ELEMENTS OF FINITE ORDER IN $V(ZA_4)$

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The conjugacy classes for all elements of finite order in the unit group $V(\mathbb{Z}A_4)$ are determined. As an application, it is shown that all normal complements to $A_4$ in $V(\mathbb{Z}A_4)$ must be torsion free.

Let $V(\mathbb{Z}G)$ denote the group of units of augmentation 1 in the integral group ring $\mathbb{Z}G$. There is considerable interest in determining whether the group $G$ has a torsion free normal complement in $V(\mathbb{Z}G)$. The authors showed in [2] that $S_3$ has two types of normal complements in $V(\mathbb{Z}S_3)$, one with torsion and one without. They have also shown (see [1]) that $A_4$ has a torsion free normal complement in $V(\mathbb{Z}A_4)$ and that $S_4$ has a normal complement in $V(\mathbb{Z}S_4)$ which includes torsion elements (see [3]). Two questions arise naturally:

1. Can $A_4$ also have a normal complement in $V(\mathbb{Z}A_4)$ which includes torsion?
2. Can $S_4$ also have a torsion free normal complement in $V(\mathbb{Z}S_4)$?

This paper gives a negative answer to Question 1 by completing the task of finding all of the conjugate classes of elements of finite order in $V(\mathbb{Z}A_4)$ and then showing that a subgroup containing any such class must also contain an element of order 2 in $A_4$. Earlier work has shown that the torsion elements of $V(\mathbb{Z}A_4)$ are of order 2 or 3 and that all elements of order 2 are conjugate [1]. Sekiguchi [4] showed that there are four conjugate classes of subgroups of $V(\mathbb{Z}A_4)$ which are isomorphic to $A_4$. It follows from his work that there are at least eight conjugate classes of elements of order 3; these classes include all of the elements of order 3 which lie in subgroups isomorphic to $A_4$. Our Theorem 1 shows that there are exactly four additional conjugate classes of elements of order 3 which do not lie in any subgroup isomorphic to $A_4$. Theorem 2 gives the answer to Question 1.

The results of [1] characterize $V(\mathbb{Z}A_4)$ as an explicit subgroup of $\text{SL}(3, \mathbb{Z})$ and thus permit us to utilize information about $\text{SL}(3, \mathbb{Z})$. The characterization relies on the following definition: If $X = [x_{ij}] \in \text{SL}(3, \mathbb{Z})$, then the pseudotracess $t_0$, $t_1$, and $t_2$ are given by $t_0 = x_{11} + x_{22} + x_{33}$, $t_1 = x_{12} + x_{23} + x_{31}$, and $t_2 = x_{13} + x_{21} + x_{32}$. Then we can
think of $A_4$ as generated by

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and of $V(ZA_4)$ as $\{X \in SL(3, Z) | X \text{ satisfies conditions (1) and (2)}\}$

where

- \textit{condition (1):} $X \equiv B^i \pmod{2}$ for some $i$
- \textit{condition (2):} two of the pseudotraces $t_j$ are 0 modulo 4.

We begin by finding the centralizer of $B$ in the ring $M_3(Q)$ of all $3 \times 3$ rational matrices.

**LEMMA 1.** Let $X \in M_3(Q)$. Then $XB = BX$ if and only if $X = \sum r_i B^i$ with $r_i \in Q$. Moreover, if $X \in SL(3, Z)$, then

(i) $XB \equiv BX \pmod{2}$ if and only if $X \equiv B^i \pmod{2}$ and

(ii) $XB = BX$ if and only if $X = B^i$.

**Proof.** If $X = \sum r_i B^i$, then it is clear that $XB = BX$. On the other hand, an inspection of the entries in the matrices $XB$ and $BX$ will show that $XB = BX$ implies that $X = \sum r_i B^i$ for some $r_i \in Q$. The remainder of the lemma follows from the fact that the group ring $R(B)$ has only trivial units if $R$ is the ring of integers modulo 2 or if $R = Z$.

Let $I_2$ and $I_4$ denote, respectively, the subgroups of $SL(3, Z)$ consisting of all matrices which are the identity modulo 2 and 4. It is clear that $I_4$ is contained in $V(ZA_4)$ and that $\langle B \rangle I_2$ is the subgroup consisting of all matrices which satisfy condition (1).

**LEMMA 2.** $V(ZA_4)$ is a normal subgroup of $\langle B \rangle I_2$. The factor group is the elementary group of order 4 with coset representatives $R_0 = I$,

$$R_1 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $R_3 = R_1 R_2$.

**Proof.** If $M \in I_2$ then one of the $MR_i$ must belong to $V(ZA_4)$ since multiplying $M$ on the right by $R_1$ or $R_2$ has the effect, modulo 4, of adding 2 to $t_1$ or $t_2$. Thus precisely one of the $MR_i$ will satisfy condition (2). Since $B \in V(ZA_4)$ it follows that the $R_i$ are a full set of coset representatives of $V(ZA_4)$ in $\langle B \rangle I_2$. The square of each of
these representatives is in $I_4$ and thus in $V(ZA_4)$. A direct calculation shows that $B^{R_1}$ and $B^{R_2}$ are in $V(ZA_4)$, thus the normality of $V(ZA_4)$ follows from the fact that $I_2$ is abelian modulo the subgroup $I_4$ of $V(ZA_4)$.


**Lemma 3.** $SL(3, Z)$ contains exactly two conjugate classes of elements of order 3. One of these classes contains $B$. The other one contains

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$  

In $SL(3, Z)$, $B$ is conjugate to $B^2$, but this cannot happen in $V(ZA_4)$, or even in $(B)I_2$, since conjugating $B$ by an element in $(B)I_2$ produces an element in $BI_2$. We shall restrict our attention to the conjugate classes in $V(ZA_4)$ of elements congruent to $B$ modulo 2; the squares of the elements in each class will be a conjugate class of elements congruent to $B^2$ modulo 2.

Lemmas 1 and 2 yield a complete description of all of the conjugate classes in $V(ZA_4)$ of elements which are conjugate in $B$ in $SL(3, Z)$ and are congruent to $B$ modulo 2. As we will see later, additional classes arise from conjugates of $W$.

**Lemma 4.** Conjugating $B$ by the four coset representatives $R_i$ of Lemma 2 produces elements of four conjugate classes in $V(ZA_4)$. Any conjugate of $B$ in $SL(3, Z)$ which is congruent to $B$ modulo 2 and belongs to $V(ZA_4)$ will lie in one of these classes.

**Proof.** Suppose that $B^{R_i} = B^{R_i M}$ for some $M \in V(ZA_4)$. Then $R_j M R_i^{-1}$ commutes with $B$ so, by Lemma 1,

$$R_j M R_i^{-1} \in \langle B \rangle.$$  

It follows from the normality of $V(ZA_4)$ that $R_i R_i^{-1} \in V(ZA_4)$, thus $R_i = R_j$. Consequently, the $B^{R_i}$ lie in distinct conjugate classes of $V(ZA_4)$.

Next, suppose that $X \equiv B$ (mod 2), that $X \in V(ZA_4)$, and that $X = B^M$ for some $M \in SL(3, Z)$. Then $B \equiv B^M$ (mod 2) so by Lemma 1, $M \equiv B^i$ (mod 2) for some $i$. It follows from Lemma 2 that $M = R_j N$ for some $j$ and some $N \in V(ZA_4)$. Thus $X$ is conjugate in $V(ZA_4)$ to $B^{R_j}$. 

There are elements of order 3 in $V(\mathbb{Z}A_4)$ which are congruent to $B$ modulo 2 and conjugate to $W$ in $SL(3, \mathbb{Z})$. (Because of Lemma 3, such elements cannot be conjugate to any of the $B^R_i$.) In fact, if

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

then

$$W^T = \begin{bmatrix} 0 & 1 & 2 \\ 0 & -2 & -3 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad W^{TR_1} = \begin{bmatrix} 0 & 5 & 8 \\ 0 & -2 & -3 \\ 1 & 4 & 2 \end{bmatrix}$$

are in $V(\mathbb{Z}A_4)$ and are congruent to $B$ modulo 2. We shall show that these two elements lie in different conjugate classes in $V(\mathbb{Z}A_4)$ and that every element of $V(\mathbb{Z}A_4)$ which is congruent to $B$ modulo 2, and conjugate to $W$ in $SL(3, \mathbb{Z})$, is conjugate in $V(\mathbb{Z}A_4)$ to one of them.

**Lemma 5.** $W^T$ and $W^{TR_1}$ are not conjugate in $V(\mathbb{Z}A_4)$.

**Proof.** We begin by observing that if $M \in V(\mathbb{Z}A_4)$ then conditions (1) and (2) imply that the sum of the entries in $M$ must be 3 modulo 4. By condition (1), the entries on one pseudotrace are $1 + e_1, 1 + e_2, 1 + e_3$ where the $e_i$ are even, and all other entries of $M$ are even. Consequently, $1 = |M| \equiv 1 + e_1 + e_2 + e_3 \pmod{4}$, so the pseudotrace with odd entries is 3 modulo 4 and it follows from condition (2) that the sum of the entries of $M$ is 3 modulo 4.

Now suppose that $W^{TM} = W^{TR_1}$ for some $M \in V(\mathbb{Z}A_4)$. Let

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

and observe that $B^P = W^T$, so $B^{PM} = B^{PR_1}$. By Lemma 1, if $X = PMR_1^{-1}P^{-1}$, then

$$X = sI + tB + uB^2$$

for some rational numbers $s$, $t$, and $u$. Each of the column sums of $X$ is $s + t + u$; thus, if we start to evaluate $|X|$ by adding the first two rows to the third, we see that $|X| = (s + t + u)(s^2 + t^2 + u^2 - st - su - tu)$. Next, note that $X^P = MR_1^{-1}$ is an integer matrix of determinant 1 which also has column sums $s + t + u$ since $I$, $B^P$, and $(B^2)^P$ have column sums of 1. Therefore, $s + t + u$ is an integer. If we start to
evaluate $|X^P|$ by adding the first two rows to the third, then factoring out $s + t + u$, we see that

$$1 = |X^P| = (s + t + u)n$$

for some integer $n$. It is now clear from the form of $|X|$ that $s + t + u$ and $s^2 + t^2 + u^2 - st - su - tu$ are both 1 or both $-1$. If

$$s^2 + t^2 + u^2 = st + su + tu - 1$$

then $st + su + tu \geq 1$. Hence

$$1 = (s + t + u)^2 = s^2 + t^2 + u^2 + 2(st + su + tu) \geq 2,$$

a contradiction. Therefore $s + t + u = 1$.

We now know that $X^P = MR_1^{-1}$ where the column sums of $X^P$ are each 1. Multiplying $X^P$ on the right by $R_1$ adds twice the first column to the second, thus the column sums of $M$ are, respectively, 1, 3, and 1. But then the sum of the entries of $M$ is 1 modulo 4, a contradiction.

We found that the $B_i$ come from four different classes. One might expect that the $W_{TR}$ would come from four new classes. Lemma 5 has shown that $W^T$ and $W_{TR_1}$ do come from different classes. These turn out to be the only new classes.

**Lemma 6.** Each $W_{TR_i}$ is conjugate in $V(ZA_4)$ either to $W^T$ or to $W_{TR_1}$.

**Proof.** It suffices to show that $W_{TR_1 R_2}$ is $W^T M$ for some $M$ in $V(ZA_4)$. It will follow that $W_{TR_1}$ and $W_{TR_2}$ are in the same class, since $R_2^2 \in V(ZA_4)$. The matrix

$$M = \begin{pmatrix} 1 & 4 & 4 \\ -2 & -7 & -6 \\ 2 & 6 & 5 \end{pmatrix}$$

has the required properties.

The next lemma shows that the 6 classes found in Lemmas 4 and 5 account for all of the elements of order 3 in $V(ZA_4)$ which are congruent to $B$ modulo 2.

**Lemma 7.** Suppose that $X \in V(ZA_4)$, that $X \equiv B \pmod{2}$, and that $X = W^M$ for some $M \in SL(3, \mathbb{Z})$. Then $X$ is conjugate in $V(ZA_4)$ to one of $W^T$ and $W_{TR_1}$. 
Proof. By hypothesis, $WM \equiv MB \pmod{2}$. If $M = [m_{ij}]$, then a comparison of the entries of $WM$ and $MB$ shows that

$$M \equiv \begin{bmatrix} m_{11} & m_{11} & m_{11} \\ m_{21} & m_{22} & m_{21} + m_{22} \\ m_{21} + m_{22} & m_{21} & m_{22} \end{bmatrix} \pmod{2}.$$ 

Since $|M| = 1$, $m_{11}$ must be odd, and not both of $m_{21}, m_{22}$ can be even. Thus, modulo 2, $M$ is one of

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We shall need the matrix $P$ such that $B^P = W^T$ (see Lemma 5), the matrix

$$U = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

which can be seen to have the property that $B^U = W$, and the matrix

$$K = sI + tW^T + u(W^2)^T = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 2 \\ 0 & -1 & -2 \end{pmatrix}$$

where $s = t = -2/3, u = 1/3$.

We now let

$$G = K^{-1}P^{-1}UM = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} M.$$

Then $G \in \text{SL}(3, \mathbb{Z})$ and it follows from our information about the form of $M$ modulo 2 that $G \in \langle B \rangle I_2$. Thus, By Lemma 2, $GR_i \in V(\mathbb{Z}A_4)$ for some $i$.

Note that

$$UM = PKG = (PKP^{-1})PG$$

where $PKP^{-1}$ commutes with powers of $B$ since it is a sum of powers of $B$. Therefore,

$$X = W^P = B^{UM} = B^{PG} = W^{TG}.$$ 

Thus, $X^{R_i} = W^{T(GR_i)}$ is a conjugate of $W^T$ in $V(\mathbb{Z}A_4)$. It follows from Lemmas 2 and 6 that $X$ is conjugate in $V(\mathbb{Z}A_4)$ either to $W^T$ or to $W^{TR_i}$.

**Theorem 1.** $V(\mathbb{Z}A_4)$ contains precisely 12 conjugacy classes of elements of order 3. The elements $B^{R_i}, i = 0, 1, 2, 3,$ together with $W^T$...
and \( W^{TR_i} \) are representatives of the 6 conjugacy classes that are congruent to \( B \) modulo 2; their squares are representatives of the other 6 classes.

**Proof.** The theorem is immediate in view of Lemmas 3–7. As we noted after stating Lemma 3, it suffices to find the classes for elements congruent to \( B \) modulo 2; there are then corresponding classes for elements congruent to \( B^2 \) modulo 2. Lemma 3 narrowed the search to conjugates of \( B \) and \( W \) in \( SL(3, \mathbb{Z}) \). Lemma 4 described the classes arising from conjugates of \( B \). Lemma 5 exhibited two distinct classes arising from conjugates of \( W \); Lemma 6 showed that these were the only new classes generated from \( W^T \) by the \( R_i \); Lemma 7 showed that any class arising from a conjugate of \( W \) has to be one produced from \( W^T \) by an \( R_i \).

The authors have shown (see [1])

**Lemma 8.** All elements of order 2 in \( V(ZA_4) \) are conjugate in \( V(ZA_4) \).

Theorem 1 and Lemma 8 account for all the conjugacy classes of elements of finite order in the unit group \( V(ZA_4) \). If \( N \) is any normal subgroup of \( V(ZA_4) \) containing an element of order 2, then it follows from Lemma 8 that \( A \in N \). Thus, a normal complement to \( A_4 \) in \( V(ZA_4) \) cannot contain an element of order 2. We shall now show that any normal subgroup containing an element of order 3 must also contain an element of order 2 and thus establish

**Theorem 2.** All normal complements to \( A_4 \) in \( V(ZA_4) \) are torsion free.

**Proof.** Let \( N \) be a normal subgroup of \( V(ZA_4) \) containing an element of order 3. In view of Theorem 1, it follows that \( N \) contains one of the \( B^{R_i} \) or \( W^T \) or \( W^{TR_i} \).

**Case 1.** Suppose \( B^{R_i} \in N \). A routine calculation shows that \( A^{R_i} \in V(ZA_4) \) for each \( i \). In \( A_4 \), the commutator \((A, B)\) is an element of order 2; therefore \((A, B)^{R_i}\) is an element of order 2 which lies in \( N \).

**Case 2.** Suppose that \( N \) contains \( W^T \) or \( W^{TR_i} \).

Let

\[
M_1 = \begin{pmatrix}
-1 & 0 & -2 \\
2 & -1 & 2 \\
-2 & 0 & -3
\end{pmatrix}
\quad \text{and} \quad
M_2 = \begin{pmatrix}
1 & 2 & 0 \\
-2 & -3 & 0 \\
2 & 2 & 1
\end{pmatrix},
\]

and note that the \( M_i \in V(ZA_4) \).
Let
\[ X = W^T W^{TM_1} W^{TM_2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 8 & 8 & 1 \end{pmatrix}. \]

Then \( X \) is an element of order 2 in \( V(ZA_4) \) which will lie in \( N \) if \( N \) contains \( W^T \). Also, since \( R_1 \) normalizes \( V(ZA_4) \), any normal sub-group containing \( W^{TR_1} \) must contain \( W^{TR_1 H_i} \) where \( H_i = M_i^{R_1} \) and thus will contain \( X^{R_1} \).

Remark. The proof for Case 1 amounted to showing that any normal subgroup containing a \( B^{R_1} \) must contain a conjugate of \( A_4 \). As Sekiguchi showed in [4], \( V(ZA_4) \) contains just 4 conjugate classes of groups isomorphic to \( A_4 \). Our elements \( W^T \) and \( W^{TR_1} \) are not contained in subgroups of \( V(ZA_4) \) which are isomorphic to \( A_4 \). For example, \( W^T = B^P \) but \( A^P \notin V(ZA_4) \) so \( \langle B^P, A^P \rangle \) is isomorphic to \( A_4 \) but it is not contained in \( V(ZA_4) \).

References


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