CHARACTERIZATION OF CROSSED PRODUCT (PROPERLY INFINITE CASE)

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Dedicated to the memory of Professor Henry Abel Dye

We characterize a crossed product of a properly infinite factor by an outer action of a finite group.

0. Introduction. We will obtain a properly infinite version of characterization of crossed product algebras as a generalization of the recent result (explained below) in the II_1-factor set-up.

In [6] Ocneanu announced classification of subfactors in the hyperfinite II_1-factor. His work is built upon Jones' index theory [2] and deep analysis in the Pimsner-Popa paper [7], and this classification is based on a newly-introduced invariant called a paragroup. His announcement contains the following intrinsic characterization of crossed products: Let M be a II_1-factor with a subfactor N, $[M;N] < \infty$. If $M \cap N' = \mathbf{C}1$, $M_1 \cap N'$ is abelian, and the depth of $M \supseteq N$ is 2, then there exist a finite group G and an outer action α of G on N such that $M = N \rtimes_\alpha G$. Here, M_1 is the basic extension of $M \supseteq N$, and the definition of a depth can be found in [6].

A proof of this beautiful characterization has not been written yet (although the result as well as its proof are implicit in [7]). The purpose of the present paper is to provide a complete proof of a properly infinite version of this characterization. Therefore, the notion of index explained in [4] has to be used here, and Ocneanu's conditions are replaced by the following (equivalent) conditions: $M \cap N' = \mathbf{C}1$, $M_1 \cap N'$ is abelian, and $\dim_{\mathbf{C}}(M_1 \cap N') = \text{Index } E$. (In particular, Index E is an integer.) Here, E is a unique normal conditional expectation from M onto N.

Our standard references on operator algebras are [8] and [9], and basic facts on index theory can be found in [2], [4], and [7]. Some auxiliary results will be proved in appendices.

1. Notations and main theorem. Throughout let M be a factor with a subfactor N, and let E be a normal conditional expectation from M onto N satisfying Index $E < \infty$.

Let H be the standard L^2-space of M so that M and N are acting on H. Once a natural cone is fixed in H, one can perform the basic construction. As usual by M_1 we denote the basic extension of $M \supseteq N$, that is, $M_1 = JN'J = M \vee \{e\}''$, where J is the modular conjugation on H and e is the projection arising from E. Let E_M be the normal conditional expectation from M_1 onto M constructed from E^{-1}. ($E_M = (\text{Index } E)^{-1} \times JE^{-1}(J \cdot J)J$) (see Appendix I for details).

LEMMA 1. *If $M_1 \cap N'$ is abelian, then* $\dim_{\mathbb{C}}(M_1 \cap N') \leq \text{Index } E$.

Proof. Let $\{p_i\}_{i=1,2,\dots,m}$ be the minimal projections in $M_1 \cap N'$. Recall that

$$\text{Index}(E \circ E_M) = (\text{Index } E)(\text{Index } E_M) = (\text{Index } E)^2.$$

Considering local indices ([4], 4), we estimate

$$(\text{Index } E)^2 \geq \sum_{i=1}^{m} E \circ E_M(p_i)^{-1} \text{Index}(E \circ E_M)_{p_i}$$

$$\geq \sum_{i=1}^{m} E \circ E_M(p_i)^{-1}.$$

Notice that $E \circ E_M(p_i)$'s are positive scalars summing up to 1. Therefore, the minimum of the above sum of the inverses is $m \times (1/m)^{-1} = m^2$ (and the minimum is attained only when all of $E \circ E_M(p_i)$ are exactly $1/m$).   □

THEOREM 2. *Assume that M is properly infinite and* Index $E = n \in \mathbb{N}_+$. *If $M \cap N' = \mathbb{C}1$, $M_1 \cap N'$ is abelian, and $\dim_{\mathbb{C}}(M_1 \cap N') = n$, then there exist a finite group G (of order n) and an outer action α of G on N such that M is the crossed product $N \rtimes_\alpha G$. (Furthermore, E is the canonical conditional expectation from $N \rtimes_\alpha G$ onto N.)*

When $M = N \rtimes_\alpha G$ from the beginning, it is easy to check that all of the above conditions are indeed satisfied. It is an amusing exercise so that we leave it to the reader. We just remark that (i) E in this case is the canonical expectation from $N \rtimes_\alpha G$ onto N, (ii) this exercise requires the characterization of an outer action obtained in [3].

2. Proof of Theorem. We still keep the assumptions at the beginning of §1. From now on, we will also assume that $M \cap N' = \mathbf{C}1$, Index $E = n \in \mathbf{N}_+$, and $M_1 \cap N'$ is abelian. (Hence, by Lemma 1, $\dim_{\mathbf{C}}(M_1 \cap N') \le n$.)

LEMMA 3. *Assume that* $\dim_{\mathbf{C}}(M_1 \cap N') = n$, *and let* $\{p_i\}_{i=1,2,\dots,n}$ *be the minimal projections in* $M_1 \cap N'$. *Then we have* $E_M(p_i) = (1/n)1$ *for each* i. *In particular, we get* $\{p_1, p_2, \dots, p_n\} = \{p \in (M_1 \cap N')_{\text{proj}};$ $E_M(p) = 1/n\}$.

Proof. Under the present assumptions, the inequalities in the proof of Lemma 1 read:

$$n^2 \ge \sum_{i=1}^{n} E \circ E_M(p_i)^{-1} \operatorname{Index}(E \circ E_M)_{p_i}$$

$$\ge \sum_{i=1}^{n} E \circ E_M(p_i)^{-1} \ge n \times \left(\frac{1}{n}\right)^{-1} = n^2.$$

We thus conclude that $E \circ E_M(p_i) = 1/n$ (and $\operatorname{Index}(E \circ E_M)_{p_i} = 1$). Since $M \cap N' = \mathbf{C}1$, $E_M(p_i)$ is already a scalar and we get

$$1/n = E \circ E_M(p_i) = E_M(p_i). \qquad \square$$

We next investigate a "Weyl group" as in [7]. The minimal projections $\{p_i\}_{i=1,2,\dots,n}$ will be fixed from now on. We may and do assume that $p_1 = e$. For each p_i, we choose a partial isometry $v_i \in M_1$ satisfying $v_i v_i^* = p_i$ and $v_i^* v_i = e$. (It is always possible if M is of type III. It is also possible if M is of type either II$_1$ or II$_\infty$. In fact, in these cases, E comes from the unique trace and Lemma 3 guarantees that p_i and e have the same trace value. See Appendix II.) By Lemma 3.3, [4], $u_i = n E_M(v_i e) \in M$ satisfies $v_i = v_i e = u_i e$. We thus get

(1) $\qquad \qquad \left\{ \begin{array}{l} p_i = u_i e u_i^*, \\ e = e u_i^* u_i e = E(u_i^* u_i)e, \text{ that is, } E(u_i^* u_i) \end{array} \right.$

(2)

Applying E_M to (1), from Lemma 3 we get

$$1/n = E_M(p_i) = E_M(u_i e u_i^*) = u_i E_M(e) u_i^* \quad (\text{since } u_i \in M)$$
$$= (1/n) u_i u_i^*.$$

Hence u_i is a coisometry. Then (2) means that the operator $1 - u_i^* u_i \ge 0$ satisfies $E(1 - u_i^* u_i) = 0$. The expectation E being faithful, we have

$1 - u_i^* u_i = 0$, and consequently u_i is a unitary in M. Since $v_i^* v_j = 0$ ($i \neq j$), we compute

$$0 = e u_i^* u_j e = E(u_i^* u_j)e, \text{ that is, } E(u_i^* u_j) = 0.$$

Let $\mathcal{N}(N)$ be the set of all unitaries u in M satisfying $u N u^* = N$. Notice that the n unitaries $u_1, u_2, \ldots, u_n$ belong to $\mathcal{N}(N)$. In fact, for each $y \in N$, we compute

$$y u_i e u_i^* = y p_i = p_i y = u_i e u_i^* y$$

because of $p_i \in M_1 \cap N'$. But this means that $u_i^* y u_i \in M$ commutes with e, that is, $u_i^* y u_i \in N$. Obviously $\mathcal{N}(N)$ forms a group (under the ordinary operator multiplication) and the unitary group $\mathcal{U}(N)$ is a normal subgroup in $\mathcal{N}(N)$.

DEFINITION 4. The quotient group $\mathcal{N}(N)/\mathcal{U}(N)$ is denoted by G (the Weyl group of $M \supseteq N$).

For each $u \in \mathcal{N}(N)$, $u e u^*$ is a projection in $M_1 \cap N'$. In fact, for each $y \in N$, we compute

$$u e u^* y = u e (u^* y u) u^* = u (u^* y u) e u^* \quad (\text{since } u^* y u \in N)$$
$$= y u e u^*.$$

We also notice that

$$E_M(u e u^*) = u E_M(e) u^* = (1/n) u u^* = 1/n.$$

Therefore, $u e u^*$ has to be one of p_i's (see Lemma 3).

LEMMA 5 (*Proposition 1.7, [7]*). *The Weyl group G is a finite group of order n. More precisely, $u \in \mathcal{N}(N) \to u e u^* \in \{p \in (M_1 \cap N')_{\mathrm{proj}}; E_M(p) = 1/n\}$ induces the bijection:*

$$[u] \in G \to u e u^* \in \{p \in (M_1 \cap N')_{\mathrm{proj}}; E_M(p) = 1/n\}.$$

Proof. Since e commutes with an arbitrary element in N, the second map is certainly well-defined. To show the injectivity, let us assume $u e u^* = \tilde{u} e \tilde{u}^*$ ($u, \tilde{u} \in \mathcal{N}(N)$). Then the unitary $\tilde{u}^* u \in M$ commutes with e, that is, $\tilde{u}^* u \in N$ and $[\tilde{u}] = [u]$. The surjectivity follows from $p_i = u_i e u_i^*$. □

We have shown that the Weyl group G (a finite group of order n) is parametrized by:

$$\{p \in (M_1 \cap N')_{\mathrm{proj}}; E_M(p) = 1/n\} = \{p_1, p_2, \ldots, p_n\}.$$

Therefore, in what follows, we shall write

$$\{p \in (M_1 \cap N')_{\mathrm{proj}}; E_M(p) = 1/n\} = \{p_g\}_{g \in G}$$
$$(p_1 = e, \text{ where 1 denotes the unit in } G)$$

and

$$g = [u_g] \in G = \mathcal{N}(N)/\mathcal{U}(N) \qquad (\text{with } u_1 = 1).$$

Notice that Ad u_g ($g \neq 1$) is an outer automorphism of N. In the above argument, a choice of a partial isometry v_i (hence that of a unitary u_g) is highly non-unique. Can we choose n representative unitaries u_g, $g \in G$, in such a way that u_g's themselves form a group (in $\mathcal{N}(N)$)? This is a cohomological question. Indeed, since $[u_g]$'s form a group, we obtain a 2-cocycle on G (whose values are in the unitary group $\mathcal{U}(N)$). The group G being finite, it is known that a 2-cocycle is a coboundary. (This is due to Sutherland. See [8] for example.) This means that by perturbing u_g by a unitary in N we can assume that u_g's form a group. Notice that this perturbation does not destroy the property $u_g e u_g^* = p_g$ (since e and p_g belong to $M_1 \cap N'$). By setting $\alpha_g = \mathrm{Ad}\, u_g|_N$ (with the perturbed u_g's), we obtain an outer action α on N.

Proof of Theorem 2. For any x in M, we compute

$$xe = 1xe = \sum_{g \in G} u_g e u_g^* xe = \sum_{g \in G} u_g E(u_g^* x) e.$$

Hence $x = \sum_{g \in G} u_g x_g$ with $x_g = E(u_g^* x) \in N$. Also the "coefficients" x_g's are uniquely determined by x since $E(u_h^* u_g) = 0$, $g \neq h$, obviously remains valid after the above-mentioned perturbation. (This is an "N-module decomposition" of M obtained in [7].) Let $N \rtimes_\alpha G$ be the crossed product generated by the usual generators $\pi_\alpha(x)$ ($x \in N$) and λ_g ($g \in G$). It is easy to see that $\sum_{g \in G} u_g x_g \to \sum_{g \in G} \lambda_g \pi_\alpha(x_g)$ gives rise to an isomorphism from M onto $N \rtimes_\alpha G$. Since $E(u_g^* y) = 0$ ($g \neq 1$, $y \in N$) and $E(u_1^* y) = y$ ($y \in N$), a generic element $x = \sum_{g \in G} u_g x_g \in M$ belongs to the subfactor N if and only if $x_g = 0$ for all $g \neq 1$. Thus the above-mentioned isomorphism from M onto $N \rtimes_\alpha G$ sends N onto $\pi_\alpha(N)$, that is, $N \subseteq M$ and $\pi_\alpha(N) \subseteq N \rtimes_\alpha G$ are conjugate. □

REMARK 6. By "dualizing" a situation, we can prove the following characterization of fixed point subalgebras: Assume Index $E = n \in N$. If $M \cap N' = \mathbf{C}1$, $M_2 \cap M'$ is abelian, and $\dim_{\mathbf{C}}(M_2 \cap M') = n$, then

there exists an outer action of a finite group G (of order n) on M such that $N = M_G$, the fixed point subalgebra. Here, M_2 denotes the basic extension of $M_1 \supseteq M$. (See [6] for the II_1-factor case.)

This result can be proved based on the general theory on actions and coactions of a group (and a "group dual"). However, we will briefly describe a direct proof in what follows.

Our previous arguments guarantee existence of n unitaries $\{u_g\}_{g \in G}$ in M_1 such that $u_g M u_g^* = M$ and u_g's form a group (in $\mathscr{U}(M_1)$) of order n. Let J_1 be the modular conjugation on $L^2(M_1)$. Then $v_g = J_1 u_g J_1$ normalizes $M_2 = J_1 M' J_1$ so that we get an action of G on M_2. This is an outer action and we have $M_2 \supseteq (M_2)_G \supseteq M_1$. Thanks to [1], the index of the (unique) expectation from M_2 onto $(M_2)_G$ is n so that we conclude

$$(3) \qquad\qquad\qquad (M_2)_G = M_1.$$

Since M and N are properly infinite, they have a common cyclic and separating vector ζ in $H = L^2(M)$. Let J_M, J_N be the modular conjugation arising from (M, ζ) and (N, ζ) respectively. Let $\gamma = \operatorname{Ad} J_N J_M$ be Longo's canonical endomorphism from M to N, [5]. It is easy to see that $N \supseteq \gamma(M) \supseteq \gamma(N)$ are downward basic extensions. Also notice that $M_1 = J_M N' J_M = J_M J_N N J_N J_M$ (acting on H with a cyclic and separating vector ζ). The modular conjugation J_1 coming from (M_1, H, ζ) is $(J_M J_N) J_N (J_M J_N)^* = J_M J_N J_M$, so that $M_2 = J_1 M' J_1 = J_M J_N M J_N J_M$. (All the upward and downward basic extensions can be constructed by using the single Hilbert space H.) Notice that $\gamma(M) \cap \gamma(N)' = \gamma(M \cap N') = \gamma(\mathbf{C}1) = \mathbf{C}1$. We also compute

$$
\begin{aligned}
M \cap \gamma(M)' &= M \cap (J_N J_M M J_M J_N)' = M \cap (J_N J_M M' J_M J_N) \\
&= J_N J_M \{(J_M J_N M J_N J_M) \cap M'\} J_M J_N \\
&= J_N J_M (M_2 \cap M') J_M J_N.
\end{aligned}
$$

So the inclusion $\gamma(M) \supseteq \gamma(N)$ satisfies the same assumptions as $M \supseteq N$. Thus, by applying (3) to the inclusion $\gamma(M) \supseteq \gamma(N)$, we obtain the result.

APPENDIX I. Natural cone and basic construction.

As mentioned in the main body of the paper, a basic construction is determined after a natural cone is fixed. Although this is a folklore for specialists, this was not explicitly explained in [4]. Thus to minimize confusion we will make this point clear in the appendix.

Let $E: M \to N$ be as the beginning of §1 and let H be a Hilbert space on which M is acting standardly. We now fix a natural cone H^+ ($\subseteq H$) and the associated modular conjugation J. The commutant N' is independent of the choice of H^+ and J is solely determined by H^+ so that obviously $M_1 = JN'J$ is determined by H^+. Actually we have more. Namely the involved projection $e = e_N$ ($M_1 = \langle M, e \rangle$) is also uniquely determined by H^+ (although e was defined after choosing a faithful normal state on N, [4]).

LEMMA A. *Let ϕ_i ($i = 1, 2$) be faithful normal states on N with $\phi_i \circ E = \omega_{\xi_i}$ ($\xi_i \in H^+$). Define the projections e_i ($i = 1, 2$) by $e_i x \xi_i = E(x)\xi_i$ ($x \in M$). Then the two projections e_i in N' are identical.*

Proof. For $x \in M$ and $t \in \mathbf{R}$, we compute
$$e_1 x (D(\phi_2 \circ E); D(\phi_1 \circ E))_t \xi_1 = E(x)(D(\phi_2 \circ E); D(\phi_1 \circ E))_t \xi_1$$
$$= E(x)(D(\phi_2 \circ E); D(\phi_1 \circ E))_t \xi_1$$
because of $(D(\phi_2 \circ E); D(\phi_1 \circ E))_t = (D\phi_2; D\phi_1)_t \in N$. The two H-valued functions
$$\begin{cases} f(t) = e_1 x (D(\phi_2 \circ E); D(\phi_1 \circ E))_t \xi_1, \\ g(t) = E(x)(D(\phi_2 \circ E); D(\phi_1 \circ E))_t \xi_1 \end{cases}$$
extend to bounded continuous functions on $-1/2 \leq \operatorname{Im} z \leq 0$ which are analytic in the interior. The previous computation means $f(t) = g(t)$, $t \in \mathbf{R}$. Using the uniqueness of analytic continuation, we get $f(z) = g(z)$ and in particular ($z = -i/2$)
$$e_1(x\xi_2) = E(x)\xi_2,$$
which means $e_1 = e_2$.   □

Next we investigate the effect coming from a different natural cone (possibly in a different Hilbert space). So let us assume that M is also acting standardly on a Hilbert space K and that K^+ ($\subseteq K$) is a natural cone with the associated modular conjugation $\tilde{J}$. Let ϕ be a faithful normal state on N with $\phi \circ E(x) = (x\xi, \xi)_H = (x\zeta, \zeta)_K$ ($\xi \in H^+$, $\zeta \in K^+$). Consider the usual projections e and $\tilde{e}$ given by
$$e\xi = E(x)\xi, \quad \tilde{e}x\zeta = E(x)\zeta \quad (x \in M).$$
(e is acting on H while $\tilde{e}$ is acting on K.) A linear map $x\zeta \in M\zeta$ ($\subseteq K$) $\to x\xi \in M\xi$ ($\subseteq H$) extends to a surjective isometry u from K onto H. For each $x \in M$ we compute
$$u\tilde{e}u^* x\xi = u\tilde{e}x\zeta = uE(x)\zeta = E(x)\xi,$$

which means $u\tilde{e}u^* = e$. It is also easy to see $uxu^* = x$ $(x \in M)$. Here, the right (resp. left) side x is understood to be acting on K (resp. H). Therefore, $\rho = \mathrm{Ad}(u)\colon B(K) \to B(H)$ sends $\langle M, \tilde{e}\rangle$ onto $\langle M, e\rangle$ in such a way that

$$\begin{cases} \rho(x) = x & (x \in M), \\ \rho(\tilde{e}) = e. \end{cases}$$

Consequently basic extensions coming from different natural cones can be identified. We recall that J and $\tilde{J}$ are also related via u. It is well known (and can be proved easily) that $u\tilde{J}u^* = J$. "Basic extensions of E" are "conjugate" via the above ρ. This follows from the definition of E_M (involving E^{-1} and a J-operator) and the above comparison of two J-operators. But we can also argue in the following way. Namely, the "basic extension" E_M is characterized as a unique normal conditional expectation satisfying $E_M(e) = 1$. Therefore, $E_M\colon \langle M, e\rangle \to M$ and $(E_M)^\sim\colon \langle M, \tilde{e}\rangle \to M$ are related by

$$(E_M)^\sim(x) = E_M(\rho(x)).$$

APPENDIX II. Type II_∞ case.

Let $E\colon M \to N$ be as before with Index $E < \infty$. In this appendix we will show that E comes from the unique trace on M if $M \cap N' = \mathbf{C}1$ and M (or equivalently N) is of type II_∞. Let tr (resp. τ) be the unique (up to a scalar) faithful semi-finite normal trace on M (resp. N). A real catch here is to show the semi-finiteness of tr on N.

LEMMA B. *Under the above assumptions, we have* $\tau \circ E = \lambda \mathrm{tr}$ *for some* $\lambda > 0$. *(Hence,* tr *is semi-finite on* N *and* E *comes from* tr.)*

Proof. Since $\tau \circ E$ is semi-finite (on M), there exists a unique positive self-adjoint operator h (the Radon-Nikodym derivative relative to tr) affiliated with M such that $\tau \circ E = \mathrm{tr}(h\cdot)$. For each $x \in N$ and $t \in \mathbf{R}$, we compute

$$\begin{aligned} h^{it}xh^{-it} &= \sigma_t^{\tau \circ E}(x) \quad \text{(since } h \text{ is the Radon-Nikodym derivative)} \\ &= \sigma_t^{\tau}(x) \quad \text{(since } x \text{ is in } N) \\ &= x. \end{aligned}$$

This means that h commutes strongly with an arbitrary element in N. The relative commutant $M \cap N'$ being trivial, we must have $h = \lambda 1$ for some $\lambda > 0$ and $\tau \circ E = \lambda \mathrm{tr}$.   □

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