FUNCTIONS IN $\mathbb{R}^2(E)$ AND POINTS OF THE FINE INTERIOR

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Let $E \subset \mathbb{C}$ be a set that is compact in the usual topology. Let $m$ denote 2-dimensional Lebesgue measure. We denote by $R_0(E)$ the algebra of rational functions with poles off $E$. For $p \geq 1$, let $L^p(E) = L^p(E, dm)$. The closure of $R_0(E)$ in $L^p(E)$ will be denoted by $R^p(E)$.

In this paper we study the behavior of functions in $R^2(E)$ at points of the fine interior of $E$. We prove that if $U \subset E$ is a finely open set of bounded point evaluations for $R^2(E)$, then there is a finely open set $V \subset U$ such that each $x \in V$ is a bounded point derivation of all orders for $R^2(E)$. We also prove that if $R^2(E) \neq L^2(E)$, there is a subset $S \subset E$ having positive measure such that if $x \in S$ each function in $\bigcup_{p>2} R^p(E)$ is approximately continuous at $x$. Moreover, this approximate continuity is uniform on the unit ball of a normed linear space that contains $\bigcup_{p>2} R^p(E)$.

1. Introduction. Let $E \subset \mathbb{C}$ be a set that is compact in the usual topology. Let $m$ denote 2-dimensional Lebesgue measure. We denote by $R_0(E)$ the algebra of rational functions with poles off $E$. For $p \geq 1$, let $L^p(E) = L^p(E, dm)$. The closure of $R_0(E)$ in $L^p(E)$ will be denoted by $R^p(E)$.

In [16] we studied the smoothness properties of functions in $R^p(E)$, $p > 2$, at bounded point evaluations. The case $p = 2$ is different. Fernström has shown in [7] that $R^2(E)$ can be unequal to $L^2(E)$ without there being any bounded point evaluations for $R^2(E)$. In this paper we use the fine topology introduced by Cartan to study the behavior of functions in $R^2(E)$ at points of the fine interior of $E$. We prove that if $U \subset E$ is a finely open set of bounded point evaluations for $R^2(E)$, then there is a finely open set $V \subset U$ such that each $x \in V$ is a bounded point derivation of all orders for $R^2(E)$. Finely open sets of this kind are contained in certain "Swiss cheese sets". We also prove that if $R^2(E) \neq L^2(E)$, there is a set $S \subset E$ having positive measure such that if $x \in S$ each function in $\bigcup_{p>2} R^p(E)$ is approximately continuous at $x$. Moreover, this approximate continuity is uniform on the unit ball of a normed linear space that contains $\bigcup_{p>2} R^p(E)$.
2. Functions in $R^2(E)$ defined on finely open sets. When $R^2(E) \neq L^2(E)$, the fine interior is non-empty. This follows from a theorem of Havin [10] that we shall now state. Let $\Delta(x, r)$ denote the open disk of radius $r$ centered at $x$. Let $C_2$ denote the Wiener capacity as defined in [11].

**Theorem 2.1 (Havin).** Let $E \subset \mathbb{C}$ be a compact set without interior in the usual topology. Then $R^2(E) \neq L^2(E)$ if and only if there is a set $S \subset E$ having positive measure such that for $x \in S$,

$$\limsup_{r \to 0} \frac{C_2(\Delta(x, r) \setminus E)}{r^2} = 0.$$ 

One way to relate this theorem to fine interior points is to use Wiener's criterion. Let

$$A_n(x) = \left\{ z : \frac{1}{2^{n+1}} \leq |z - x| \leq \frac{1}{2^n} \right\}.$$ 

Then $x$ is a fine interior point of $E$ if and only if

$$\sum_{n=1}^{\infty} nC_2(A_n(x) \setminus E) < \infty.$$ 

For a proof see [11, p. 220]. It follows from Wiener's criterion and Theorem 2.1 that if $R^2(E) \neq L^2(E)$, the fine interior has positive measure.

Each point of the fine interior has a system of fine neighborhoods that are compact in the usual topology (see [2]). Debiard and Gaveau observed in [5] that if the fine interior of $E$ is nonempty, it satisfies the Baire property: The intersection of a countable number of open dense sets in $E$ is always dense in $E$. We give the following proof.

**Proposition 2.1.** If $E$ is a set having non-empty fine interior $E'$, then $E'$ satisfies the Baire property.

**Proof.** Let $D_1, D_2, \ldots$ be a sequence of finely open dense sets in $E$. We must show that for each finely open set $U \subset E'$, $U \cap (\bigcap_{i=1}^{\infty} D_i) \neq \emptyset$. Now $U \cap D_1 \neq \emptyset$ because $D_1$ is dense. Pick $x_1 \in U \cap D_1$ and a fine neighborhood $B_1$ of $x_1$ such that $B_1$ is compact in the usual topology. Since $D_2$ is dense, there exists $x_2 \in B_1 \cap D_2$ and a fine neighborhood $B_2$ of $x_2$ compact in the usual topology such that $B_2 \subset B_1 \cap D_2$. Continuing in this way, we get a sequence $\{B_n\}$ of compact finely open sets such that $B_n \subset B_{n-1} \cap D_n$. Since $B_1$ is compact, the finite intersection
property implies that $\bigcap_1^\infty B_n \neq \emptyset$. Hence $\bigcap_1^\infty D_n \neq \emptyset$, and $E'$ satisfies the Baire property.

Each point of $E$ is a point of full area density for $E$ (see [6, p. 170]). Moreover, one can use results in [1, p. 43], due to Beurling to show that any finely open subset of $\mathbb{C}$ includes circles of arbitrarily small radii centered at each of its points. Next we define those points of the fine interior at which the functions in $R^2(E)$ may have smoothness properties.

**Definition 2.1.** A point $x \in E$ is a bounded point evaluation (BPE) for $R^2(E)$ if there exists a constant $C$ such that

$$|f(x)| \leq C \|f\|_{L^2(E)}$$

for all $f \in R_0(E)$.

**Definition 2.2.** A point $x \in E$ is a bounded point derivation (BPD) of order $s$ for $R^2(E)$ if there exists a constant $C$ such that

$$|f^{(s)}(x)| \leq C \|f\|_{L^2(E)}$$

for all $f \in R_0(E)$.

If $x$ is a BPE for $R^2(E)$, the map $f \mapsto f(x)$ extends from $R_0(E)$ to a bounded linear functional on $R^2(E)$. Let $N(x)$ equal the norm of this linear functional. We will need the following lemma and proposition.

**Lemma 2.1.** The function $N$ is lower semi-continuous on the set of BPE's for $R^2(E)$.

For the proof see [16, p. 72].

The proof of the next statement is in [15, p. 148].

**Proposition 2.2.** Let $f: X \rightarrow \mathbb{R}$ be a lower semi-continuous function on a Baire space $X$. Every non-empty open set in $X$ contains a non-empty open set on which $f$ is uniformly bounded.

If $X \subseteq \mathbb{C}$ is compact in the usual topology, we let $R(X)$ denote the closure of $R_0(X)$ in the sup norm on $X$.

**Theorem 2.2.** Suppose that $U \subseteq E'$ is a finely open set such that every point of $U$ is a BPE for $R^2(E)$. Then there is a compact set $X \subseteq U$ such that $X$ has non-empty fine interior, and for each $f \in R^2(E)$, $f|_X \in R(X)$.

**Proof.** Let $U \subseteq E'$ be a finely open set of BPE's for $R^2(E)$. By Proposition 2.2 there is a finely open set $V \subseteq U$ on which the $R^2(E)$
norm of "evaluation at \( x \)" is bounded. Let \( X \subset V \) be a set that is compact in the usual topology and that contains a finely open set. Let \( f \in R^2(E) \). Then there is a sequence \( \{ f_n \} \) in \( R_0(E) \) such that \( \| f_n - f \|_{L^2(E)} \to 0 \). By the choice of \( X \) there is a constant \( C \) such that

\[
\sup_{z \in X} |f_n(z) - f_m(z)| \leq C \| f_n - f_m \|_{L^2(E)}.
\]

Thus the sequence obtained by restricting the \( f_n \)'s to \( X \) converges in \( R(X) \) to the restriction of \( f \) to \( X \). We conclude that \( f|_X \in R(X) \).

**Let \( X \) be as in the above theorem.**

**COROLLARY 2.2.** Every point of \( X \) is a BPD of all orders for \( R^2(E) \).

**Proof.** Let \( x \in X \), and let \( s \) be a positive integer. By [4], \( x \) is a BPD of all orders for \( R(X) \). Hence there is a constant \( C \) such that if \( f \in R_0(E), \| f^{(s)}(x) \| \leq C \| f \|_X \) where \( \| \|_X \) denotes the sup norm on \( X \). By the choice of \( X \) (see the proof of Theorem 2.2), there is another constant \( C' \) such that \( \| f \|_X \leq C' \| f \|_{L^2(E)} \). Taken together these inequalities imply that \( x \) is a BPD of order \( s \) for \( R^2(E) \).

There do exist examples of compact nowhere dense sets \( E \) that contain finely open subsets of BPE's for \( R^2(E) \).

**3. The case of no BPE's for \( R^2(E) \).** In this section we show that whenever \( R^2(E) \neq L^2(E) \), there is a subset of \( E \) on which functions in \( R^2(E) \) that are not continuous may still have smoothness properties. To describe this set of points we begin by letting \( \phi \) be a positive function defined on \((0, \infty)\) such that \( \phi \) is decreasing and \( \lim_{r \to 0^+} \phi(r) = \infty \).

**DEFINITION 3.1.** A point \( x \in E \) is a BPE of type \( \phi \) for \( R^2(E) \) if there is a constant \( C \) such that

\[
|f(x)| \leq C \left\{ \int_E |f(z)|^2 \phi(|z - x|) \, dm(z) \right\}^{1/2}
\]

for all \( f \in R_0(E) \).

Fernström introduced BPE's of type \( \phi(r) = \log^\beta 1/r \) for \( \beta > 1 \) in [7]. The proof of the following theorem is similar to the proof of Theorem 3 in [8].

**THEOREM 3.1.** Let \( E \subset \mathbb{C} \) be compact. Then \( x \) is a BPE of type \( \phi \) for \( R^2(E) \) if and only if

\[
\sum_{n=1}^{\infty} \phi^{-1}(2^{-n}) 2^{2n} C_2(A_n(x) \setminus E) < \infty.
\]
For certain $\phi$'s the above series will converge on a set of positive measure whenever $R^2(E) \neq L^2(E)$.

**Definition 3.2.** A non-negative, real-valued function $\phi$ defined on $(0, \infty)$ is *nice* if it satisfies the following conditions:

(i) There is an $r_0 > 0$ such that $\phi$ is decreasing on $(0, r_0)$, and $\lim_{r \to 0^+} \phi(r) = +\infty$.

(ii) $\lim_{r \to 0^+} r \cdot \phi(r) = 0$, and there is an $s_0 > 0$ such that $1/(r \cdot \phi(r))$ is decreasing on $(0, s_0)$; and

(iii) there is a $t_0 > 0$ such that $\int_{t_0}^{r_0} (1/(r \cdot \phi(r))) \, dr < \infty$.

**Examples.**

(1) $\phi(r) = \frac{1}{r^\alpha}, \quad 0 < \alpha < 1$.

(2) $\phi(r) = \log^\beta \frac{1}{r}, \quad \beta > 1, \quad 0 < r \leq 1, \quad \phi(r) = 0 \quad \text{for } r > 1$.

(3) $\phi(r) = \left( \log \frac{1}{r} \right) \left[ \log \left( \log \frac{1}{r} \right) \right]^\beta, \quad \beta > 1, \quad 0 < r \leq 1/2,

$\phi(r) = (\log 2) \cdot [\log(\log 2)]^\beta, \quad \text{for } r > 1/2$.

Condition (iii) of Definition 3.2 combined with Theorem 2.1 and Theorem 3.1 imply the following:

**Theorem 3.2.** Let $E \subset \mathbb{C}$ be a compact set without interior in the usual topology. Let $\phi$ be nice. Then if $R^2(E) \neq L^2(E)$ the set of BPE's of type $\phi$ has positive measure.

Let $S$ denote the set of $x \in E$ such that $\limsup_{r \to 0} C_2(\Delta(x, r) \setminus E)/r^2 = 0$. Suppose that $x \in E$ is a BPE of type $\phi$. We define a norm $\| \|_\phi$ on functions in $L^2(E)$ as follows:

$$\| f \|_\phi = \sup_{y \in S} \| f \cdot \phi(|z - y|) \cdot \phi(|z - x|) \|_{L^2(E)}$$

where $f$ is a function of $z$. Let $R^\phi(E)$ be the closure of $R_0(E)$ in this norm. For certain $\phi$ such as $\phi(r) = \log^\beta 1/r$, $\beta > 0$, Hölder's inequality implies that $\bigcup_{p \geq 2} R^p(E) \subset R^\phi(E)$.

Now suppose that $x$ is a BPE of type $\phi$. Let $L^2(E, \phi \, dm)$ be the space of all complex measurable functions $f$ defined on $E$ such that
\[
\{ \int_E |f^2(z)| \cdot \varphi(|z - x|) \, dm(z) \}^{1/2} < \infty.
\]
By a well known theorem [14], there is a function \( g \in L^2(E, \varphi \, dm) \) such that
\[
f(x) = \int_E f \cdot g \cdot \varphi(|z - x|) \, dm(z)
\]
for all \( f \in R_0(E) \). We have the following theorem.

**Theorem 3.3.** Let \( \varphi \) be a nice function such that \( \int_0^1 \varphi^3(r) r \, dr < \infty \). Suppose that \( x \in E \) is a BPE of type \( \varphi \). Let \( \varepsilon > 0 \). Then there is a set \( A \subset E \) having full area density at \( x \) such that if \( y \in A \) and \( f \in R_0(E) \),
\[
|f(y) - f(x)| \leq \varepsilon \|f\|_\varphi.
\]

We will give an outline of the proof. For more details see [16].

**Outline of Proof of Theorem 3.3.** Let \( \varepsilon > 0 \). Let \( g \in L^2(E, \varphi \, dm) \) be the representing function for \( x \) as defined above. Then if
\[
c(y) = \int_E \frac{z - x}{z - y} g(z) \cdot \varphi(|z - x|) \, dm(z)
\]
is defined and \( \neq 0 \),
\[
\frac{1}{c(y)} \frac{z - x}{z - y} g(z) \cdot \varphi(|z - x|)
\]
is a representing function for \( y \). Among the points where \( c(y) \) is defined are those in the set \( A_1 \) of the following lemma:

**Lemma 3.1.** For each \( \delta > 0 \), the sets
\[
A_1 = \left\{ y \in \mathbb{C} : |y - x| \int_E \frac{|g(z)| \cdot \varphi(|z - x|)}{|z - y|} \, dm(z) < \delta \right\}
\]
and
\[
A_2 = \left\{ y \in \mathbb{C} : |y - x| \left[ \int_E \frac{|g(z)|^2 \cdot \varphi(|z - x|)}{|z - y|^2 \cdot \varphi^2(|z - y|)} \, dm(z) \right]^{1/2} < \delta \right\}
\]
have full area density at \( x \).

The proof uses the properties of the nice function \( \varphi \) and is similar to that of Lemma 3.3 in [16]. Now if \( c(y) \) is defined and \( \neq 0 \), and if
\[ f \in R_0(E), \text{ we have} \]
\[
\begin{align*}
  f(y) - f(x) &= \frac{1}{c(y)} \int_E \frac{[f(z) - f(x)] \cdot (z - x)}{(z - y)} g(z) \cdot \varphi(|z - x|) \, dm(z) \\
  &= \frac{1}{c(y)} \int_E \left[ f(z) - f(x) \right] \left[ 1 + \frac{y - x}{z - y} \right] g(z) \cdot \varphi(|z - x|) \, dm(z) \\
  &= \frac{y - x}{c(y)} \int_E \left[ f(z) - f(x) \right] \frac{\varphi(|z - y|)}{\varphi(|z - y|)} g(z) \cdot \varphi(|z - x|) \, dm(z).
\end{align*}
\]

From Hölder's inequality, the assumption that \( x \) is a BPE of type \( \varphi \), and the assumption that \( \int_0^1 \varphi^3(r) \, dr < \infty \), it follows that
\[
|f(y) - f(x)| \leq C \frac{|y - x|}{c(y)} \|f\|_\varphi \left\{ \int_E \frac{|g(z)|^2 \cdot \varphi(|z - x|)}{|z - y|^2 \cdot \varphi^2(|z - y|)} \, dm(z) \right\}^{1/2}
\]
where \( C \) is independent of \( f \).

Choose \( \delta > 0 \) so small that if \( y \in A_1 \cap A_2 \) (see Lemma 3.1), then
\[
\frac{C}{c(y)} |y - x| \left\{ \int_E \frac{|g(z)|^2 \cdot \varphi(|z - x|)}{|z - y|^2 \cdot \varphi^2(|z - y|)} \, dm(z) \right\}^{1/2} < \varepsilon.
\]

Lemma 3.1 implies that the set \( A = A_1 \cap A_2 \) has full area density at \( x \). Moreover, if \( y \in A \) and \( f \in R_0(E) \),
\[
|f(y) - f(x)| \leq \varepsilon \|f\|_\varphi.
\]

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P. D. Allenby and M. Sears, Extension of flows via discontinuous functions .................................................. 209
Arthur William Apter and Moti Gitik, Some results on Specker’s problem .................................................. 227
Shiu-Yuen Cheng and Johan Tysk, An index characterization of the catenoid and index bounds for minimal surfaces in $\mathbb{R}^4$ ......................... 251
Mikihiro Hayashi and Mitsuru Nakai, Point separation by bounded analytic functions of a covering Riemann surface ............................ 261
Charles Philip Lanski, Differential identities, Lie ideals, and Posner’s theorems ........................................... 275
Erich Miersemann, Asymptotic expansion at a corner for the capillary problem .............................................. 299
Dietrich W. Paul, Theory of bounded groups and their bounded cohomology ............................................. 313
Ibrahim Salama, Topological entropy and recurrence of countable chains ..................................................... 325
Zbigniew Slodkowski, Pseudoconvex classes of functions. I. Pseudoconcave and pseudoconvex sets .................. 343
Alfons Van Daele, $K$-theory for graded Banach algebras. II ...................... 377
Edwin Wolf, Functions in $R^2(E)$ and points of the fine interior .................. 393