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**ON THE FINEST LEBESGUE TOPOLOGY ON THE SPACE OF
ESSENTIALLY BOUNDED MEASURABLE FUNCTIONS**

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Let (Ω, Σ, μ) be a σ -finite measure space and let \mathcal{T}_0 and \mathcal{T}_∞ denote the usual metrizable topologies on L^0 and L^∞ , respectively. In this paper the space L^∞ with the mixed topology $\gamma(\mathcal{T}_\infty, \mathcal{T}_0|_{L^\infty})$ is examined. It is proved that $\gamma(\mathcal{T}_\infty, \mathcal{T}_0|_{L^\infty})$ is the finest Lebesgue topology on L^∞ , and that it coincides with the Mackey topology $\tau(L^\infty, L^1)$.

1. Introduction. For notation and terminology concerning Riesz spaces and locally solid topologies we refer to [1].

Let (Ω, Σ, μ) be a σ -finite measure space, and let L^0 denote the set of equivalence classes of all real valued μ -measurable functions defined and finite a.e. on Ω . Then L^0 is a super Dedekind complete Riesz space under the ordering $x \leq y$, whenever $x(t) \leq y(t)$ a.e. on Ω . The Riesz F -norm

$$\|x\|_0 = \int_{\Omega} |x(t)|(1 + |x(t)|)^{-1} f(t) d\mu \quad \text{for } x \in L^0,$$

where a function $f: \Omega \rightarrow (0, \infty)$ is μ -measurable with $\int_{\Omega} f(t) d\mu = 1$, determines a Lebesgue topology on L^0 , which we will denote by \mathcal{T}_0 (see [7, I, §6], [1, Theorem 24.67]). This topology generates convergence in measure on the measurable subsets of Ω whose measure is finite. We will denote by \mathcal{T}_∞ the topology on L^∞ generated by the usual B -norm

$$\|x\|_\infty = \operatorname{ess\,sup}_{t \in \Omega} |x(t)|.$$

Moreover, we denote by $\sigma(L^\infty, L^1)$, $\tau(L^\infty, L^1)$ and $\beta(L^\infty, L^1)$ the weak, Mackey and strong topologies on L^∞ respectively, with respect to the dual pair $(L^\infty, L^1, \langle \cdot, \cdot \rangle)$, where

$$\langle x, y \rangle = \int_{\Omega} x(t)y(t) d\mu \quad \text{for } x \in L^\infty, y \in L^1.$$

In this paper we shall examine the space L^∞ with the mixed topology $\gamma(\mathcal{T}_\infty, \mathcal{T}_0|_{L^\infty})$. This topology is defined as follows. Take a sequence

(ε_n) of positive numbers, a number $r > 0$ and let

$$W((\varepsilon_n), r) = \bigcup_{N=1}^{\infty} \left(\sum_{n=1}^N V(\varepsilon_n) \cap nB(r) \right),$$

where $B(r) = \{x \in L^\infty : \|x\|_\infty \leq r\}$ and $V(\varepsilon_n) = \{x \in L^\infty : \|x\|_0 \leq \varepsilon_n\}$. Then the family of all such $W((\varepsilon_n), r)$ forms a base of neighbourhoods of zero for $\gamma(\mathcal{T}_\infty, \mathcal{T}_0|_{L^\infty})$ (see [11, p. 49]). In view of [11, Theorem 2.2.2] $\gamma(\mathcal{T}_\infty, \mathcal{T}_0|_{L^\infty})$ is the finest linear topology on L^∞ which agrees with $\mathcal{T}_0|_{L^\infty}$ on $\|\cdot\|$ -bounded sets. Henceforth, we will write briefly γ instead of $\gamma(\mathcal{T}_\infty, \mathcal{T}_0|_{L^\infty})$.

The space of bounded sequences l^∞ with the mixed topology γ has been investigated in [4], where among other things, the results from Theorems 5, 6 and 8 below are obtained. The mixed topology γ on l^∞ is the same as the strict topology β [3] on $C(S)$, where $S = \mathbb{N} =$ the set of all natural numbers.

2. The mixed topology γ on L^∞ . It is well known that the norm topology \mathcal{T}_∞ on L^∞ satisfies both the Fatou property and the Levi property (see [7, IV, §3] and [7, X, §4]), and that \mathcal{T}_∞ does not satisfy the Lebesgue property if Ω does not consist of only finite number of atoms (see [7, IV, §3]). We shall show that the mixed topology γ is the finest Hausdorff Lebesgue topology on L^∞ . We start by giving some characterization of sequential convergence in (L^∞, γ) .

THEOREM 1. *For a sequence (x_n) in L^∞ , $x_n \rightarrow 0$ for γ if and only if $\|x_n\|_0 \rightarrow 0$ and $\|x_n\|_\infty < M$ for some $M > 0$ and all $n = 1, 2, \dots$*

Proof. Since the balls $B(r) = \{x \in L^\infty : \|x\|_\infty \leq r\}$, $r > 0$ are closed in \mathcal{T}_0 (see [7, IV, §3, Lemma 5]) the result follows from [11, Theorem 2.3.1].

We now are able to prove the basic property of γ .

THEOREM 2. *The mixed topology γ is the finest Hausdorff Lebesgue topology on L^∞ .*

Proof. Using [1, Theorem 1.2] it is easy to show that γ is a locally solid topology. In order to show that γ is a Lebesgue topology, let us assume that $x_\alpha \downarrow 0$ holds in L^∞ and let (ε_n) be a sequence of positive numbers and $r > 0$. Then there exists an increasing sequence of indices $\{\alpha_n\} \subset \{\alpha\}$ such that $x_{\alpha_n} \downarrow 0$ holds in L^∞ , because

L^∞ has the countable sup property (see [9, Proposition 5.20]). Since \mathcal{T}_0 is a Lebesgue topology, we have $x_{\alpha_n} \rightarrow 0$ for γ by Theorem 1. Then there exists a natural number n_0 such that $x_{\alpha_{n_0}} \in W((\varepsilon_n), \tau)$, so $x_\alpha \in W((\varepsilon_n), r)$ for $\alpha \geq \alpha_{n_0}$, and hence $x_\alpha \rightarrow 0$ for γ . Now let ξ be a Hausdorff Lebesgue topology on L^∞ . Then by [1, Theorem 12.9] we have $\xi|_{[-x,x]} = \mathcal{T}_0|_{[-x,x]}$ for every $0 < x \in L^\infty$. Hence, by [11, Theorem 2.2.2] the inclusion $\xi \subset \gamma$ holds, and thus the proof is finished.

REMARK. It is known that L^∞ has no minimal topology, if the measure μ is atomless [2].

We now consider the problem of separableness of the space (L^∞, γ) . First, we recall some definition. Let \sim be the following equivalence relation in Σ : $A \sim B$ if and only if $\mu(A \dot{-} B) = 0$ ($\dot{-}$ denotes the symmetric difference). Denote by Σ/\sim the set of equivalence classes and by $[A]$ the equivalence class of A . Then on Σ/\sim one can define a metric function $\rho([A], [B]) = \|\chi_A - \chi_B\|_0$. (χ_A denotes the characteristic function of the set A .) The measure μ is said to be separable if the metric space $(\Sigma/\sim, \rho)$ is separable (see [7, I, §6]).

THEOREM 3. *The space (L^∞, γ) is separable if and only if the measure μ is separable.*

Proof. Assume that the space (L^∞, γ) is separable and let $0 < x \in L^0$. Let $x_n = x \wedge ne$, where e denotes the constant function one. Then $0 \leq x_n \uparrow x$ holds in L^0 , so $x_n \rightarrow x$ for \mathcal{T}_0 . Thus L^∞ is dense in (L^0, \mathcal{T}_0) , hence (L^0, \mathcal{T}_0) is separable by hypothesis [7, I, §6]. By [7, I, §6, Theorem 16] the measure μ is separable.

Next, assume that the measure μ is separable. Let

$$\mathcal{P} = \left\{ \sum_{k=1}^m c_k \chi_{A_k} : A_k \in \Sigma, \mu(A_k) < \infty, \right. \\ \left. A_{k_1} \cap A_{k_2} = \emptyset \text{ for } k_1 \neq k_2, c_k \in \mathbf{R}, m \in \mathbf{N} \right\}$$

where \mathbf{R} denotes the set of real numbers. Then $\mathcal{P} \subset L^\infty$ and using Theorem 1, by usual argument one can show that the set \mathcal{P} is dense in (L^∞, γ) . Let Σ_0 be a countable subset of Σ/\sim , which is dense in $(\Sigma/\sim, \rho)$. Let $\mathcal{P}_0 = \{ \sum_{k=1}^m r_k \chi_{A_k} \in \mathcal{P} : [A_k] \in \Sigma_0, r_k \in \mathbf{Q} \}$, where \mathbf{Q} denotes the set of rational numbers. Let $0 \leq x = \sum_{k=1}^m c_k \chi_{A_k} \in \mathcal{P}$. Then, by hypothesis, for every $k = 1, \dots, m$ there exist a sequence

($[A_k^n]$) in Σ_0 and a sequence (r_k^n) of positive rational numbers such that $\|\chi_{A_k^n} - \chi_{A_k}\|_0 \rightarrow 0$ as $n \rightarrow \infty$ and $0 \leq r_k^n \uparrow_n c_k$ for $k = 1, \dots, m$. Putting $x_n = \sum_{k=1}^m r_k^n \chi_{A_k^n}$ for $n = 1, 2, \dots$, we have $\|x_n - x\|_0 \rightarrow 0$ and $|x_n(t)| \leq \max_{1 \leq k \leq m} c_k$ a.e. on Ω . Thus, by Theorem 1, $x_n \rightarrow x$ for γ . It follows that the set \mathcal{P}_0 is dense in $(\mathcal{P}, \gamma|_{\mathcal{P}})$, so \mathcal{P}_0 is dense also in (L^∞, γ) . Thus the space (L^∞, γ) is separable, because the set \mathcal{P}_0 is countable.

The next theorem describes the topological dual of (L^∞, γ) .

THEOREM 4. *For a linear functional f on L^∞ the following statements are equivalent:*

- (i) f is continuous for γ .
- (ii) f is sequentially continuous for γ .
- (iii) There exists a unique $y \in L^1$ such that

$$f(x) = \int_{\Omega} x(t)y(t) d\mu \quad \text{for } x \in L^\infty.$$

Proof. (i) \Leftrightarrow (ii) It follows from [11, Theorem 2.6.1].

(ii) \Leftrightarrow (iii) By Theorem 1, the functional f is sequentially continuous for γ if and only if it is sequentially order star-continuous, and if and only if it is sequentially order continuous (cf. [6, VII, §2]). Thus, in view of [7, VI, §2, Theorem 1] the proof is finished.

As an application of Theorems 2 and 4 we get the following important property of γ .

THEOREM 5. *The mixed topology γ on L^∞ is a Mackey topology, i.e., $\gamma = \tau(L^\infty, L^1)$.*

Proof. Since the Mackey topology $\tau(L^\infty, L^1)$ is a Lebesgue topology (see [1, Ex. 4, p. 163] and [1, Theorem 9.1]), by Theorem 2 we have $\tau(L^\infty, L^1) \subset \gamma$. According to Theorem 4, it suffices to show that γ is a locally convex topology. Indeed, let us put $x_n(t) = n$ for $t \in \Omega$ and $n = 1, 2, \dots$. Let \mathcal{F}_I be the generalized inductive limit topology of $(L^\infty, \tau(L^\infty, L^1), j_n, [-x_n, x_n])$ (see [5, p. 2]), i.e., \mathcal{F}_I is the finest of all locally convex topologies ξ on L^∞ under which the inclusion maps

$$j_n: ([-x_n, x_n], \tau(L^\infty, L^1)|_{[-x_n, x_n]}) \rightarrow (L^\infty, \xi)$$

are continuous for $n = 1, 2, \dots$. By [5, Proposition 5] \mathcal{F}_I is also the finest of all linear topologies ξ on L^∞ under which each of the maps j_n

is continuous. Since γ and $\tau(L^\infty, L^1)$ are Hausdorff Lebesgue topologies, by [1, Theorem 12.9] we have

$$\gamma|_{[-x_n, x_n]} = \tau(L^\infty, L^1)|_{[-x_n, x_n]} \quad \text{for } n = 1, 2, \dots$$

Thus $\gamma \subset \mathcal{I}$. On the other hand, since

$$\mathcal{I}|_{[-x_n, x_n]} \subset \tau(L^\infty, L^1)|_{[-x_n, x_n]} = \mathcal{I}_0|_{[-x_n, x_n]} \quad \text{for } n = 1, 2, \dots,$$

by [11, Theorem 2.2.2] we get $\mathcal{I} \subset \gamma$. Thus $\mathcal{I} = \gamma$; hence γ is locally convex. Therefore, we have $\gamma \subset \tau(L^\infty, L^1)$. Thus the proof is finished.

For a linear topology \mathcal{F} on L^∞ , we will denote by $\text{Bd}(\mathcal{F})$ the collection of all \mathcal{F} -bounded subsets of L^∞ .

Additional properties of γ are included in the next theorem.

THEOREM 6. *The space L^∞ endowed with γ is complete.*

Proof. Since γ is a Lebesgue topology, in view of [1, Theorem 13.9] it suffices to show that γ is a Levi topology. But $\text{Bd}(\gamma) = \text{Bd}(\mathcal{I}_\infty)$ [11, Theorem 2.4.1], so γ is a Levi topology, because we know that \mathcal{I}_∞ is a Levi topology.

COROLLARY 7. *The mixed topology γ is not metrizable.*

Locally convex Hausdorff space (X, ξ) is called sequentially barreled if every $\sigma(X^*, X)$ -convergent to zero sequence in the topological dual $X^* = (X, \xi)^*$ is equicontinuous [10].

THEOREM 8. *The space (L^∞, γ) is sequentially barreled.*

Proof. Combining Theorem 4 and Theorem 5, we have $\gamma = \tau(L^\infty, (L^\infty, \gamma)^+)$, where $(L^\infty, \gamma)^+$ denotes the sequential topological dual of (L^∞, γ) . Since the space (L^∞, γ) is complete, according to [10, Proposition 4.3] the space (L^∞, γ) is sequentially barreled.

Since L^∞ is the norm dual of L^1 we have $\beta(L^\infty, L^1) = \mathcal{I}_\infty$. Therefore, according to Theorem 4 and Corollary 7 we obtain that the space (L^∞, γ) is not barreled.

Additional characterizations of sequential convergence in (L^∞, γ) are included in the next theorem.

THEOREM 9. *For a sequence (x_n) in L^∞ the following statements are equivalent:*

- (i) $x_n \rightarrow 0$ for γ .
- (ii) $x_n \rightarrow 0$ for the absolutely weak topology $|\sigma|(L^\infty, L^1)$.
- (iii) $\int_\Omega |x_n(t)y(t)| d\mu \rightarrow 0$ for every $y \in L^1$.

Proof. (i) \Leftrightarrow (ii) Since $|\sigma|(L^\infty, L^1) \subset \tau(L^\infty, L^1)$ (see [1, Theorem 6.7], assume that $x_n \rightarrow 0$ for $|\sigma|(L^\infty, L^1)$. By [1, Theorem 12.9] we have $|\sigma|(L^\infty, L^1)|_{[-x, x]} = \mathcal{T}_0|_{[-x, x]}$ for every $0 < x \in L^\infty$, because $|\sigma|(L, L^1)$ is a Hausdorff Lebesgue topology. Since the set $\{x_n\}$ is $\sigma(L, L^1)$ -bounded and $\text{Bd}(\sigma(L^\infty, L^1)) = \text{Bd}(\tau(L^\infty, L^1)) = \text{Bd}(\tau_\infty)$ we obtain that $\{x_n\} \subset [-x, x]$ for some $0 < x \in L^\infty$. Thus $\|x_n\|_0 \rightarrow 0$, and in view of Theorem 1 we have $x_n \rightarrow 0$ for γ .

(ii) \Leftrightarrow (iii) Obvious.

The next theorem gives criteria for the compactness of sets in (L^∞, γ) .

THEOREM 10. *For a subset Z of L^∞ the following statements are equivalent:*

- (i) Z is relatively compact for \mathcal{T}_0 and $\|x\|_\infty < M$ for some $M > 0$ and every $x \in Z$.
- (ii) Z is relatively compact for γ .
- (iii) Z is relatively compact for $|\sigma|(L^\infty, L^1)$.

Proof. (i) \Leftrightarrow (ii) Obvious, because we know that $\text{Bd}(\mathcal{T}_\infty) = \text{Bd}(\gamma)$ and the topologies γ and \mathcal{T}_0 coincide on order intervals of L^∞ .

(ii) \Rightarrow (iii) Obvious, because $|\sigma|(L^\infty, L^1) \subset \gamma$.

(iii) \Rightarrow (ii) Combining [8, I, §3, Lemma 11] and Theorem 9, Z is relatively compact for γ .

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Pacific Journal of Mathematics

Vol. 140, No. 1 September, 1989

Michel Brestovski , Algebraic independence of solutions of differential equations of the second order	1
Bohumil Cenk , Cohomology operations from higher products in the de Rham complex	21
Gustavo Corach and Daniel Suarez , Generalized rational convexity in Banach algebras	35
Keresztély Corrádi and Sándor Szabó , A new proof of Rédei's theorem	53
Steven R. Costenoble and Stefan Waner , Equivariant orientations and G -bordism theory	63
Angel Granja , Apéry basis and polar invariants of plane curve singularities	85
Young Soo Jo , Isometries of tridiagonal algebras	97
Ronald Leslie Lipsman , Harmonic analysis on exponential solvable homogeneous spaces: the algebraic or symmetric cases	117
Erich Miersemann , On the behaviour of capillaries at a corner	149
Marian Nowak , On the finest Lebesgue topology on the space of essentially bounded measurable functions	155
Pascal J. Thomas , Hardy interpolating sequences of hyperplanes	163
H. Bevan Thompson , Differentiability properties of subfunctions for second order ordinary differential equations	181