APÉRY BASIS AND POLAR INVARIANTS OF PLANE CURVE SINGULARITIES

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Let $C$ be an irreducible plane algebroid curve singularity over an algebraically closed field $K$, defined by a power series $f \in K[[X, Y]]$. In this paper, we study those power series $h \in K[[X, Y]]$ for which the intersection multiplicity $(f \cdot h) = \dim_K(K[[X, Y]]/(f, y))$ is an element of the Apéry basis of the value semigroup for $C$. We prove a factorization theorem for these power series, obtaining strong properties of their irreducible factors. In particular we show that some results by M. Merle and R. Ephraim are a special case of this theorem.

**Introduction.** In this paper we denote by $K$ an algebraically closed field of arbitrary characteristic.

Let $C$ be an irreducible plane algebroid curve over $K$ (i.e. $C = \text{Spec}(R)$, where $R = K[[X, Y]]/(f)$, with $f$ irreducible). We will suppose $f \not\in YK[[X, Y]]$ and we will write $n = \text{Ord}_x(f(X, 0))$.

We will denote by $S(C)$ the semigroup of values of $C$ (see [2], 11.0.1 and [3], 4.3.1), by $A_n = \{0 = a_0 < a_1 < \cdots < a_{n-1}\} = \{\min(S(C))n(k + n\mathbb{Z}_+); 0 \leq k \leq n - 1\}$ the Apéry basis of $S(C)$ relative to $n$ (see [2], 1.1.1) and by $\{v_0, \ldots, v_r\}$ the $n$-sequence in $S(C)$, where $v_0 = n$, and $v_i = \min\{v \in S(C); \gcd(v_0, v_1, \ldots, v_{i-1}) > \gcd(v_0, v_1, \ldots, v_{i-1}, v)\}, 1 \leq i \leq r$ (see [1], 6.6, [2], 1.3.2 and [6]). (Note that $\gcd(v_0, \ldots, v_r) = 1$.)

The main objective of this work is the proof of the following theorem.

**Factorization Theorem.** Let $h \in K[[X, Y]]$ be such that $0 \leq k = \text{Ord}_x(h(X, 0)) \leq n - 1$. Then $(f \cdot h) \leq a_k$. Suppose $(f \cdot h) = a_k$. If $k = \sum_{0 \leq q \leq r} s_q(n/d_{q-1})$, where $d_q = \gcd(v_0, \ldots, v_q), (d_0 = v_0 = n, d_r = 1), 0 \leq s_q \leq r$ and $0 \leq s_q \leq d_{q-1}/d_q$, then

$$h = \prod_{1 \leq i \leq r} h_i \quad \text{and} \quad h_i = \prod_{1 \leq j \leq m_i} h_{ij},$$

with $h_{ij}$ either irreducible or unit in $K[[X, Y]]$, $1 \leq j \leq m_i$, $1 \leq i \leq r$, and

$$1 \leq j \leq m_i \quad \text{Ord}_x(h_j(X, 0)) = s_i(n/d_{i-1}), 1 \leq i \leq r.$$
Here \( (f \cdot h) \) denotes, for two power series \( f \) and \( h \), the intersection multiplicity of the algebroid cycles defined, respectively, by \( f \) and \( h \).

In the fourth section we see that the polars of an irreducible complex analytic germ of a plane curve singularity satisfy the hypotheses of the above theorem for \( k = n - 1 \). Thus, the Theorem 3.1 of [5] and Lemma 1.6 of [4] follow from the above Factorization Theorem.

1. Apéry basis and the \( n \)-sequence. In this section we will summarize some properties of the Apéry basis. For other properties you can see [2] and [6].

**Proposition 1.** If \( M_j = K[[Y]] + K[[Y]]X + \cdots + K[[Y]]X^j, 0 \leq j \leq n - 1 \), then:

1. \( \{a_j\} = v(M_{j-1} + X^j) - v(M_{j-1}), 1 \leq j \leq n - 1 \),
2. \( v(M_j) = \bigcup_{0 \leq i \leq j} (a_i + n\mathbb{Z}_+), 0 \leq j \leq n - 1 \),
3. \( a_i + a_j \leq a_{i+j}, 0 \leq i + j \leq n - 1 \),

where \( v(M_i) = \{(f \cdot g); g \in M_i - \{0\}\}, 0 \leq i \leq n - 1 \) and \( v(M_{i-1} + X^i) = \{(f \cdot (g + X^i)); g \in M_{i-1}\}, 1 \leq i \leq n - 1 \).

**Proof.** See [2], Satz 3 and [6], Proposition 2.

**Remark 2.** Note that in the above proposition \( a_j \geq (f \cdot (g + X^j)) \) for each \( g \in M_{j-1}, 1 \leq j \leq n - 1 \). (If \( (f \cdot (g + X^j)) > a_j \), then there exists \( g_{j-1} \in M_{j-1} \) such that \( (f \cdot (g_{j-1} + X^j)) = a_j \), so \( a_j = (f \cdot (g - g_{j-1})) \) and we get a contradiction.)

**Proposition 3.** One has

\[
as_1(d/d_0) + \cdots + s_j(d/d_{j-1}) = s_1v_1 + \cdots + s_jv_j,
\]

and \( v_{j+1} > (d_{j-1}/d_j)v_j, 0 \leq j \leq r - 1, \) with \( 0 \leq s_i \leq (d_{i-1}/d_i), 1 \leq i \leq r \).

**Proof.** See [2], Satz 2 and [6], Proposition 1.

**Remark 4.** Note that \( v_j = a_{d/d_j}, 1 < j < r \) and

\[
A_n = \{as_1(d/d_0) + \cdots + s_r(d/d_{r-1}); 0 \leq s_i < (d_{i-1}/d_i), 1 < i < r\}.
\]

**Example 5.** Here we give some examples of different possibilities for the Apéry basis and \( n \)-sequences. Let us consider the curves
\( C_i = \text{Spec}(K[[X, Y]]/(f_i)), 1 \leq i \leq 3, \) where \( f_1 = X^2 + Y^5, f_2 = (Y + X^2)^2 + X^5 \) and \( f_3 = Y^2 + X^5. \) It is easy to check that
\[
S(C_1) = S(C_2) = S(C_3) = \{0, 2, 4, 5, 6, 7, 8, \ldots \},
\]
and one has \( f_i \notin YK[[X, Y]], 1 \leq i \leq 3, \) and \( \text{Ord}_X(f_1(X, 0)) = 2, \)
\( \text{Ord}_X(f_2(X, 0)) = 4 \) and \( \text{Ord}_X(f_3(X, 0)) = 5. \) So \( A_2 = \{0 = a_0, a_1 = 5\}. \) The 2-sequence is \( \{v_0 = 2, v_1 = 5\}, a_1 = (f_1 \cdot X), d_0 = d = 2 \) and \( d_1 = 1. \) \( A_4 = \{0 = a_0, a_1 = 2, a_2 = 5, a_3 = 7\}. \) The 4-sequence is \( \{v_0 = 4, v_1 = 2, v_3 = 5\}, a_1 = (f_2 \cdot X), a_2 = (f_2 \cdot (Y + X^2)), a_3 = (f_2 \cdot (Y + X^2))X, d_0 = d = 4, d_1 = 2 \) and \( d_2 = 1. \) And \( A_5 = \{0 = a_0, a_1 = 2, a_2 = 4, a_3 = 6, a_4 = 8\}. \) The 5-sequence is \( \{v_0 = 5, v_1 = 2\}, a_i = (f_3 \cdot X^i), 1 \leq i \leq 4, d_0 = d = 5 \) and \( d_1 = 1. \)

2. \( n \)-sequences and Hamburger-Noether expansions. Let \( x \) and \( y \) be, respectively, the residue classes of \( X \) and \( Y \) in \( R. \) Assume that \( n_0 = (f \cdot X) \leq (f \cdot Y) = n, \) that is, \( X \) is a generic coordinate (or \( x \) is a transversal parameter of \( C, \) see [3]) and \( Y \) could be generic, or have maximal contact with \( f, \) or any thing in between. In this form, we can study all of these possibilities for \( Y \) simultaneously. This is the point of taking the Apéry basis with respect to a general \( n, \) rather than \( n = n_0. \) If \( n = n_0 \) then \( Y \) should be generic.

Let
\[
y = a_{01}x + \cdots + a_{0h_0}x^{h_0} + x^{h_0}z_1,
\]
\[
x = z_1^{h_1}z_2,
\]

\[
z_{s_1-1} = a_{s_1k_1}z_{s_1}^{k_1} + \cdots + a_{s_1h_{s_1}}z_{s_1}^{h_{s_1}} + z_{s_1}^{h_{s_1}}z_{s_1+1},
\]

\[
z_{s_g-1} = a_{s_gk_g}z_{s_g}^{k_g} + \cdots
\]

be the Hamburger-Noether expansion of \( C \) in the basis \((x, y)\) (see [3], 2.2.2 and 3.3.4), and let \( n_i = \text{Ord}_{z_{s_g}}(z_i), 0 \leq i \leq s_g (z_0 = x), \)
\( (1 = n_{s_g} < n_{s_g-1} < \cdots < n_0 \leq n = \text{Ord}_{z_{s_g}}(y), \) see [3], 2.2.5).

Note that the Hamburger-Noether expansion is nothing but an explicit description of the minimal resolution of singularities \( C \) of \( C \) by a sequence of point blowing-ups. \( z_i, z_i-1 \) are the regular parameters of the ambient plane at the \( h_0 + \cdots + h_i \)th blowing up. \( z_{s_g} \) is a regular parameter of \( C. \) In particular, for any \( h \in K[[X, Y]] \) such that \( f \) does not divide \( h \)
\[
(f \cdot h) = \text{Ord}_{z_{s_g}}(h).
\]
The following proposition is an easy consequence of the Hamburger-Noether expansion and the formula for Zariski exponents of a plane curve (see [3] 4.2.7 and 4.3.10).

**Proposition 6.** With the above notations one has:

1. \( n_0 = \min(S(C) - \{0\}) \),
2. \( n_0 \leq n = v_0 \leq h_0 n_0 + n_1 \),
3. (i) If \( v_0 \leq v_1 \), then \( r = g \), \( v_0 = n_0 \) and
   \[
   v_{i+1} = \left(1/n_{s_i}\right) \sum_{0 \leq j \leq s_i} h_j n_j^2 + n_{s_i+1},
   \]
   \( 0 \leq i \leq r - 1 \), \( (s_0 = 0) \). Moreover \( a_{01} \neq 0 \).

   (ii) If \( v_0 > v_1 \) and \( d_1 = v_1 \), then \( r = g + 1 \), \( v_0 = k_0 v_1 \), \( k_0 \geq 2 \), \( v_1 = n_0 \) and
   \[
   v_{i+2} = \left(1/n_{s_i}\right) \sum_{0 \leq j \leq s_i} h_j n_j^2 + n_{s_i+1},
   \]
   \( 0 \leq i \leq r - 1 \), \( (s_0 = 0) \). Moreover \( a_{0j} = 0 \), \( 1 \leq j < k_0 \) and \( a_{1k_0} \neq 0 \).

   (iii) If \( v_0 > v_1 \) and \( d_1 < v_1 \), then \( r = g \), \( v_1 = n_0 \), \( v_0 = h_0 n_0 + n_1 \) and
   \[
   v_{i+1} = \left(1/n_{s_i}\right) \sum_{0 \leq j \leq s_i} h_j n_j^2 + n_{s_i+1},
   \]
   \( 0 \leq i \leq r - 1 \), \( (s_0 = 0) \). Moreover \( a_{0j} = 0 \), \( 1 \leq j \leq h_0 \).

**Proof.** (1) and (2) are obvious from the Hamburger-Noether expansions. We must only prove (3).

For this, if one writes \( \overline{\beta}_0 = n_0 \) and
\[
\overline{\beta}_i = \left(1/n_{s_i}\right) \sum_{0 \leq j \leq s_i} h_j n_j^2 + n_{s_i+1},
\]
\( 0 \leq i \leq g - 1 \), then one has

(I) \( \overline{\beta}_i = \min(S(C) - \{0\}) \) and \( \overline{\beta}_i = \min\{\overline{\beta} \in S(C); \gcd(\overline{\beta}_0, \ldots, \overline{\beta}_{i-1}) \>
\]
\( \gcd(\overline{\beta}_0, \ldots, \overline{\beta}_{i-1}, \overline{\beta})\}, 1 \leq i \leq g \) (see [3], 4.2.7 and 4.3.10).

On the other hand, note that one has the equalities

(II) \( v_0 = n \) and \( v_i = \min\{v \in S(C); \gcd(v_0, \ldots, v_{i-1}) \>
\]
\( \gcd(v_0, \ldots, v_{i-1}, v)\}, 1 \leq i \leq r \).

We distinguish the following three possibilities:

(i) \( n_0 = n < h_0 n_0 + n_1 \). In that case \( a_{01} \neq 0 \), \( v_0 = n_0 \) and it follows from (I) and (II) that \( r = g \) and \( v_i = \overline{\beta}_i \), \( 1 \leq i \leq g \).

(ii) \( n_0 < n = k_0 n_0 < h_0 n_0 + n_1 \). Then \( a_{0j} = 0 \), \( 1 \leq j \leq k_0 \), \( a_{0k_0} \neq 0 \), \( v_0 = k_0 n_0 \), \( v_1 = n_0 \) and it follows from (I) and (II) that \( r = g + 1 \) and \( v_{i+1} = \overline{\beta}_i \), \( 1 \leq i \leq r - 1 \).
(iii) $n_0 < n = h_0 n_0 + n_1$. Now $a_{0j} = 0$, $1 \leq j \leq h_0$, $v_0 = h_0 n_0 + n_1$, $v_1 = n_0$ and it follows from (I) and (II) that $r = g$ and $v_i = \beta_i$, $2 \leq i \leq r$.

3. Infinitely near points and intersection multiplicity. Now consider another irreducible plane algebroid curve over $K$, $C' = \text{Spec}(R')$, with $R' = K[[X, Y]]/(f')$, $C' \neq C$ and $f' \notin YK[[X, Y]]$. Let $x'$ and $y'$ be the residue classes of $X$ and $Y$, respectively, in $R'$. We denote by

$$y' = a_{01}' x' + \cdots + a_{0h_0}' x'^{h_0} + x'^{h_0} z'_1,$$

$$x' = z'_1 z'_2,$$

$$z'_{s_i'-1} = a_{s_i'k_i'} z'^{k_i'} s'_i + \cdots + a_{s_i'k_i'} z'^{k_i'} s'_i + z'_1 z'_1 z'_1 z'_{s_i'-1},$$

the Hamburger-Noether expansion of $C$ in the basis $(x', y')$. We also put $n'_i = \text{Ord}_{z'_i}(z'_i)$, $0 \leq i \leq s'_i$, $(x' = z'_0)$ and $n' = \text{Ord}_x(f'(X, 0)) = \text{Ord}_{x'}(y')$.

Let $N$ be the number of infinitely near points that $C$ and $C'$ have in common (i.e. $N = h_0 + h_1 + \cdots + h_{s-1} + i - 1$, $s$ being the largest integer for which $h_q = h'_q$, $0 \leq q \leq s - 1$, and $a_{jk} = a'_{jk}$, $i \leq k \leq h_j$, $0 \leq j \leq s - 1$, and $i$ being the least index such that $a_{si} \neq a'_{si}$ $(i \leq h_s + i, i \leq h'_s + 1)$) (see [3] 2.3.2).

**Proposition 7.** If

$$\sum_{0 \leq q \leq s_i - 1} h_q + k_i - 1 \leq N \leq \sum_{0 \leq q \leq s_i - 1} h_q + k_i - 1,$$

$1 \leq i \leq g$, $(s_0 = 0)$, then $(f \cdot f') \leq n'd_{j-1} v_j / n$, where $j = i$ if $v_0 < v_1$ or $v_0 > v_1$, $d_1 < v_1$, and $j = i + 1$ if $v_0 > v_1$, $d_1 = v_1$. Furthermore, if $(f \cdot f') < n'd_{j-1} v_j / n$, then $d_{j-1}$ divides $(f \cdot f')$.

**Proof.** One has $n = h_{q+1} n_{q+1} + n_{q+2}$, $s_j \leq q \leq s_{j+1} - 2$, $n_{s_{j+1}-1} = k_{j+1} n_{s_{j+1}}$, $0 < j \leq g - 1$, and $n'_p = h'_{p+1} n'_{p+1} + n'_{p+2}$, $s'_j \leq p \leq s'_{j+1} - 2$, $n'_{s'_{j+1}-1} = k'_{j+1} n'_{s'_{j+1}}$, $0 < j \leq g' - 1$. 
So \( n_{s_i} \) divides \( n_i \), and \( n'_{s_i} \) divides \( n'_k \) for \( i < s_j \) and \( k < s'_j \). On the other hand, since

\[
\sum_{0 \leq q \leq s_{i-1} - 1} h_q + k_{i-1} \leq N
\]

then \( h_q = h'_q \), \( 0 \leq q \leq s_{i-1} - 1 \) and \( k_{i-1} = k'_{i-1} \), so

(III) \( n/n_{s_{i-1}}, n_q/n_{s_{i-1}} = n'_q/n'_{s_{i-1}}, 0 \leq q \leq s_{i-1} \).

From Proposition 5 we see that

(IV) \( d_{j-1} = n_{s_{i-1}} \).

Thus, one can compute \((f \cdot f')\) in terms of the possible values of \( N \) (see [3], 2.3.2 and 2.3.3). Namely, one has the following possibilities:

(A) \( N = \sum_{0 \leq q \leq s_{i-1} - 1} h_q + k_{i-1} \), with \( k_{i-1} < k < \min(h_{s_{i-1}}, h'_{s_{i-1}}) \).

In that case one has

\[
(f \cdot f') = \sum_{0 \leq q < s_{i-1} - 1} h_q n_q n'_q + k n_{s_{i-1} - 1} n'_{s_{i-1}}
\]

so \( d_{j-1} \) divides \((f \cdot f')\) by (IV), and \( \alpha = n'd_{j-1}v_j/n \), by (III), (IV) and Proposition 6.

(B) \( N = \sum_{0 \leq q \leq s} h_q \), with \( s_{i-1} \leq s < \min(s_i, s'_i) \) and \( h_s < h'_s \).

Now one has

\[
(f \cdot f') = \sum_{0 \leq q \leq s} h_q n_q n'_q + n_{s+1} n'_s
\]

\[
< \sum_{0 \leq q \leq s-1} h_q n_q n'_q + h'_s n sn'_s + n sn'_s+1 = \beta.
\]

(Note that \( h_s < h'_s \), so \( n_{s-1} n'_s = h_s n sn'_s + n_{s+1} n'_s < (h_s + 1)n_s n'_s \leq h'_s n sn'_s < h'_s n sn'_s + n sn'_s+1 \).) By (III), (IV) and Proposition 6, it follows that

\[
(f \cdot f') = \sum_{0 \leq q < s_{i-1}} h_q n_q n'_q + n_{s_{i-1}+1} n_{s_{i-1}} = n'd_{j-1}v_j/n, \quad \text{or}
\]

\[
(f \cdot f') = \sum_{0 \leq q < s_{i-1}} h_q n_q n'_q + n_{s_{i-1}} n'_{s_{i-1}+1} < \beta = n'd_{j-1}v_j/n,
\]

and \( d_{j-1} \) divides \((f \cdot f')\).

The other cases can be proved in a similar way:

(B') \( N = \sum_{0 \leq q \leq s-1} h_q + h'_s \), with \( s_{i-1} \leq s < \min(s_i, s'_i) \) and \( h'_s < h_s \).

(C.1) \( N = \sum_{0 \leq q \leq s_{i-1}} h_q + k_i - 1 \), with \( s_i < s'_i \) and \( k_i < h'_{s_i} \).

(C.2) \( N = \sum_{0 \leq q \leq s_{i-1}} h_q + h'_{s_i} \), with \( s_i < s'_i \) and \( h'_{s_i} < k_i \).
(C') \( N = \sum_{0 \leq q \leq s_i - 1} h_q + k_i - 1 \), with \( s'_i < s_i \) and \( k'_i < h_{s'_i} \).

(C.2) \( N = \sum_{0 \leq q \leq s_i - 1} h_q + h_{s'_i} \), with \( s'_i < s_i \) and \( h_{s'_i} < k'_i \).

(D) \( N = \sum_{0 \leq q < s_i - 1} h_q + k_i - 1 \), with \( s_i = s'_i \) and \( k_i < k'_i \).

(D') \( N = \sum_{0 \leq q < s_i - 1} h_q + k_i - 1 \), with \( s_i = s'_i \), \( k_i = k'_i \) and \( a_{s_i, k_i} \neq a'_{s_i, k_i} \).

**Corollary 8.** For each nonnegative integer \( j \), \( 1 \leq j \leq r \), the following statements are equivalent:

(1) \( (f \cdot f') > n'd_{j-1}v_j/n \),

(2) \( N = \sum_{0 \leq q < s_i - 1} h_q + k_i - 1 \),

where \( i = j \) if \( v_0 < v_1 \) or \( v_0 > v_1 \) and \( d_1 < v_1 \), and \( i = j - 1 \), \( k_0 = v_0/v_1 \) if \( v_0 > v_1 \) and \( d_1 = v_1 \). In particular, if either (1) or (2) is true then \( n' = n's_n/d_j \).

**Proof.** (1) \( \Rightarrow \) (2). If \( v_0 > v_1 \), \( d_1 = v_1 \) and \( (f \cdot f') > n'v_1 \) then \( N > k_0 - 1 \). Indeed, suppose \( N \leq k_0 - 1 \). Then \( a_{0q} = a'_{0q} \), for \( q \leq N \) and \( a_{0N+1} \neq a'_{0N+1} \). If \( a'_{0N+1} \neq 0 \) then \( (N + 1)n_0 = n' \) and if \( a'_{0N+1} = 0 \) then \( N + 1 = k_0 \) and \( (N + 1)n_0 \leq n' \), so in any case \( (f \cdot f') = (N + 1)n_0n'_0 \leq n'v_1 \) and we get a contradiction.

Now suppose \( (f \cdot f') > n'd_{j-1}v_j/n \) and

\[
\sum_{0 \leq q < s_i - 1} h_q + k_i - 1 < N
\]

with \( j \geq 1 \) if \( v_0 < v_1 \) or \( v_0 > v_1 \) and \( d_1 < v_1 \), and with \( j \geq 2 \) if \( v_0 > v_1 \) and \( d_1 = v_1 \). Then we can assume

\[
\sum_{0 \leq q \leq s_{p-1} - 1} h_q + k_{p-1} < N \leq \sum_{0 \leq q \leq s_{p-1} - 1} h_q + k_p - 1,
\]

with \( 1 \leq i \leq p \). It follows from Proposition 7 that \( (f \cdot f') \leq n'd_{s-1}v_s/n \), with \( s \leq j \) and \( d_{s-1}v_s \leq d_{j-1}v_j \) (see [2], Satz 2) which is a contradiction.

(2) \( \Rightarrow \) (1). If \( v_0 > v_1 \), \( d_1 = v_1 \) and \( N > k_0 - 1 \), then \( (f \cdot f') > k_0n_0n'_0 \), and \( n' = k_0n'_0 \), \( (a_{0k_0} = a'_{0k_0}) \), so one has \( (f \cdot f') > n'v_1 \) \( (n_0 = v_1) \).

Now if

\[
\sum_{0 \leq q \leq s_i - 1} h_q + k_i - 1 < N
\]
with \(i \geq 1\) then \(n/n_{s_i} = n'/n'_{s_i}\), \(n_q/n_{s_i} = n'_q/n'_{s_i}\), \(0 \leq q \leq s_i\) and
\[
(f \cdot f') = \sum_{0 \leq q \leq s_i-1} h_q n_q n'_q + k_i n_{s_i} n'_{s_i} = \gamma.
\]

By Proposition 6
\[
(n'/n)d_{j-1} v_j = (n'_{s_i-1}/n_{s_i-1}) \left( \sum_{0 \leq q \leq s_i-1} h_q n_q^2 + n_{s_i-1+1} n_{s_i-1} \right).
\]

Now
\[
\gamma = \sum_{0 \leq q \leq s_i-1} h_q n_q n'_q + k_i n_{s_i} n'_{s_i} = (n_{s_i-1}/n_{s_i-1}) \left( \sum_{0 \leq q \leq s_i-1} h_q n_q^2 + k_i n_{s_i}^2 \right).
\]

Thus we have to show that
\[
\sum_{0 \leq q \leq s_i-1} h_q n_q^2 + n_{s_i-1+1} n_{s_i-1} = \sum_{0 \leq q \leq s_i-1} h_q n_q^2 + k_i n_{s_i}^2.
\]

But this follows by repeated application of the identities \(n_{q-1} = h_q n_q + n_{q+1}\), since \(k_i n_{s_i} = n_{s_i-1}\).

**Corollary 9.** For \(1 \leq j \leq r\), if \((f \cdot f') < n'd_{j-1} v_j/n\), then \(d_{j-1}\) divides \((f \cdot f')\).

**Proof.** If \(v_0 > v_1\), \(d_1 = v_1\) and \((f \cdot f') < n'v_1\) then \(N \leq k_0 - 1\) (Corollary 8). Thus, if \(a_{0q} = a'_{0q}\), \(1 \leq q \leq N\), and \(a_{0N+1} \neq a'_{0N+1}\) then \(N + 1 = k_0\) and \((f \cdot f') = (N + 1)n_0 n'_0 = n_0' v_0\). (For if \(N + 1 < k_0\) then \((f \cdot f') = n'v_1\) which is a contradiction.)

Now we can assume \((f \cdot f') < n'd_{j-1} v_j/n\), with \(j \geq 1\) if \(v_0 < v_1\) or \(v_0 > v_1\) and \(d_1 < v_1\), and \(j \geq 2\) if \(v_0 > v_1\) and \(d_1 = v_1\). By Corollary 8 one has
\[
\sum_{0 \leq q \leq s_i-1} h_q + k_i - 1 \geq N
\]
with \(i = j\) if \(v_0 < v_1\) or \(v_0 > v_1\) and \(d_1 < v_1\), and with \(i = j - 1\) if \(v_0 > v_1\) and \(d_1 = v_1\). So, by Proposition 7, \(d_{j-1}\) divides \((f \cdot f')\).

**4. Proof of the Factorization Theorem.** As \(\text{Ord}_x(h(X,0)) = k\) we can write \(h = uh'\), with \(h' \in M_{k-1} + X^k\) and \(u \in K[[X, Y]]\) being a unit. So \((f \cdot h) = (f \cdot h') \leq a_k\).

Also, we can write \(a_k = \sum_{0 \leq q \leq s} s_q v_q\) and \(k = \sum_{0 \leq q \leq r} s_q (d/d_q)\), with \(0 \leq s_q < d_{q-1}/d_q\) (see Remark 4). Let \(q\) be the greatest index such that \(s_q \neq 0\) and let
\[
h = \prod_{0 \leq j \leq m} h_j
\]
be the factorization of $h$ as a product of irreducible elements in $K[[X,Y]]$.

If for any $j$

$$\frac{(f \cdot h_j)}{\text{Ord}_x(h_j(X,0))} > d_{q-1}v_q/n$$

then, by Corollary 8, $\text{Ord}_x(h_j(X,0)) = an/d_q$ ($a \neq 0$), but $k < n/d_q$ which is a contradiction. (Note that $s_p = 0$ for $p > q$ and

$$k \leq \sum_{1 \leq p \leq q} \left(\frac{(d_{p-1}/d_p)}{d_{p-1}} - 1\right) = \frac{d}{d_{q-1}} - 1 < \frac{d}{d_q} = n/d_q.$$)

On the other hand, if for $1 \leq j \leq m$

$$\frac{(f \cdot h_j)}{\text{Ord}_x(h_j(X,0))} < d_{q-1}v_q/n$$

then $d_{q-1}$ divides $(f \cdot h)$ by Corollary 9. So $d_{q-1}/d_q$ divides $s_q$, and hence $s_q = 0$ since $0 \leq s_q < d_{q-1}/d_q$, and we get a contradiction.

Thus, there exists $h_{j_0}$ such that

$$\frac{(f \cdot h_{j_0})}{\text{Ord}_x(h_{j_0}(X,0))} = d_{q-1}v_q/n.$$  

Moreover, if $q \geq 2$ then $\text{Ord}_x(h_{j_0}(X,0)) = an/d_{q-1}$ by Corollary 8, as $d_{q-1}v_q > d_qv_{q-1}$ (see Proposition 3). If $q = 1$ then $(f \cdot h_{j_0}) = \text{Ord}_x(h_{j_0}(X,0)) = an/d_{q-1}$. In any case $\text{Ord}_x(h_{j_0}(X,0)) = an/d_{q-1}$ with $0 \leq a \leq s_q$.

(Note that $k \leq \sum_{1 \leq p \leq q} (\frac{(d_{p-1} - 1)}{d_{p-1}} + s_q d/d_{q-1} - (d/d_{q-1}) + s_q d/d_{q-1} = (s_q + 1)d/d_{q-1} = (s_q + 1)n/d_{q-1}.)$

So $h' = h/h_{j_0}$ satisfies $\text{Ord}_x(h'(X,0)) = k' = k - an/d_{q-1}$ and

\[(f \cdot h') = a_k - a(n/d_{q-1})d_{q-1}v_q/n = a_k - av_q = a_k';\] hence the Theorem follows by iterating the above reasoning using $h'$ instead of $h$ in the next step.

5. **The complex analytic case.** In this section, $C$ is assumed to be an irreducible complex analytic germ at $0 \in C^2$ of a plane curve singularity.

Let $n$ be the multiplicity of $C$ and let $P(C)$ be a general polar of $C$ (i.e. $P(C)$ is defined by a reduced element $h = \lambda(\partial f/\partial X) - \mu(\partial f/\partial Y)$ of $C\{X,Y\}$, and $n - 1$ is the multiplicity of $P(C)$). M. Merle in [5] has proved that $P(C)$ decomposes into $g$ curves $\Gamma\{1\}, \ldots, \Gamma\{g\}$, where $\Gamma\{g\} (1 \leq q \leq g)$ is such that

1. its multiplicity is $(n/e_{q-1})((e_{q-1}/e_q) - 1)$,
2. every irreducible component of $\Gamma\{q\}$, $\Gamma\{q\}$, has a contact of order $\beta_q$ with $C$ and $\frac{(\Gamma\{q\}) \cdot C)\cap (\Gamma\{q\}) = \beta_q/(n/e)$.


Here \( \{\beta_0, \ldots, \beta_g\} \) is the minimal system of generators of \( S(C) \), \( e_q = \gcd(\beta_0, \ldots, \beta_q) \), \( 0 \leq q \leq g \), \( \beta_0 < \beta_1 < \cdots < \beta_g \) are the Puiseux exponents and \( m(\Gamma_{(q,i)}) \) denotes the multiplicity of \( \Gamma_{(q,i)} \).

Without loss of generality, we may assume that \( n = \text{Ord}_x(f(X,0)) \), and therefore \( n - 1 = \text{Ord}_x(h(X,0)) \).

On the other hand,
\[
(f \cdot h) = \sum_{0 \leq q \leq g} ((e_{q-1}/e_q) - 1)\beta_q.
\]
and hence \( f \cdot h = a_{n-1} \), since \( \{\beta_0, \ldots, \beta_g\} \) is the \( n \)-sequence in \( S(C) \) (see [2], Satz 2 and [5], Prop. 1.1).

Thus, \( h \) satisfies the hypotheses of the Factorization Theorem for \( k = n - 1 \), and the above Theorem 3.1 of [5] is a special case of ours. (Note that \( \Gamma_{(q,i)} \) has a contact of order \( \beta_q \) with \( C \) if and only if \( (\Gamma_{(q,i)} \cdot C)/m(\Gamma_{(q,i)}) = \beta_q/(n/e_{q-1}) \), see [5], Prop. 2.4.)

In general, if \( M \) is a smooth germ of a plane curve singularity defined by \( z \in C\{X, Y\} \), then the polar of \( C \) with respect to \( M \) is the (possibly nonreduced) germ whose defining ideal is generated by the Jacobian \( J(f, z) = \partial(f, z)/\partial(X, Y) \) (see [4]). In particular, a general polar \( P(C) \) of \( C \) is defined by \( h = J(f, \lambda X + \mu Y) \) with \( (\lambda, \mu) \) general.

Thus, without loss of generality, we may assume that \( z = Y \) (since \( M \) is smooth) and \( J(f, z) = \partial f/\partial X \).

**Proposition 10.** Keeping the above notations, one has
(a) \( \text{Ord}_x((\partial f/\partial X)(X,0)) = \text{Ord}_x(f(X,0)) - 1 = n - 1 \).
(b) \( (f(\partial f/\partial X)) = a_{n-1} \).

**Proof.** (a) It is obvious.
(b) If \( n = \text{Ord}_x(f(X,0)) \geq \text{Ord}_y(f(0,y)) = m \) then one has a Puiseux type parametrization of \( C \)
\[
X = t^m, \quad Y = \Psi(t)
\]
and we can write (up to multiplication by a unit)
\[
f(X, Y) = \prod_{0 \leq q \leq m} (X - \Psi(W^q X^{1/m})),
\]
Thus,
\[
(f \cdot (\partial f/\partial X)) = \text{Ord}_t((\partial f/\partial X)(t^m, \Psi(t)))
\]
\[
= \text{Ord}_t(\Psi^1(t^m)) + \text{Ord}_t \left( \prod_{1 \leq q \leq m-1} (\Psi(t) - \Psi(W^q t)) \right).
\]
where \( \Psi^1(X^{1/m}) = \partial/\partial X(\Psi(X^{1/m})) \).
On the other hand, we can write
\[ \Psi(X^{1/m}) = \sum_{1 \leq j \leq i_0} a_{0j} X^{jn/m} + \sum_{0 \leq j \leq i_1} a_{1j} X^{(\beta_1 + je_1)/m} + \cdots + \sum_{0 \leq j} a_{gj} X^{(\beta_g + je_g)/m}, \]
where \( m = \beta_0 < \beta_1 < \cdots < \beta_g \) are the Puiseux exponents of \( C \) and \( e_i = \gcd(\beta_0, \ldots, \beta_i), 1 \leq i \leq g. \)

Then we have \( \text{Ord}_t \Psi(X^{1/n}) = n - m, \) and
\[ \text{Ord} \left( \prod_{1 \leq q \leq m-1} (\Psi(t) - \Psi(w^qt)) \right) = \sum_{1 \leq q \leq g} (e_{i-1} - e_i) \beta_i. \]
(Note that \( \text{Ord}_t(\Psi(t) - \Psi(w^qt)) = \beta_j, \) if
\[ q \in \{k(e_{j-2}/e_{j-1}); 1 \leq k < e_{j-1}\} - \{k(e_{j-1}/e_j); 1 \leq k < e_j\}, \]
\[ 1 \leq j \leq g \quad (e_{-1} = e_0 = m). \]

Now
\[ \sum_{1 \leq i \leq g} (e_{i-1} - e_i) \beta_i = c + m - 1, \]
where \( c \) is the conductor of \( S(C) \) (i.e. \( c = \min\{d \in S(C); d + \mathbb{Z}_+ \subset S(C)\} \), see [3], 4.4) and \( c + n - 1 = a_{n-1}, \) since
\[ A_n = \{\min(S(C) \cap (j + n\mathbb{Z}_+); 0 \leq j \leq n - 1\}. \]

Finally, a similar argument shows that \((f \cdot \partial f/\partial X) = c + n - 1, \) if \( n = \text{Ord}_x(f(X,0)) < \text{Ord}_y(f(0, Y)). \)

**Remark 11.** Proposition 10 shows that if \( h \) defines the polar of \( C \) with respect to \( M \) then \( h \) satisfies the hypotheses in the Factorization Theorem for \( k = n - 1, \) so Lemma 1.6 of [4] is also a special case of (2) in the Factorization Theorem.

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**References**


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