

# Pacific Journal of Mathematics

**CCR-RINGS**

IRVING KAPLANSKY

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IRVING KAPLANSKY

*Dedicated to the memory of Henry Dye*

**The class of CCR-rings is introduced, parallel to CCR-algebras in the theory of  $C^*$ -algebras. It is proved that in a semisimple  $\pi$ -regular ring  $R$  there exists an idempotent  $e$  such that  $eRe$  is strongly regular.**

It is fitting that in this tribute to Henry Dye I return to the topic of CCR-algebras. He was one of that able group that made for a golden age of  $C^*$ -algebras at Chicago in the 1950's.

In the ensuing thirty-five years I have more than once thought of the fact that the CCR property has an evident purely algebraic analogue, and that, sooner or later, it would get attention from ring-theorists. In this note I shall give the definition and prove one theorem. I use the same designation "CCR", although this is a slight abuse of language.

**DEFINITION.** A ring is CCR if every primitive homomorphic image is simple with a minimal one-sided ideal.

**REMARKS.** 1. I invite any reader who so wishes to replace "CCR" by Dixmier's "liminaire".

2. Primitivity is not left-right symmetric and so (perhaps) the same is true for the CCR property. For definiteness, let it be agreed that I mean left CCR.

3. There is of course the more general notion of GCR: every primitive image has a minimal one-sided ideal. In this case the passage from GCR  $C^*$ -algebras to GCR-rings is a bona fide generalization rather than an analogue. If the theory develops further it will probably encompass GCR-rings.

The theorem is an analogue of [2, Lemma 3] and can be regarded as a globalization of the fact that a simple ring  $T$  with a minimal one-sided ideal possesses an idempotent  $g$  such that  $gTg$  is a division ring.

**THEOREM.** *Let  $R$  be a  $\pi$ -regular semisimple CCR-ring. Then there exists in  $R$  a nonzero idempotent  $e$  such that  $eRe$  is strongly regular.*

I shall take the space to give two definitions. A ring is  $\pi$ -regular if for every  $a$  there exist an element  $x$  and an integer  $n$  such that  $a^n x a^n = a^n$ . The  $\pi$ -regular property is a weakening of von Neumann regularity, in which  $n$  is always 1;  $\pi$ -regularity has the merit of being satisfied in any algebraic algebra. A ring is strongly regular if for every  $a$  there exists  $x$  with  $a^2 x = a$ . Although it is not immediately apparent, it is true that strong regularity is left-right symmetric. There are numerous equivalent conditions, one of which is the following: a ring is strongly regular if and only if it is von Neumann regular and has no nonzero nilpotent elements.

The proof of the theorem divides into three parts.

(1) Take a nonzero idempotent  $h$  in  $R$ . (One exists since otherwise  $R$  would be a nil ring, contradicting semisimplicity.) We propose to transfer the problem from  $R$  to  $hRh$ . It is known that  $\pi$ -regularity and semisimplicity survive. So does the CCR property; indeed it survives in a strengthened form, and that is the purpose of this first step of the proof. The strengthened statement is that any primitive image of  $hRh$  is simple Artinian (i.e. a complete matrix ring over a division ring). This follows from the fact that the primitive ideals in  $hRh$  have the obvious form (see Theorem 3.1 in [1]), together with the further fact that the corner created by an idempotent in a simple ring with a minimal one-sided ideal is simple Artinian.

To simplify notation we replace  $hRh$  by  $R$ . So we are starting over, with the added knowledge that every primitive image of  $R$  is simple Artinian.

(2) There is now an opportunity for the category argument of [3] to make a repeat appearance. Let  $X$  be the structure space of  $R$  (the primitive ideals of  $R$  in the Stone-Jacobson-Zariski topology). Let  $X_m$  be the set of all primitive ideals  $P$  such that the size of the matrices in  $R/P$  is at most  $m$ . We shall shortly argue that  $X_m$  is a closed subset of  $X$ . Granted this, we have  $X$  expressed as a countable union of closed sets. Since  $X$  is of the second category [3, Theorem 10.2] one of the  $X_m$ 's has a nonempty interior.

The fact that  $X_m$  is closed can virtually be quoted from [1]. The setup that needs to be analyzed is as follows. We are given a  $\pi$ -regular ring  $T$  which possesses a set  $\{J_r\}$  of two-sided ideals such that  $\bigcap J_r = 0$  and each  $T/J_r$  is the ring of all  $n$  by  $n$  matrices over a division ring, with  $n \leq m$  where  $m$  is a given integer. Suppose that  $J$  is another two-sided ideal in  $T$  and that  $T/J$  consists of all  $k$  by  $k$  matrices over a division ring. We are to prove that  $k \leq m$ . It follows from our

hypothesis that the nilpotent elements of  $T$  have index bounded by  $m$ . For, if  $x \in T$  is nilpotent, then  $x^m = 0$  in every  $T/J_r$ . Hence  $x^m \in \bigcap J_r$ ,  $x^m = 0$ . Now it suffices to cite [1, Theorem 2.3].

(3) Suppose that  $X_i$  contains the nonempty open set  $U$ . Let  $H$  be the intersection of the primitive ideals comprising  $U$ . Then  $H$  is again  $\pi$ -regular and semisimple. What we have gained is that  $H$  is of bounded index, in the terminology of [1]. As noted at the beginning of §4 of [1],  $H$  is built out of a finite number of pieces, each of which is homogeneous in the sense that its image matrix rings all have the same size. In particular, the bottom layer (say  $K$ ) is itself homogeneous. According to Theorem 4.1 of [1],  $K$  has a direct summand  $L$  which has a unit element, and then by Theorem 4.2 of [1],  $L$  is a total matrix ring over a strongly regular ring. With this the desired idempotent for the theorem is visible.

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