma implies then:

\[ \left\| \frac{f}{\|f\|_K} + \lambda g \right\|_K \leq \varepsilon/2 \quad \text{or} \quad \left\| \frac{f}{\|f\|_K} - \lambda g \right\|_K \leq \varepsilon/2. \]

(Phelps' lemma is applied to the space \((\text{Sp}K, j_K)\): the linear space generated by \(K\) equipped with the gauge (or the Minkowski functional) of \(K\).)

But \(f(u)/\|f\|_K \geq f(u) \geq 1 - \varepsilon/2 > \varepsilon/2\) (if \(\varepsilon < 1\)) and \(\lambda g(u) \geq 0\), so we have necessarily \(\|f/\|f\|_K - \lambda g\|_K \leq \varepsilon/2\).

Then

\[ \|f - \lambda g\|_K \leq \frac{\varepsilon}{2} + \left\| \frac{f}{\|f\|_K} - f \right\|_K \leq \varepsilon. \]

(ii) \(\Rightarrow\) (i). (Our proof is simpler than the one given by Whitfield and Zizler [9].) Let \(K\) be a compact convex subset of \(X\) not containing 0. By (ii) and the Hahn-Banach theorem there exists \(g \in \text{Ext}(B(X^*))\) such that \(\inf_K g > 0\).

Let us first note the following easy fact:

On bounded subsets of \(X^*\), the \(w^*\)-topology coincides with the topology \(\mathcal{T}\) of uniform convergence on compact sets of \(X\).

From the extremality of \(g\), we deduce that there exists an \(x \in S(X), \delta > 0\), such that:

\[ g \in S(B(X^*); x, \delta) \quad \text{and} \quad \text{diam}_{\|f\|_K}[S(B(X^*); x, \delta)] \leq \varepsilon, \]

where \(\varepsilon\) is defined by \(3\varepsilon = \inf_K g\).

Let us consider now the increasing family of balls (for \(r > 1\)): \(D_r = B(\text{r}x, (r - 1)\varepsilon)\), and let us show that \(K \subset D_r\) for some \(r\).

If not, let \(y \in \bigcap_{r>0}(K \setminus D_r)\), and let \(g_r \in S(X^*)\) be such that

\[ g_r(\text{r}x - y) = \|\text{r}x - y\| \geq (r - 1)\varepsilon. \]

Then \(g_r(x) \to 1\), and

\[ (g - g_r)(y) = g(y) + g_r(\text{r}x - y) - \varepsilon g_r(x) \geq 3\varepsilon + (r - 1)\varepsilon - \varepsilon g_r(x) = 2\varepsilon + \varepsilon(1 - g_r(x)) \geq 2\varepsilon, \]

which is a contradiction to the choice of \(x\) and \(\delta\).

\[ \square \]

**Remark.** Let us show that property (\(CI\)) is the "natural" intersection property which is associated to Gateaux-smoothness. In order to do this, we will describe the similarities between the dual characterizations of properties (\(I\)) and (\(CI\)).
Recall first that $X$ has property (I) if and only if the set of $w^*$-denting points of $B(X^*)$ is norm dense in $S(X^*)$ [3]. And observe that the definition of $w^*$-denting points (resp. extreme points) is obtained from the one of $w^*$-strongly exposed points (resp. $w^*$-exposed points) by allowing the $w^*$-slices not to be parallel.

2. A “Transfer Theorem” for property (CI). In this section we will prove a “transfer theorem” which is analogous to the corresponding one for property (I) [2]. For other “transfer theorems” see [4], [5].

In this paper all the linear operators we consider are assumed to be bounded.

**Theorem 2.** Let $T: X \to Y$ be a linear operator such that $T$ and $T^*$ are injective.

If $Y$ has an equivalent norm with property (CI), then $X$ has an equivalent norm with property (CI).

**Proof.** Recall that we denote by $\mathcal{T} (= \mathcal{T}_X)$ the topology on $X^*$ of uniform convergence on compact sets of $X$.

We decompose the proof into three steps:

1. If $T: X \to Y$ is a linear operator, then $T^* : Y^* \to X^*$ is $\mathcal{T}_Y - \mathcal{T}_X$ continuous.

   Indeed, let $\varepsilon > 0$ and let $K$ be a compact subset of $X$. Then $T(K)$ is a compact subset of $Y$, and:

   $$ T^*(\{y^* \in Y^* : \sup_{T(K)} y^* < \varepsilon\}) \subset \{x^* \in X^* : \sup_{K} x^* < \varepsilon\}. $$

2. $X$ is the dual of $(X^*, \mathcal{T})$.

   Indeed, every $x \in X$ is $w^*$-continuous on $X^*$, hence $\mathcal{T}$-continuous. On the other hand, if $\xi \in (X^*, \mathcal{T})^*$, then $\xi$ is continuous on $(B(X^*), \mathcal{T}) = (B(X^*), w^*)$, so $\xi \in X$. (Another way to see this is to observe that $\mathcal{T}$ is coarser than the Mackey topology associated to $w^*$.)

   It is now easy to deduce the following:

   **Claim.** If $H$ is a subspace of $X^*$ which is $w^*$-dense in $X^*$, then $H$ is $\mathcal{T}$-dense in $X^*$.

3. If $T: X \to Y$ is such that $T^*$ is injective, then $X$ has an equivalent norm for which $T^*(\text{Ext}(Y^*)) \subset \text{Ext}(X^*)$.

   Indeed, let $\| \cdot \|$ be the original norm of $X$, and $C = T^*(B(Y^*))$.

   Define on $X^*$ a convex $w^*$-lower-semicontinuous function by:

   $$ \psi(x^*) = \|x^*\|^* + \int_{0}^{\infty} e^{-t} \text{dist}(x^*, tC) \, dt, $$
and define the new norm on $X$ by:

$$B_{1,1}(x^*) = \{x^*: \psi(x^*) \leq 1\}.$$  

**REMARKS.** (i) To see that $\psi$ is $w^*$-lower semicontinuous ($w^*$-l.s.c.) it suffices to observe the easy (and well known) fact that for a $w^*$-compact subset $K$ of $X^*$ the function $x^* \to \text{dist}(x^*, K)$ is $w^*$-l.s.c.

(ii) The functional $\psi(x^*)$ is symmetric, i.e.: $\psi(x^*) = \psi(-x^*)$, since $C$ is, and satisfies $\|x^*\| \leq \psi(x^*) \leq 2\|x^*\|$; hence the set $\{\psi(x^*) \leq 1\}$ is the unit ball of a dual equivalent norm on $X^*$, which is simply the gauge of the set $\{\psi(x^*) \leq 1\}$.

Let $y^*_0 \in \text{Ext}(Y^*)$, and choose $t_0 > 0$ such that $|t_0 T^*(y^*_0)|^* = 1$. We want to prove that $t_0 T^*(y^*_0) = x^*_0 \in \text{Ext}(B_{1,1}(X^*))$.

Let $x^*_1, x^*_2$ be such that $2x^*_0 = x^*_1 + x^*_2$, $|x^*_1|^* = |x^*_2|^* = 1$. Then $\psi(x^*_0) = \psi(x^*_1) = \psi(x^*_2) = 1$, and by a convexity argument, and the fact that $t \to \text{dist}(x^*, tC)$ is continuous for every $x^* \in X^*$, we deduce that for every $t$, we have $2\text{dist}(x^*_0, tC) = \text{dist}(x^*_1, tC) + \text{dist}(x^*_2, tC)$.

So $\text{dist}(x^*_1, t_0 C) = \text{dist}(x^*_2, t_0 C) = 0$, but $C$ is norm closed, then $x^*_1 \in t_0 C$ and $x^*_2 \in t_0 C$.

By injectivity of $T^*$, and extremality of $y^*_0$, we deduce that $x^*_0$ is extremal.

The theorem is now an easy consequence of the above three facts. Indeed, give $X$ and $Y$ equivalent norms for which $\text{Ext}(Y^*)$ is $\mathcal{F}$-dense in $Y^*$, and $T^*(\text{Ext}(Y^*)) \subset \text{Ext}(X^*)$. Then $T^*(\text{Ext}(Y^*))$ is $\mathcal{F}$-dense in $T^*(Y^*)$ which is itself $\mathcal{F}$-dense in $X^*$. The conclusion follows.  

**REMARKS.** (i). Property $(CI)$ is hereditary (a subspace of a space with an equivalent $(CI)$-norm, has an equivalent $(CI)$-norm) if and only if the above “transfer theorem” is valid without the hypothesis “$T^*$ injective”.

The if part is trivial.

Suppose $(CI)$ is hereditary. Let $T: X \to Y$ be an injective operator. If we factorize $T$ by its image:

$$\begin{array}{ccc}
X & \xrightarrow{\ T\ } & Y \\
\downarrow S & & \uparrow T^* \\
Z = T(X) & \xleftarrow{T^*}\ & \end{array}$$

the heredity of property $(CI)$, and Theorem 2, implies that $X$ has an equivalent $(CI)$-norm if $Y$ does.

The same remark applies to Deville’s “transfer theorem” for Property $(I)$: Let $T: X \to Y$ be such that $T^*$ and $T^{**}$ are injective; then $X$ has an equivalent $(I)$-norm if $Y$ does.
(ii) It was proved in [3], that if the norm of $X$ is locally uniformly convex, then its dual norm on $X^*$ satisfies property (*I): every $w^*$-compact set is an intersection of balls.

In particular spaces $l^\infty(\Gamma)$ have equivalent $(CI)$-norms. Then, if property $(CI)$ is hereditary, every Banach space will have an equivalent $(CI)$-norm (since the spaces $l^1(\Gamma)$ have equivalent l.u.c. norms, and every Banach space is a subspace of some $l^\infty(\Gamma)$-space).

3. Applications. In [9], Whitfield and Zizler proved that every Banach space with a transfinite Schauder basis has an equivalent norm with property $(CI)$.

In [2], Deville uses his "transfer theorem" for property $(I)$ to prove that the James’ spaces $J(\eta)$ have equivalent norms with property $(I)$.

We give here a "unified" proof of these results which is simpler than Whitfield-Zizler’s proof, and give a generalization of Deville’s result on $J(\eta)$ spaces.

Recall first that a family of projections $(P_\alpha)_{0 \leq \alpha \leq \mu}$, $\mu$ ordinal, is a transfinite Schauder decomposition of the Banach space $X$ if:

(i) $P_0 = 0, P_\mu = \text{id}_X$

(ii) $P_\alpha P_\beta = P_{\text{min}(\alpha, \beta)}$ for every $\alpha, \beta \leq \mu$

(iii) $\Phi: [0, \mu] \times X \to X: \Phi(\alpha, x) = P_\alpha x$ is separately continuous.

Such a decomposition is said to be shrinking if

$$X^* = \overline{\bigcup_{\alpha < \mu} (P_{\alpha+1} - P_\alpha)(X^*)}.$$

The following theorem should be compared with Zizler’s theorem on l.u.c. renormings [10].

**Theorem 3.** Let $(P_\alpha)_{0 \leq \alpha \leq \mu}$ be a Schauder decomposition [resp. a shrinking Schauder decomposition] of the Banach space $X$. Suppose that for every $\alpha, 0 \leq \alpha < \mu$, the space $X_\alpha = (P_{\alpha+1} - P_\alpha)(X)$ has an equivalent norm with property $(CI)$ [resp. with property $(I)$]. Then the space $X$ has an equivalent norm with property $(CI)$ [resp. with property $(I)$].

"Transfer theorems" for properties $(I)$ and $(CI)$ permit the proof of the theorem to be reduced to the following special case:

**Proposition 4.** Let $(X_\alpha, \| \cdot \|_\alpha)_{\alpha \in \Gamma}$ be a family of spaces with property $(CI)$ [resp. with property $(I)$], then the space $X = (\bigoplus_{\alpha \in \Gamma} X_\alpha)_{c_0}$ has an equivalent norm with property $(CI)$ [resp. with property $(I)$].
Proof. Let \( \| \cdot \| \) be an equivalent lattice norm on \( c_0(\Gamma) \) which is \( C^\infty \) \cite{1}. (Lattice norms on \( c_0(\Gamma) \) are norms satisfying the following property: If two elements \( x = (x_\alpha)_{\alpha \in \Gamma}, \) and \( y = (y_\alpha)_{\alpha \in \Gamma} \) are such that \( |x_\alpha| \leq |y_\alpha| \) for every \( \alpha \in \Gamma, \) then \( \|x\| \leq \|y\|. \) \( C^\infty \) stands for infinitely Fréchet-differentiable.)

Define on \( X \) an equivalent norm by:

\[
\|(x_\alpha)_{\alpha \in \Gamma}\| = \|(\|x_\alpha\|_\alpha)_{\alpha \in \Gamma}\|.
\]

A direct computation shows that its dual norm on \( X^* = (\bigoplus_{\alpha \in \Gamma} X_\alpha)^* \) is given by

\[
\|(x_\alpha^*)_{\alpha \in \Gamma}\|^* = \|(\|x_\alpha^*\|_\alpha^*)_{\alpha \in \Gamma}\|^*.
\]

Let \( A \) be such that for every \( (a_\alpha)_{\alpha \in \Gamma} \in c_0(\Gamma) \) we have

\[
\frac{1}{A} \sup_{\alpha \in \Gamma} |a_\alpha| \leq \|(a_\alpha)_{\alpha \in \Gamma}\| \leq A \sup_{\alpha \in \Gamma} |a_\alpha|.
\]

First case. Property (CI).

Step 1. We first show the following:

Claim. If \( x^* = (x_\alpha^*)_{\alpha \in \Gamma} \in X^* \) is such that \( x_\alpha^* \in \text{Ext}(X_\alpha^*) \) for every \( \alpha \in \Gamma, \) and \( \|(x_\alpha^*)^*\|_{\alpha \in \Gamma} \) is a \( w^* \)-exposed point of \( l^1(\Gamma) \), then \( x^* \in \text{Ext}(X^*) \).

Proof. Let \( (a_\alpha)_{\alpha \in \Gamma} \) be an element of \( c_0(\Gamma) \) which exposes \( \|(x_\alpha^*)^*\|_{\alpha \in \Gamma} \):

\[
\|(a_\alpha)_{\alpha \in \Gamma}\| = \|(\|x_\alpha^*\|_\alpha^*)_{\alpha \in \Gamma}\|^* = \sum_{\alpha \in \Gamma} a_\alpha \|x_\alpha^*\|_\alpha^* = 1;
\]

then \( a_\alpha \geq 0 \) for every \( \alpha \in \Gamma. \)

If \( 2x^* = x_1^* + x_2^* \), and \( |x_1^*|^* = |x_2^*|^* = 1, \) then

\[
2 = 2 \sum_{\alpha \in \Gamma} a_\alpha \|x_\alpha^*\|_\alpha = \sum_{\alpha \in \Gamma} a_\alpha \|x_{1,\alpha}^*\|_\alpha - \sum_{\alpha \in \Gamma} a_\alpha \|x_{2,\alpha}^*\|_\alpha \leq 2.
\]

So \( \sum_{\alpha \in \Gamma} a_\alpha \|x_{1,\alpha}^*\|_\alpha = \sum_{\alpha \in \Gamma} a_\alpha \|x_{2,\alpha}^*\|_\alpha = 1. \)

Since \( (a_\alpha)_{\alpha \in \Gamma} \) exposes \( \|(x_\alpha^*)^*\|_{\alpha \in \Gamma} \), we have: \( \|x_{1,\alpha}^*\|_\alpha = \|x_{2,\alpha}^*\|_\alpha = \|x_\alpha^*\|_\alpha \) for every \( \alpha \in \Gamma. \) And by the extremality of \( x_\alpha^* \) for every \( \alpha, \) we have \( x^* = x_1^* = x_2^*. \)

Step 2. We will prove that the set of extreme points described in Step 1 is \( \mathcal{F} \)-dense in \( X^*. \)

Let \( \varepsilon > 0, K \subset B(X) \) be a compact subset of \( X, x^* \in X^*, |x^*|^* = 1. \)

Suppose \( K \) is convex and symmetric.

Put \( a_\alpha^* = \|x_\alpha^*\|_\alpha^*, K_\alpha = \pi_\alpha(K), \) where \( \pi_\alpha \) is the natural projection of \( X \) onto \( X_\alpha. \) Then \( K_\alpha \subset AB(X_\alpha). \)
For each $\alpha \in \Gamma$, choose $\tilde{x}_\alpha^* \in \text{Ext}(X_\alpha^*)$, $\|\tilde{x}_\alpha^*\|_\alpha = 1$, $\mu_\alpha^* \geq 0$, such that $\|\mu_\alpha^*\tilde{x}_\alpha^* - x_\alpha^*\|_{K_\alpha} \leq \epsilon a_\alpha^*$.

Choose $\Gamma_0 \subset \Gamma$, $\Gamma_0$ finite, such that $\sum_{\alpha \in \Gamma \setminus \Gamma_0} a_\alpha^* \leq \epsilon$.

For $\alpha \in \Gamma_0$, put $\lambda_\alpha^* = \mu_\alpha^*$, and for $\alpha \in \Gamma \setminus \Gamma_0$, put $\lambda_\alpha^* = a_\alpha^*$. Then $(\lambda_\alpha^*)_{\alpha \in \Gamma} \in l^1(\Gamma)$.

Choose $(\tilde{\lambda}_\alpha^*)_{\alpha \in \Gamma}$ to be a $w^*$-exposed point of $l^1(\Gamma)$ such that:

$$\|(\tilde{\lambda}_\alpha^*)_{\alpha \in \Gamma}\|^* = \|(\lambda_\alpha^*)_{\alpha \in \Gamma}\|^* \quad \text{and} \quad \sum_{\alpha \in \Gamma} |\tilde{\lambda}_\alpha^* - \lambda_\alpha^*| \leq \epsilon.$$ 

By Step 1, $(\tilde{\lambda}_\alpha^*\tilde{x}_\alpha^*)_{\alpha \in \Gamma}$ is an extreme point of $X^*$, and

$$\|(\tilde{\lambda}_\alpha^*\tilde{x}_\alpha^* - x_\alpha^*)_{\alpha \in \Gamma}\|_{K} \leq \sum_{\alpha \in \Gamma} \|(\tilde{\lambda}_\alpha^*\tilde{x}_\alpha^* - x_\alpha^*)\|_{K_\alpha}$$

$$\leq \sum_{\alpha \in \Gamma_0} \{A|\tilde{\lambda}_\alpha^* - \lambda_\alpha^*| + \|\tilde{\lambda}_\alpha^*\tilde{x}_\alpha^* - x_\alpha^*\|_{K_\alpha}\} + A \sum_{\alpha \in \Gamma \setminus \Gamma_0} \|\tilde{\lambda}_\alpha^*\tilde{x}_\alpha^* - x_\alpha^*\|_{\alpha}$$

$$\leq 2A\epsilon + A \sum_{\alpha \in \Gamma \setminus \Gamma_0} \{\tilde{\lambda}_\alpha^* - \lambda_\alpha^*| + \|\tilde{\lambda}_\alpha^*\tilde{x}_\alpha^*\|_{\alpha} + \|x_\alpha^*\|_{\alpha}\} \leq 5A\epsilon.$$

**Second case. Property (I).** Recall first that a Banach space has property (I) if and only if the set of $w^*$-denting points of $B(X^*)$ is norm dense in $S(X^*)$ [3].

**Step 1.** We will show the following:

*Claim.* If $x^* = (x_\alpha^*)_{\alpha \in \Gamma} \in X^*$ is such that $x_\alpha^* \in w^*$-dent($X_\alpha^*$) for every $\alpha \in \Gamma$, and $(\|x_\alpha^*\|_{\alpha})_{\alpha \in \Gamma}$ is a $w^*$-strongly exposed point of $l^1(\Gamma)$, then $x^* \in w^*$-dent($X^*$).

*Proof.* Put $a_\alpha^* = \|x_\alpha^*\|_{\alpha}$, and let $(a_\alpha)_{\alpha \in \Gamma}$ be such that $\|(a_\alpha)_{\alpha \in \Gamma}\| = \|(a_\alpha)_{\alpha \in \Gamma}\|^* = \sum_{\alpha \in \Gamma} a_\alpha a_\alpha^* = 1$; then $a_\alpha \geq 0$ for every $\alpha$.

Let $\epsilon > 0$, and choose $\Gamma_0 \subset \Gamma$, $\Gamma_0$ finite such that $\sum_{\alpha \in \Gamma \setminus \Gamma_0} a_\alpha^* \leq \epsilon$ and $\inf_{\Gamma_0} a_\alpha^* = \delta > 0$.

Choose $\eta_1 > 0$, and $x_\alpha \in X_\alpha$, for every $\alpha \in \Gamma_0$, such that $\|x_\alpha\|_{\alpha} = 1$, and

$$x_\alpha(y_\alpha^*) \geq a_\alpha^*(1 - \eta_1) \quad \Rightarrow \quad \|y_\alpha^* - x_\alpha^*\|_{\alpha} \leq \epsilon a_\alpha^*.$$ 

For $\alpha \in \Gamma \setminus \Gamma_0$, pick any $x_\alpha \in X_\alpha$, $\|x_\alpha\|_{\alpha} = 1$.

Choose $\epsilon' \leq \epsilon$, such that $1 - \eta_1 \leq (1 - \epsilon'/\delta)/(1 + \epsilon'/\delta)$, and let $\eta_2 > 0$ be such that

$$\sum_{\alpha \in \Gamma} a_\alpha b_\alpha^* \geq 1 - \eta_2 \quad \Rightarrow \quad \sum_{\alpha \in \Gamma} |b_\alpha^* - a_\alpha^*| \leq \epsilon'.$$


Now if $y^* = (y^*_a)_{a \in \Gamma}$ is such that:
\[
\sum_{a \in \Gamma} a_\alpha x_\alpha(y^*_a) \geq 1 - \eta_2 \quad \text{and} \quad |y^*|^* = \|\|(y^*_a)^*\|_{\alpha}\|_{\alpha}^* \leq 1,
\]
then
\[
\sum_{a \in \Gamma} a_\alpha \|y^*_a\|_{\alpha}^* \geq 1 - \eta_2 \quad \text{and} \quad \|(x_\alpha(y^*_a))_{a \in \Gamma}\|^* \leq 1.
\]
So we have
\[
\sum_{a \in \Gamma} |a_\alpha^* - \|y^*_a\|_{\alpha}^*| \leq \varepsilon' \quad \text{and} \quad \sum_{a \in \Gamma} |a_\alpha^* - x_\alpha(y^*_a)| \leq \varepsilon'.
\]
For $\alpha \in \Gamma_0$, we have:
\[
x_\alpha \left( \frac{y^*_a}{\|y^*_a\|_{\alpha}} \right) \geq \frac{a_\alpha^* - \varepsilon'}{a_\alpha^* + \varepsilon'} \geq \frac{1 - \varepsilon'/\delta}{1 + \varepsilon'/\delta} \geq 1 - \eta_1
\]
from this we deduce $\|y^*_a - x_\alpha^*\|_{\alpha}^* \leq \varepsilon a_\alpha^* + |a_\alpha^* - \|y^*_a\|_{\alpha}^*|.$
Then
\[
\sum_{a \in \Gamma} \|y^*_a - x_\alpha^*\|_{\alpha}^* \\
\leq \sum_{a \in \Gamma_0} \{ \varepsilon a_\alpha^* + |a_\alpha^* - \|y^*_a\|_{\alpha}^*| \} + \sum_{a \in \Gamma \setminus \Gamma_0} \{ \|x_\alpha^*\|_{\alpha}^* + \|y^*_a\|_{\alpha}^* \} \\
\leq A \varepsilon + \varepsilon + \sum_{a \in \Gamma \setminus \Gamma_0} \{ \|y^*_a\|_{\alpha}^* - a_\alpha^* + a_\alpha^* \} \leq (A + 4) \varepsilon
\]
which concludes the proof of $x^* \in w^*-\text{dent}(X^*)$.

**Step 2.** We will show that the set of $w^*$-denting points described in Step 1 is norm dense in $X^*$.

Let $\varepsilon > 0$, and $x^* = (x^*_a)_{a \in \Gamma} \in X^*$, $|x^*|^* = 1$. Put $a_\alpha^* = \|x_\alpha^*\|_{\alpha}^*.$

For every $\alpha \in \Gamma$, choose $\tilde{x}_\alpha^* \in w^*-\text{dent}(X^*_a)$ such that $\|\tilde{x}_\alpha^*\|_{\alpha}^* = 1$ and $\|a_\alpha^*\tilde{x}_\alpha^* - x_\alpha^*\|_{\alpha}^* \leq \varepsilon a_\alpha^*.$

Choose a $w^*$-strongly exposed point $(\tilde{a}_\alpha^*)_{a \in \Gamma}$ of $l^1(\Gamma)$ such that $\|(\tilde{a}_\alpha^*)_{a \in \Gamma}\|^* = 1$ and $\sum_{a \in \Gamma} |a_\alpha^* - \tilde{a}_\alpha^*| \leq \varepsilon$. We can suppose $\tilde{a}_\alpha^* \geq 0$ for every $\alpha$.

Then $\tilde{x}^* = (a_\alpha^*\tilde{x}_\alpha^*)_{a \in \Gamma}$ is a $w^*$-denting point of $X^*$, $|\tilde{x}^*|^* = 1$, and
\[
\sum_{a \in \Gamma} \|\tilde{a}_\alpha^*\tilde{x}_\alpha^* - x_\alpha^*\|_{\alpha} \leq \sum_{a \in \Gamma} |\tilde{a}_\alpha^* - a_\alpha^*| + \|a_\alpha^*\tilde{x}_\alpha^* - x_\alpha^*\|_{\alpha} \leq (A + 1) \varepsilon.
\]
This achieves the proof of Proposition 4. \qed

**Proof of Theorem 3.** For every $\alpha, 0 \leq \alpha < \mu$, denote by $\pi_\alpha$ the operator $(P_{\alpha+1} - P_\alpha)$ when considered as an operator from $X$ into $(P_{\alpha+1} - P_\alpha)(X) = X_\alpha$. 

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Standard argument shows that for every $x \in X$

$$(\|P_{\alpha+1}x - P_\alpha x\|)_{0 \leq \alpha < \mu} \in c_0([0, \mu]).$$

Let $\| \cdot \|_\alpha$ be an equivalent norm on $X_\alpha$ with property (CI) [resp. with property (I)]. We can suppose $\| \cdot \|_\alpha \leq \| \cdot \|$ on $X_\alpha$, for each $\alpha$, where $\| \cdot \|$ is the norm induced by $X$ on $X_\alpha$.

Let

$$T: X \to Y = \left( \bigoplus_{0 \leq \alpha < \mu} (X_\alpha, \| \cdot \|_\alpha) \right) \subset c_0 : Tx = (\pi_\alpha(x))_{0 \leq \alpha < \mu}.$$ 

Then $T$ is continuous and injective.

The operator $T^*: Y^* \to X^*$ is given by

$$T^*((x_\alpha^*)_{0 \leq \alpha < \mu}) = \sum_{0 \leq \alpha < \mu} \pi_\alpha^*(x_\alpha^*).$$

Then $T^*$ is injective.

Moreover, $T^*(Y^*)$ is norm dense in $X^*$ when the decomposition is shrinking [since $\pi_\alpha^*(X^*_\alpha) = (P_{\alpha+1}^* - P_\alpha^*)(X^*)$].

The theorem follows in case of property (CI) by our "transfer theorem", and in case of property (I) by Deville’s “transfer theorem” [2].

Using techniques of [8], it can be proved.

**PROPOSITION 5.** Let $X$ be a Banach space with a transfinite Schauder basis, and $Y$ a space with an equivalent norm with property (CI). Then the space $X \hat{\otimes}_\rho Y$ has an equivalent norm with property (CI), for every tensor norm $\rho$.

The idea of the proof is to show that if $(P_\alpha)_{0 \leq \alpha \leq \mu}$ is a Schauder basis of $X$, then the family $(P_\alpha \otimes \text{Id}_Y)_{0 \leq \alpha \leq \mu}$ is a Schauder decomposition of $X \hat{\otimes}_\rho Y$, and to apply Theorem 3.

**REMARK.** If $(X_n)_{n \geq 1}$ is a sequence of Banach spaces with equivalent (CI)-norms, then $\bigoplus_{n=1}^\infty X_n$ has an equivalent (CI)-norm. Indeed, consider the operator $T: (\bigoplus_{n=1}^\infty X_n)_{l^\infty} \to (\bigoplus_{n=1}^\infty X_n)_{c_0}$.

It is not clear whether the family of spaces with equivalent (CI)-norms is stable under (uncountable) $l^\infty$-sums.
Acknowledgment. I want to thank Robert Deville for several helpful discussions concerning this work.

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Received February 12, 1987 and in revised form June 22, 1987.

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