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Dedicated to the memory of Henry Dye

The class of CCR-rings is introduced, parallel to CCR-algebras in the theory of C^*-algebras. It is proved that in a semisimple π-regular ring R there exists an idempotent e such that eRe is strongly regular.

It is fitting that in this tribute to Henry Dye I return to the topic of CCR-algebras. He was one of that able group that made for a golden age of C^*-algebras at Chicago in the 1950's.

In the ensuing thirty-five years I have more than once thought of the fact that the CCR property has an evident purely algebraic analogue, and that, sooner or later, it would get attention from ring-theorists. In this note I shall give the definition and prove one theorem. I use the same designation "CCR", although this is a slight abuse of language.

DEFINITION. A ring is CCR if every primitive homomorphic image is simple with a minimal one-sided ideal.

REMARKS. 1. I invite any reader who so wishes to replace "CCR" by Dixmier's "liminaire".

2. Primitivity is not left-right symmetric and so (perhaps) the same is true for the CCR property. For definiteness, let it be agreed that I mean left CCR.

3. There is of course the more general notion of GCR: every primitive image has a minimal one-sided ideal. In this case the passage from GCR C^*-algebras to GCR-rings is a bona fide generalization rather than an analogue. If the theory develops further it will probably encompass GCR-rings.

The theorem is an analogue of [2, Lemma 3] and can be regarded as a globalization of the fact that a simple ring T with a minimal one-sided ideal possesses an idempotent g such that gTg is a division ring.

THEOREM. *Let R be a π-regular semisimple CCR-ring. Then there exists in R a nonzero idempotent e such that eRe is strongly regular.*

I shall take the space to give two definitions. A ring is π-regular if for every a there exist an element x and an integer n such that $a^n x a^n = a^n$. The π-regular property is a weakening of von Neumann regularity, in which n is always 1; π-regularity has the merit of being satisfied in any algebraic algebra. A ring is strongly regular if for every a there exists x with $a^2 x = a$. Although it is not immediately apparent, it is true that strong regularity is left-right symmetric. There are numerous equivalent conditions, one of which is the following: a ring is strongly regular if and only if it is von Neumann regular and has no nonzero nilpotent elements.

The proof of the theorem divides into three parts.

(1) Take a nonzero idempotent h in R. (One exists since otherwise R would be a nil ring, contradicting semisimplicity.) We propose to transfer the problem from R to hRh. It is known that π-regularity and semisimplicity survive. So does the CCR property; indeed it survives in a strengthened form, and that is the purpose of this first step of the proof. The strengthened statement is that any primitive image of hRh is simple Artinian (i.e. a complete matrix ring over a division ring). This follows from the fact that the primitive ideals in hRh have the obvious form (see Theorem 3.1 in [**1**]), together with the further fact that the corner created by an idempotent in a simple ring with a minimal one-sided ideal is simple Artinian.

To simplify notation we replace hRh by R. So we are starting over, with the added knowledge that every primitive image of R is simple Artinian.

(2) There is now an opportunity for the category argument of [**3**] to make a repeat appearance. Let X be the structure space of R (the primitive ideals of R in the Stone-Jacobson-Zariski topology). Let X_m be the set of all primitive ideals P such that the size of the matrices in R/P is at most m. We shall shortly argue that X_m is a closed subset of X. Granted this, we have X expressed as a countable union of closed sets. Since X is of the second category [**3**, Theorem 10.2] one of the X_m's has a nonempty interior.

The fact that X_m is closed can virtually be quoted from [**1**]. The setup that needs to be analyzed is as follows. We are given a π-regular ring T which possesses a set $\{J_r\}$ of two-sided ideals such that $\bigcap J_r = 0$ and each T/J_r is the ring of all n by n matrices over a division ring, with $n \leq m$ where m is a given integer. Suppose that J is another two-sided ideal in T and that T/J consists of all k by k matrices over a division ring. We are to prove that $k \leq m$. It follows from our hypothesis that the nilpotent elements of T have index bounded by m. For, if $x \in T$ is nilpotent, then $x^m = 0$ in every T/J_r. Hence $x^m \in \bigcap J_r$, $x^m = 0$. Now it suffices to cite [**1**, Theorem 2.3].

(3) Suppose that X_i contains the nonempty open set U. Let H be the intersection of the primitive ideals comprising U. Then H is again π-regular and semisimple. What we have gained is that H is of bounded index, in the terminology of [**1**]. As noted at the beginning of §4 of [**1**], H is built out of a finite number of pieces, each of which is homogeneous in the sense that its image matrix rings all have the same size. In particular, the bottom layer (say K) is itself homogeneous. According to Theorem 4.1 of [**1**], K has a direct summand L which has a unit element, and then by Theorem 4.2 of [**1**], L is a total matrix ring over a strongly regular ring. With this the desired idempotent for the theorem is visible.

References

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