Pacific Journal of Mathematics

FUNCTIONS IN $R^2(E)$ AND POINTS OF THE FINE INTERIOR

EDWIN WOLF

Vol. 134, No. 2 June 1988

PACIFIC JOURNAL OF MATHEMATICS
Vol. 134, No. 2, 1988

FUNCTIONS IN $R^2(E)$ AND POINTS OF THE FINE INTERIOR

EDWIN WOLF

Let $E \subset \mathbb{C}$ be a set that is compact in the usual topology. Let m denote 2-dimensional Lebesgue measure. We denote by $R_0(E)$ the algebra of rational functions with poles off E. For $p \geq 1$, let $L^p(E) = L^p(E, dm)$. The closure of $R_0(E)$ in $L^p(E)$ will be denoted by $R^p(E)$.

In this paper we study the behavior of functions in $R^2(E)$ at points of the fine interior of E. We prove that if $U \subset E$ is a finely open set of bounded point evaluations for $R^2(E)$, then there is a finely open set $V \subset U$ such that each $x \in V$ is a bounded point derivation of all orders for $R^2(E)$. We also prove that if $R^2(E) \neq L^2(E)$, there is a subset $S \subset E$ having positive measure such that if $x \in S$ each function in $\bigcup_{p>2} R^p(E)$ is approximately continuous at x. Moreover, this approximate continuity is uniform on the unit ball of a normed linear space that contains $\bigcup_{p>2} R^p(E)$.

1. Introduction. Let $E \subset \mathbb{C}$ be a set that is compact in the usual topology. Let m denote 2-dimensional Lebesgue measure. We denote by $R_0(E)$ the algebra of rational functions with poles off E. For $p \geq 1$, let $L^p(E) = L^p(E, dm)$. The closure of $R_0(E)$ in $L^p(E)$ will be denoted by $R^p(E)$.

In **[16]** we studied the smoothness properties of functions in $R^p(E)$, $p > 2$, at bounded point evaluations. The case $p = 2$ is different. Fernström has shown in [7] that $R^2(E)$ can be unequal to $L^2(E)$ without there being any bounded point evaluations for $R^2(E)$. In this paper we use the fine topology introduced by Cartan to study the behavior of functions in $R^2(E)$ at points of the fine interior of E. We prove that if $U \subset E$ is a finely open set of bounded point evaluations for $R^2(E)$, then there is a finely open set $V \subset U$ such that each $x \in V$ is a bounded point derivation of all orders for $R^2(E)$. Finely open sets of this kind are contained in certain "Swiss cheese sets". We also prove that if $R^2(E) \neq L^2(E)$, there is a set $S \subset E$ having positive measure such that if $x \in S$ each function in $\bigcup_{p>2} R^p(E)$ is approximately continuous at x. Moreover, this approximate continuity is uniform on the unit ball of a normed linear space that contains $\bigcup_{p>2} R^p(E)$.

2. Functions in $R^2(E)$ defined on finely open sets. When $R^2(E) \neq L^2(E)$, the fine interior is non-empty. This follows from a theorem of Havin [**10**] that we shall now state. Let $\Delta(x, r)$ denote the open disk of radius r centered at x. Let C_2 denote the Wiener capacity as defined in [**11**].

THEOREM 2.1 (*Havin*). *Let $E \subset \mathbb{C}$ be a compact set without interior in the usual topology. Then $R^2(E) \neq L^2(E)$ if and only if there is a set $S \subset E$ having positive measure such that for $x \in S$,*

$$\limsup_{r \to 0} \frac{C_2(\Delta(x, r)\backslash E)}{r^2} = 0.$$

One way to relate this theorem to fine interior points is to use Wiener's criterion. Let

$$A_n(x) = \left\{ z : \frac{1}{2^{n+1}} \leq |z - x| \leq \frac{1}{2^n} \right\}.$$

Then x is a fine interior point of E if and only if

$$\sum_{n=1}^{\infty} n C_2(A_n(x)\backslash E) < \infty.$$

For a proof see [**11**, p. 220]. It follows from Wiener's criterion and Theorem 2.1 that if $R^2(E) \neq L^2(E)$, the fine interior has positive measure.

Each point of the fine interior has a system of fine neighborhoods that are compact in the usual topology (see [**2**]). Debiard and Gaveau observed in [**5**] that if the fine interior of E is nonempty, it satisfies the Baire property: The intersection of a countable number of open dense sets in E is always dense in E. We give the following proof.

PROPOSITION 2.1. *If E is a set having non-empty fine interior E', then E' satisfies the Baire property.*

Proof. Let $D_1, D_2, \ldots$ be a sequence of finely open dense sets in E. We must show that for each finely open set $U \subset E'$, $U \cap (\bigcap_1^\infty D_i) \neq \varnothing$. Now $U \cap D_1 \neq \varnothing$ because D_1 is dense. Pick $x_1 \in U \cap D_1$ and a fine neighborhood B_1 of x_1 such that B_1 is compact in the usual topology. Since D_2 is dense, there exists $x_2 \in B_1 \cap D_2$ and a fine neighborhood B_2 of x_2 compact in the usual topology such that $B_2 \subset B_1 \cap D_2$. Continuing in this way, we get a sequence $\{B_n\}$ of compact finely open sets such that $B_n \subset B_{n-1} \cap D_n$. Since B_1 is compact, the finite intersection property implies that $\bigcap_1^\infty B_n \neq \emptyset$. Hence $\bigcap_1^\infty D_n \neq \emptyset$, and E' satisfies the Baire property.

Each point of E is a point of full area density for E (see [**6**, p. 170]). Moreover, one can use results in [**1**, p. 43], due to Beurling to show that any finely open subset of $\mathbb{C}$ includes circles of arbitrarily small radii centered at each of its points. Next we define those points of the fine interior at which the functions in $R^2(E)$ may have smoothness properties.

DEFINITION 2.1. A point $x \in E$ is a bounded point evaluation (BPE) for $R^2(E)$ if there exists a constant C such that

$$|f(x)| \leq C\|f\|_{L^2(E)}$$

for all $f \in R_0(E)$.

DEFINITION 2.2. A point $x \in E$ is a bounded point derivation (BPD) of order s for $R^2(E)$ if there exists a constant C such that

$$|f^{(s)}(x)| \leq C\|f\|_{L^2(E)}$$

for all $f \in R_0(E)$.

If x is a BPE for $R^2(E)$, the map $f \mapsto f(x)$ extends from $R_0(E)$ to a bounded linear functional on $R^2(E)$. Let $N(x)$ equal the norm of this linear functional. We will need the following lemma and proposition.

LEMMA 2.1. *The function N is lower semi-continuous on the set of BPE's for $R^2(E)$.*

For the proof see [**16**, p. 72].

The proof of the next statement is in [**15**, p. 148].

PROPOSITION 2.2. *Let $f: X \mapsto \mathbb{R}$ be a lower semi-continuous function on a Baire space X. Every non-empty open set in X contains a non-empty open set on which f is uniformly bounded.*

If $X \subset \mathbb{C}$ is compact in the usual topology, we let $R(X)$ denote the closure of $R_0(X)$ in the sup norm on X.

THEOREM 2.2. *Suppose that $U \subset E'$ is a finely open set such that every point of U is a BPE for $R^2(E)$. Then there is a compact set $X \subset U$ such that X has non-empty fine interior, and for each $f \in R^2(E)$, $f|_X \in R(X)$.*

Proof. Let $U \subset E'$ be a finely open set of BPE's for $R^2(E)$. By Proposition 2.2 there is a finely open set $V \subset U$ on which the $R^2(E)$

norm of "evaluation at x" is bounded. Let $X \subset V$ be a set that is compact in the usual topology and that contains a finely open set. Let $f \in R^2(E)$. Then there is a sequence $\{f_n\}$ in $R_0(E)$ such that $||f_n - f||_{L^2(E)} \to 0$. By the choice of X there is a constant C such that

$$\sup_{z \in X} |f_n(z) - f_m(z)| \leq C||f_n - f_m||_{L^2(E)}.$$

Thus the sequence obtained by restricting the f_n's to X converges in $R(X)$ to the restriction of f to X. We conclude that $f|_X \in R(X)$.

Let X be as in the above theorem.

COROLLARY 2.2. *Every point of X is a BPD of all orders for $R^2(E)$.*

Proof. Let $x \in X$, and let s be a positive integer. By **[4]**, x is a BPD of all orders for $R(X)$. Hence there is a constant C such that if $f \in R_0(E)$, $|f^{(s)}(x)| \leq C||f||_X$ where $|| \ ||_X$ denotes the sup norm on X. By the choice of X (see the proof of Theorem 2.2), there is another constant C' such that $||f||_X \leq C'||f||_{L^2(E)}$. Taken together these inequalities imply that x is a BPD of order s for $R^2(E)$.

There do exist examples of compact nowhere dense sets E that contain finely open subsets of BPE's for $R^2(E)$.

3. The case of no BPE's **for** $R^2(E)$. In this section we show that whenever $R^2(E) \neq L^2(E)$, there is a subset of E on which functions in $R^2(E)$ that are not continuous may still have smoothness properties. To describe this set of points we begin by letting φ be a positive function defined on $(0, \infty)$ such that φ is decreasing and $\lim_{r \downarrow 0^+} \varphi(r) = \infty$.

DEFINITION 3.1. A point $x \in E$ is a BPE of type φ for $R^2(E)$ if there is a constant C such that

$$|f(x)| \leq C\left\{\int_E |f(z)|^2 \varphi(|z - x|)\, dm(z)\right\}^{1/2}$$

for all $f \in R_0(E)$.

Fernström introduced BPE's of type $\varphi(r) = \log^\beta 1/r$ for $\beta > 1$ in **[7]**. The proof of the following theorem is similar to the proof of Theorem 3 in **[8]**.

THEOREM 3.1. *Let $E \subset \mathbb{C}$ be compact. Then x is a BPE of type φ for $R^2(E)$ if and only if*

$$\sum_{n=1}^{\infty} \varphi^{-1}(2^{-n}) 2^{2n} C_2(A_n(x) \backslash E) < \infty.$$

For certain φ's the above series will converge on a set of positive measure whenever $R^2(E) \neq L^2(E)$.

DEFINITION 3.2. A non-negative, real-valued function φ defined on $(0, \infty)$ is *nice* if it satisfies the following conditions:

(i) There is an $r_0 > 0$ such that φ is decreasing on $(0, r_0)$, and $\lim_{r\downarrow 0^+} \varphi(r) = +\infty$.

(ii) $\lim_{r\downarrow 0^+} r \cdot \varphi(r) = 0$, and there is an $s_0 > 0$ such that $1/(r \cdot \varphi(r))$ is decreasing on $(0, s_0)$; and

(iii) there is a $t_0 > 0$ such that $\int_0^{t_0} (1/(r \cdot \varphi(r)))\, dr < \infty$.

EXAMPLES.

$$(1) \qquad \varphi(r) = \frac{1}{r^\alpha}, \qquad 0 < \alpha < 1.$$

$$(2) \quad \varphi(r) = \log^\beta \frac{1}{r}, \quad \beta > 1, \quad \text{for } 0 < r \leq 1, \quad \varphi(r) = 0 \quad \text{for } r > 1.$$

$$(3) \quad \varphi(r) = \left(\log \frac{1}{r}\right) \left[\log \left(\log \frac{1}{r}\right)\right]^\beta, \ \beta > 1, \quad \text{for } 0 < r \leq 1/2,$$
$$\varphi(r) = (\log 2) \cdot [\log(\log 2)]^\beta, \quad \text{for } r > 1/2.$$

Condition (iii) of Definition 3.2 combined with Theorem 2.1 and Theorem 3.1 imply the following:

THEOREM 3.2. *Let $E \subset \mathbb{C}$ be a compact set without interior in the usual topology. Let φ be nice. Then if $R^2(E) \neq L^2(E)$ the set of BPE's of type φ has positive measure.*

Let S denote the set of $x \in E$ such that $\limsup_{r\to 0} C_2(\Delta(x, r)\backslash E)/r^2 = 0$. Suppose that $x \in E$ is a BPE of type φ. We define a norm $\|\ \|_\varphi$ on functions in $L^2(E)$ as follows:

$$\|f\|_\varphi = \sup_{y\in S} \|f \cdot \varphi(|z-y|) \cdot \varphi(|z-x|)\|_{L^2(E)}$$

where f is a function of z. Let $R^\varphi(E)$ be the closure of $R_0(E)$ in this norm. For certain φ such as $\varphi(r) = \log^\beta 1/r$, $\beta > 0$, Hölder's inequality implies that $\bigcup_{p>2} R^p(E) \subset R^\varphi(E)$.

Now suppose that x is a BPE of type φ. Let $L^2(E, \varphi\, dm)$ be the space of all complex measurable functions f defined on E such that

$\{\int_E |f^2(z)| \cdot \varphi(|z-x|)\, dm(z)\}^{1/2} < \infty$. By a well known theorem **[14]**, there is a function $g \in L^2(E, \varphi\, dm)$ such that

$$f(x) = \int_E f \cdot g \cdot \varphi(|z-x|)\, dm(z)$$

for all $f \in R_0(E)$. We have the following theorem.

THEOREM 3.3. *Let φ be a nice function such that $\int_0^1 \varphi^3(r) r\, dr < \infty$. Suppose that $x \in E$ is a BPE of type φ. Let $\varepsilon > 0$. Then there is a set $A \subset E$ having full area density at x such that if $y \in A$ and $f \in R_0(E)$,*

$$|f(y) - f(x)| \leq \varepsilon \|f\|_\varphi.$$

We will give an outline of the proof. For more details see **[16]**.

Outline of Proof of Theorem 3.3. Let $\varepsilon > 0$. Let $g \in L^2(E, \varphi\, dm)$ be the representing function for x as defined above. Then if

$$c(y) = \int_E \frac{z-x}{z-y} g(z) \cdot \varphi(|z-x|)\, dm(z)$$

is defined and $\neq 0$,

$$\frac{1}{c(y)} \frac{z-x}{z-y} g(z) \cdot \varphi(|z-x|)$$

is a representing function for y. Among the points where $c(y)$ is defined are those in the set A_1 of the following lemma:

LEMMA 3.1. *For each $\delta > 0$, the sets*

$$A_1 = \left\{ y \in \mathbb{C} \colon |y-x| \int_E \frac{|g(z)| \cdot \varphi(|z-x|)}{|z-y|}\, dm(z) < \delta \right\} \quad \textit{and}$$

$$A_2 = \left\{ y \in \mathbb{C} \colon |y-x| \left[\int_E \frac{|g(z)|^2 \cdot \varphi(|z-x|)}{|z-y|^2 \cdot \varphi^2(|z-y|)}\, dm(z) \right]^{1/2} < \delta \right\}$$

have full area density at x.

The proof uses the properties of the nice function φ and is similar to that of Lemma 3.3 in **[16]**. Now if $c(y)$ is defined and $\neq 0$, and if

$f \in R_0(E)$, we have

$$f(y) - f(x) = \frac{1}{c(y)} \int_E \frac{[f(z) - f(x)] \cdot (z - x)}{(z - y)} g(z) \cdot \varphi(|z - x|)\, dm(z)$$
$$= \frac{1}{c(y)} \int_E [f(z) - f(x)] \left[1 + \frac{y - x}{z - y}\right] g(z) \cdot \varphi(|z - x|)\, dm(z)$$
$$= \frac{y - x}{c(y)} \int_E \left[\frac{f(z) - f(x)}{z - y}\right] g(z) \cdot \varphi(|z - x|)\, dm(z)$$
$$= \frac{y - x}{c(y)} \int_E \frac{f(z) - f(x)}{z - y} \frac{\varphi(|z - y|)}{\varphi(|z - y|)} g(z) \cdot \varphi(|z - x|)\, dm(z).$$

From Hölder's inequality, the assumption that x is a BPE of type φ, and the assumption that $\int_0^1 \varphi^3(r) r\, dr < \infty$, it follows that

$$|f(y) - f(x)| \le \frac{C|y - x|}{c(y)} \|f\|_\varphi \left\{\int_E \frac{|g(z)|^2 \cdot \varphi(|z - x|)}{|z - y|^2 \cdot \varphi^2(|z - y|)}\, dm(z)\right\}^{1/2}$$

where C is independent of f.

Choose $\delta > 0$ so small that if $y \in A_1 \cap A_2$ (see Lemma 3.1), then

$$\frac{C}{c(y)} |y - x| \left\{\int_E \frac{|g(z)|^2 \cdot \varphi(|z - x|)}{|z - y|^2 \cdot \varphi^2(|z - y|)}\, dm(z)\right\}^{1/2} < \varepsilon.$$

Lemma 3.1 implies that the set $A = A_1 \cap A_2$ has full area density at x. Moreover, if $y \in A$ and $f \in R_0(E)$,

$$|f(y) - f(x)| \le \varepsilon \|f\|_\varphi.$$

The author is grateful to the referee for helpful comments and suggestions.

References

[1] L. Ahlfors, *Conformal Invariants*, McGraw-Hill, New York, 1973.

[2] R. M. Blumenthal and R. K. Getoor, *Markov Processes and Potential Theory*, Academic Press, New York, 1968.

[3] J. E. Brennan, *Approximation in the mean and quasi-analyticity*, J. Functional Analysis, **12** (1973), 307–320.

[4] A. Debiard and B. Gaveau, *Potential fin et algèbres de fonctions analytiques* I, J. Functional Analysis, **16** (1974), 289–304.

[5] ———, *Potential fin et algèbres de fonctions analytiques* II, J. Functional Analysis, **17** (1974), 296–310.

[6] J. Deny, *Les potentials d'énergie finie*, Acta Math., **82** (1950), 107–183.

[7] C. Fernström, *Some remarks on the space $R^2(E)$*, Internat. J. Math. Math. Sci., **6** no. 3, (1983), 459–466.

[8] C. Fernström and J. C. Polking, *Bounded point evaluations and approximation in L^p by solutions of elliptic partial differential equations*, J. Functional Analysis, **28** (1978), 1–20.

[9] L. I. Hedberg, *Non-linear potentials and approximation in the mean by analytic functions*, Math. Z., **129** (1972), 299–319.

[10] ———, *Approximation in the mean by analytic functions*, Trans. Amer. Math. Soc., **163** (1972), 157–171.

[11] L. L. Helms, *Introduction to Potential Theory*, John Wiley and Sons, Inc., 1969.

[12] N. S. Landkof, *Foundations of Modern Potential Theory*, translated from the Russian by A. P. Doohovsky, Springer-Verlag, 1972.

[13] B. Oksendal, *A Wiener test for integrals of Brownian motion and the existence of smooth curves in nowhere dense sets*, J. Functional Analysis, **36** (1980), 72–87.

[14] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, 1966.

[15] H. Schubert, *Topology*, translated from the German by Siegfried Moron, Allyn and Bacon, Inc., Boston, 1968.

[16] E. M. Wolf, *Bounded point evaluations and smoothness properties of functions in $R^p(X)$*, Trans. Amer. Math. Soc., **238** (1978), 71–88.

[17] ———, *Smoothness properties of function in $R^2(E)$ at certain boundary points*, Internat. J. Math. and Math. Sci., **2**, No. 3 (1979), 415–426.

Received March 24, 1987 and in revised form December 28, 1987.

UNIVERSITY OF LOWELL
LOWELL, MA 01854

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California 90024.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 50 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$190.00 a year (5 Vols., 10 issues). Special rate: \$95.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924 (ISSN 0030-8730) publishes 5 volumes per year. Application to mail at Second-class postage rates is pending at Carmel Valley, California, and additional mailing offices. Postmaster: send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Pacific Journal of Mathematics

Vol. 134, No. 2 June, 1988