HARDY INTERPOLATING SEQUENCES OF HYPERPLANES

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A sufficient condition is given on unions of complex hyperplanes in the unit ball of $C^n$ so that they allow extension of functions in the Hardy $H^1$ space. The result is compared to Varopoulos' theorem about zeros of $H^p$ functions.

1. Notations and definitions. For $z, w \in \mathbb{C}^n$,
\[ z \cdot w = \sum_{i=1}^{n} z_i \bar{w}_i, \]
\[ B^n = \{ z \in \mathbb{C}^n : |z|^2 = z \cdot \bar{z} < 1 \}. \]

For $a_k \in B^n$, $a_k \neq 0$,
\[ a_k^* = \frac{a_k}{|a_k|^2}. \]

$\lambda_p = p$ real-dimensional Lebesgue measure. For instance, on $\mathbb{C}$,
\[ -\frac{1}{2} d\mathbb{C} = d\mathbb{C} = d\lambda_2. \]

Automorphisms of the ball.
\[ \phi_k(z) := \phi_{a_k}(z) := \frac{a_k - P_k(z) - s_k Q_k(z)}{1 - z \cdot a_k} \]

where $P_k(z) := \frac{z a_k}{|a_k|^2} a_k$ is the projection onto the complex line through $a_k$, $Q_k(z) := z - P_k(z)$ is the projection onto the complex hyperplane perpendicular to $a_k$, $s_k^2 := 1 - |a_k|^2$.

The map $\phi_k$ is an involution of the ball (see Rudin [4]). Note that
\[ Q_k(B^n) = \{ z : P_k(z) = 0 \} = \{ z : z \cdot a_k = 0 \}. \]

We write
\[ d_G(z, w)^2 := |\phi_w(z)|^2 = 1 - \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - z \cdot \bar{w}|^2}. \]

This is an invariant distance: if $\phi$ is an automorphism of the ball (i.e. any composition of unitary transformations and the above involutions), $d_G(\phi(z), \phi(w)) = d_G(z, w)$. 

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We will study hyperplanes in the ball, denoted by:

\[ V_j := \{ z \in \mathbb{B}^n : z \cdot a_j = |a_j|^2 \}. \]

The point \( a_j \) is the point in \( V_j \) closest to the origin. It is also the center of the \( n - 1 \)-complex-dimensional ball which \( V_j \) defines inside \( \mathbb{B}^n \). This definition makes no sense when \( a_j = 0 \), so we will not consider that case. However, the problem we will consider is automorphism-invariant and if there is a hyperplane going through the origin, applying to the whole sequence an automorphism \( \phi_a \), with \( |a| \) small enough, will preserve the hypotheses (at the expense of a change in the value of \( \delta \), see below) and yield the conclusion. We define \( c_{jk}^0 \) to be the "center" of the hyperplane \( \phi_k(V_j) \), i.e.

\[ \phi_k(V_j) = \phi_k^{-1}(V_j) = \{ z \in \mathbb{B}^n : z \cdot c_{jk}^0 = |c_{jk}^0|^2 \}. \]

We further consider the angle between \( \phi_k(V_j) \) and \( V_k \):

\[ \cos \theta_{jk} := \frac{|c_{jk}^0 \cdot a_k|}{|c_{jk}^0||a_k|}. \]

**Lemma 1.**

\[ c_{jk}^0 = \frac{l_{jk} c_{jk}}{|c_{jk}|^2} \]

where

\[ c_{jk} := \left( (1 - s_k) \frac{a_j^* \cdot a_k}{|a_k|^2} - |a_j| \right) a_k + s_k a_j^*, \]

\[ l_{jk} := a_k \cdot a_j^* - |a_j|^2 = (a_k - a_j) \cdot a_j^*; \]

\[ |c_{jk}|^2 = |l_{jk}|^2 + (1 - |a_j|^2)(1 - |a_k|^2). \]

\[ \cos^2 \theta_{jk} = \left( \frac{|c_{jk} \cdot a_k|}{|c_{jk}||a_k|} \right)^2 = \frac{|a_k^* \cdot a_j^* - |a_j||a_k|^2}{|l_{jk}|^2 + (1 - |a_j|^2)(1 - |a_k|^2)}, \]

\[ |c_{jk}^0|^2 = \frac{|(a_k - a_j) \cdot a_j|^2}{|(a_k - a_j) \cdot a_j|^2 + |a_j|^2(1 - |a_j|^2)(1 - |a_k|^2)}, \]

\[ 1 - |c_{jk}^0|^2 = \frac{(1 - |a_j|^2)(1 - |a_k|^2)}{|l_{jk}|^2 + (1 - |a_j|^2)(1 - |a_k|^2)}. \]

The proofs of all lemmas are deferred until §4.
The interpolation problem. The Hardy space $H^p(B^n)$ is the space of functions $f$ holomorphic on the ball and verifying

$$\|f\|_{H^p}^p := \sup_{r<1} \int_{\partial B^n} |f(r\zeta)|^p \, d\sigma(\zeta) < \infty,$$

where $\sigma$ is $2n - 1$-dimensional Lebesgue measure on $\partial B^n$.

The Bergman space $A^p(V_k)$ is the space of functions $\alpha$ holomorphic on the hyperplane $V_k$ and verifying

$$\|\alpha\|_{A^p(V_k)}^p := \int_{V_k} |\alpha(z)|^p \, d\lambda_{2n-2}(z) < \infty.$$

**Definition.** $l^p(A^p(V_k), 1 - |a_k|^2)$ is the product of the Bergman spaces on each hyperplane, endowed with the following norm: if $\alpha = \{\alpha_k\}_{k \in \mathbb{Z}^+}$, where $\alpha_k$ is a function defined and holomorphic on $V_k$,

$$\|\alpha\|_{l^p(A^p(V_k), 1 - |a_k|^2)}^p = \sum_k (1 - |a_k|^2)^n \|\alpha_k\|_{A^p(V_k)}^p.$$

Notice that $\phi_k |_{V_k}$ is just an affine map from $V_k$ to $Q_k(B^n) \simeq B^{n-1}$, so that we can rewrite

$$\|\alpha\|_{l^p(B^n), 1 - |a_k|^2)}^p = \sum_k (1 - |a_k|^2)^n \int_{Q_k(B^n)} |\alpha_k \circ \phi_k(w)|^p \, d\lambda_{2n-2}(w).$$

Given a function $f \in H(B^n)$, the space of holomorphic functions, we consider the following map

$$T: H(B^n) \to \prod_{i=1}^{\infty} H(V_i),$$

$$f \mapsto \{f|_{V_i}\}_{i \geq 1}.$$

**Definition.** We say that $\{V_j\}_{j \in \mathbb{Z}^+}$ is an $H^p$-interpolating sequence of hyperplanes if $T$ maps $H^p(B^n)$ onto $l^p(A^p(V_k), 1 - |a_k|^2)$.

Equivalently, given $\{\alpha_k\}$ a sequence of functions holomorphic on $V_k$, such that

$$\sum_k (1 - |a_k|^2) \int_{V_k} |\alpha_k(z)|^p \, d\lambda_{2n-2}(z) < \infty,$$

there exists $f \in H^p(B^n)$ such that

$$f|_{V_k} = \alpha_k.$$
This definition is the one given by Amar [1] and reduces in the case \( n = 1 \) to that of Shapiro and Shields [5].

**Remark.** With this definition, if a sequence of hyperplanes is \( \mathcal{H}^p \)-interpolating and we take points \( b_k \in V_k, \forall k \), then the sequence \( \{b_k\} \) is \( \mathcal{H}^p \)-interpolating (in the sense of [2]).

**Proof.** If we are given a sequence of complex numbers \( \{\beta_k\} \) such that
\[
\sum_k (1 - |b_k|^2)^n |\beta_k|^p < \infty,
\]
then define
\[
\alpha_k(z) = \left( \frac{1 - |b_k|^2}{1 - z \cdot b_k} \right)^n \beta_k.
\]
Then
\[
\int_{V_k} |\alpha_k(z)|^p d\lambda_{2n-2}(z) = \int_{Q_k(B^n)} |\beta_k|^p \left| \frac{1 - |b_k|^2}{1 - \psi(w) \cdot b_k} \right|^n |J_\psi(w)| d\lambda_{2n-2}(w),
\]
where \( \psi(w) = a_k + s_k w \).
\[
|J_\psi(w)| = s_k^{2n-2} = (1 - |a_k|^2)^{n-1}, \quad \text{and} \quad 1 - \psi(w) \cdot \psi(w') = (1 - |a_k|^2)(1 - w \cdot w'),
\]
so, setting \( b'_k = \psi^{-1}(b_k) \), we get
\[
\int_{V_k} |\alpha_k(z)|^p d\lambda_{2n-2}(z) = |\beta_k|^p (1 - |a_k|^2)^{n-1} \int_{Q_k(B^n)} \left| \frac{1 - |b'_k|^2}{1 - w \cdot b'_k} \right|^n d\lambda_{2n-2}(w)
\]
\[
\leq C |\beta_k|^p (1 - |a_k|^2)^{n-1}(1 - |b'_k|^2)^n \quad \text{because} \quad np > n - 1,
\]
\[
= C |\beta_k|^p \frac{(1 - |b_k|^2)^n}{1 - |a_k|^2}.
\]
It follows that
\[
\sum_k (1 - |a_k|^2) \int_{V_k} |\alpha_k(z)|^p d\lambda_{2n-2}(z) \leq C \sum_k (1 - |b_k|^2)^n |\beta_k|^p,
\]
and the function \( f \in \mathcal{H}^p \) which we get by interpolating the \( \alpha_k \) on the hyperplanes verifies \( f(b_k) = \alpha_k(b_k) = \beta_k \).
Taking $b_k = a_k$, we get from [8] (for $p \geq 1$) the following necessary condition:

$$\sup_k \sum_j \left( \frac{(1 - |a_k|^2)(1 - |a_j|^2)}{|1 - a_k \cdot a_j|^2} \right)^n < \infty.$$  

We also get that any sequence $\{b_k\}$ must be separated in the Gleason distance; thus there exists $\delta > 0$ such that if $j \neq k$, then

$$d_G(V_j, V_k) = \inf\{d_G(z, w), z \in V_j, w \in V_k\} \geq \delta > 0.$$  

We say that the hyperplanes are separated.

2. The main result. We are looking for a sufficient geometric condition to ensure that a sequence of hyperplanes be $H^1$-interpolating. To do so, we define another family of neighborhoods for the hyperplanes.

**DEFINITION.** Given $\delta$ a positive number, we call tube around $V_k$ the following open subset of $B^n$:

$$T_\delta(V_k) := \{z \in B^n : |(z - a_k) \cdot a_k^*| < \delta(1 - |a_k|^2)\}.$$  

Those neighborhoods of the hyperplanes will be larger than those given by separatedness in the Gleason distance. This will follow from:

**LEMMA 2.** (1) Given any $z \in B^n$,

$$d_G(z, V_k)^2 = \frac{|P_k \circ \phi_k(z)|^2}{|P_k \circ \phi_k(z)|^2 + (1 - |\phi_k(z)|^2)}.$$  

(2) $\overline{V_j} \cap \overline{V_k} = \emptyset \Leftrightarrow \cos^2 \theta_{jk} > (1 - |c_{jk}^0|^2).$

(3) If (2) is satisfied,

$$1 - d_G^2(V_j, V_k) = \frac{(1 - |a_j|^2)(1 - |a_k|^2)}{|a_k^* \cdot a_j^* - |a_i||a_k|^2|} = \frac{(1 - |c_{jk}^0|^2)}{\cos^2 \theta_{jk}}.$$  

(4)  

$$d_G(V_j, V_k) \geq \delta_1 > 0 \Leftrightarrow (1 - \delta_1^2) \cos^2 \theta_{jk} \geq (1 - |c_{jk}^0|^2).$$

From this we can prove that all points of the ball which are close enough to $V_k$ in the invariant distance must be within the tube. Indeed,
by applying Lemma 2(1) and the fact that
\[ P_k \circ \phi_k(z) = \frac{(z - a_k) \cdot \bar{a}_k}{1 - z \cdot \bar{a}_k} \frac{a_k}{|a_k|^2}, \]
we see that
\[ d_G(z, V_k)^2 = \frac{|(z - a_k) \cdot \bar{a}_k|^2}{|(z - a_k) \cdot \bar{a}_k|^2 + |a_k|^2 (1 - |a_k|^2) (1 - |z|^2)}. \]
Clearly then, if \( z \in \partial T_\delta(V_k), \)
\[ d_G(z, V_k)^2 = \frac{\delta^2}{\delta^2 + |a_k|^2 \frac{1 - |z|^2}{1 - |a_k|^2}} > \frac{\delta^2}{\delta^2 + 2(1 + \delta)}, \]
which shows the inequality holds for \( z \notin T_\delta(V_k). \)

**THEOREM.** There exists a number \( c_0 = c_0(\delta) > 0 \) such that if
\[ (ii) \quad \text{for any } j \neq k, \quad T_\delta(V_j) \cap T_\delta(V_k) = \emptyset, \]
then \( \{V_k\}_{k \in \mathbb{Z}^+} \) is an \( H^1(B^n) \)-interpolating sequence of hyperplanes.

**REMARKS.** (1) It was proved in [6] that (ii) together with
\[ (i) \quad \sup_k \sum_{j: j \neq k} \left( \frac{(1 - |a_k|^2) (1 - |a_j|^2)}{|1 - a_k \cdot \bar{a}_j|^2} \right)^n < c_0 \]
and
\[ (ii) \quad \text{for any } j \neq k, \quad T_\delta(V_j) \cap T_\delta(V_k) = \emptyset, \]
then \( \{V_k\}_{k \in \mathbb{Z}^+} \) is an \( H^1(B^n) \)-interpolating sequence of hyperplanes.

(2) A similar result holds for a sequence of points, but condition (i) is enough, with any constant \( c_0 < 1 \) [8]. Here \( c_0 \) will have to be even smaller; therefore condition (i) by itself is enough to ensure separation of the points, since in particular each term of the sum must be less than \( c_0 \).

**Proof of the Theorem.** We will construct an approximate extension, i.e. an operator
\[ \hat{E} : l^1(A^1(V_k), 1 - |a_k|^2) \rightarrow H^1(B^n) \]
such that
\[(E1) \quad \|\hat{E}\|_{\text{op}} < \infty\]
and
\[(E2) \quad \|T\hat{E} - I\|_{\text{op}} < 1.\]

Then \(T\hat{E}\) is invertible, and one can write a true extension by letting \(E = \hat{E}(T\hat{E})^{-1}\). The operator \(TE\) will be the identity map on \(l^1\) and for \(\alpha \in l^1\), \(E(\alpha)\) will be a solution to the interpolation problem.

Let
\[\hat{E}(\alpha)(z) = \sum_{k \in \mathbb{Z}^+} \left( \frac{1 - |a_k|^2}{1 - z \cdot \alpha_k} \right)^{2n} \hat{\alpha}_k(z),\]
where \(\hat{\alpha}_k = \alpha_k \circ \Phi_k \circ Q_k \circ \phi_k\) is an extension of \(\alpha_k\) to \(B^n\). Note that for \(z \in V_j\), the \(j\)th term in the sum is exactly \(12^n \hat{\alpha}_j(z) = \alpha_j(z)\). (E1) is easily checked, for the coefficient of \(\hat{\alpha}_k(z)\) is bounded and it follows from the computations in [6] that
\[
\int_{\partial B^n} \left| \left( \frac{1 - |a_k|^2}{1 - z \cdot \alpha_k} \right)^n \hat{\alpha}_k(z) \right| d\sigma(z)
\leq C(1 - |a_k|^2) \int_{V_k} |\alpha_k(z)| d\lambda_{2n-2}(z).
\]

This step fails for \(p > 1\), and prevents us from proving \(H^p\) results for hyperplanes similar to those for points in [8].

The theorem reduces to:

**Main Lemma.** *For \(c_0\) small enough, there exists \(c_1 < 1\) such that for any \(\alpha \in l^1(A^1(V_k), 1 - |a_k|^2)\),

\[
\sum_j (1 - |a_j|^2) \int_{V_j} \left| \sum_{k: k \neq j} \left( \frac{1 - |a_k|^2}{1 - z \cdot \alpha_k} \right)^{2n} \hat{\alpha}_k(z) \right| d\lambda_{2n-2}(z)
\leq c_1 \sum_k (1 - |a_k|^2) \int_{V_k} |\alpha_k(z)| d\lambda_{2n-2}(z).
\]

**Comparison with zero-set results.** Clearly, if \(\{V_k\}_{k \in \mathbb{Z}^+}\) satisfy the hypotheses of the theorem, then their union will be a subset of a zero set for \(H^1\) functions. To see it, simply adjoin to the sequence a hyperplane \(V_0\) such that (i) and (ii) still hold (this can be achieved by taking
$a_0^*$ on $\partial B^n \setminus \bigcup_{k \leq 1} T_{2\delta}(V_k)$ and $|a_0|$ very close to 1); then interpolate 1 on $V_0$ and 0 everywhere else.

This needs to be compared to the results of N. Th. Varopoulos, at least in the special case of a divisor made up of a countable union of complex hyperplanes [9, §8]. In that case, he showed:

**Proposition 8.2.** There exist constants $C_1, \ldots, C_4$ such that if

$$\sum_{j: |1-\alpha_j| \leq C_1(1-|a_k|^2)} (1-|a_j|^2)^n \leq C_2(1-|a_k|^2)^n$$

and

$$\text{Card}\{j: V_j \cap K_h(\zeta) \neq \emptyset, V_j \notin K_{C_3h}(\zeta)\} \leq C_4$$

where $K_h(\zeta) := \{z \in B^n: |1-z \cdot \zeta| < h\}$, then there exists $p > 0$ such that $\bigcup_k V_k$ is a zero set for $H^p(B^n)$.

It can be shown (see e.g. [3]) that (8.18), which is a Carleson measure condition, is equivalent to

$$\sup_k \sum_j \left(\frac{(1-|a_k|^2)(1-|a_j|^2)}{|1-a_k \cdot \bar{a}_j|^2}\right)^n < \infty.$$ 

On the other hand, if we assume separatedness in the invariant distance, (8.19) is satisfied in the following stronger form:

$$\exists C_5 > 0 \text{ such that } \text{Card}\{j: V_j \cap K_h(\zeta) \neq \emptyset, V_j \notin K_{C_3h}(\zeta)\} \leq 1.$$ 

Note that the above set is non-empty only when $h \leq 2/C_5$.

The idea of the proof is first to use the triangle inequality for the Koranyi distance to reduce oneself to the case where $\zeta \in V_j \cap \partial B^n$; then to apply an automorphism to bring $V_j$ to $\phi_j(V_j)$, which is a hyperplane through the center of $B^n$. The region $K_h(\zeta)$ is transformed into a similar region, because $a_j$, by the assumption that $j$ is in the above set, is far enough away from $\zeta$. If another index $k$ was also in the set, the hyperplane $\phi_j(V_k)$ would pass through $\phi_j(K_h(\zeta))$, and thus its projection onto $\phi_j(V_j)$ would come too close to the boundary, violating the conclusion of Lemma 5, given below.

Varopoulos’ theorem, as he pointed out, provides no control over the value of $p$ (which could indeed be very small, if one works out the constants involved). This is essentially because the norm of the Carleson measure supported by the divisor cannot be made arbitrarily small. For this very special structure of the divisor $\bigcup_j V_j$, our result provides additional control on the exponent, although the actual zero
set involved could be much larger than $\bigcup_j V_j$. Namely:

**Proposition.** If $\{V_k\}_{k \in \mathbb{Z}^+}$ satisfies

$$(i_M) \quad \sup_k \sum_{j \neq k} \left( \frac{|1 - |a_k|^2| (1 - |a_j|^2)}{|1 - a_k \cdot \overline{a_j}|^2} \right)^n < 2^M c_0$$

and

$$(ii_N) \quad \text{for any } k, \quad \text{Card}\{j: T_\delta(V_j) \cap T_\delta(V_k) \neq \emptyset\} \leq N,$$

where $M \geq 0$, $N \geq 0$, are integers, then there exists $f \neq 0$, $f \in H^{1/2M(N+1)}(B^n)$, such that $f \big|_{V_k} \equiv 0$ for all $k$.

**Proof.** An elementary combinatorial argument shows that under $(ii_N)$, the sequence can be split into $N + 1$ subsequences, each of which satisfies $(ii)$, and of course $(i_M)$. Then Mills’ Lemma [8] allows us to split each such subsequence into $2^M$ further subsequences verifying $(i)$. Thus we are reduced to the case $M = 0$, $N = 0$, i.e. the assumptions of the theorem; by the argument given at the beginning of this section, each subsequence has a nonzero $H^1$ function vanishing on it. Taking the product of the annihilating functions, we find $f \in H^{1/2M(N+1)}(B^n)$.

3. **Proof of the main lemma.** For convenience, we shall introduce the notation $A_k = \alpha_k \circ \phi_k$. Thus $A_k$ is a function defined on $A_k(B^n) \cong B^{n-1}$, and

$$(1 - |a_k|^2)^n \int_{Q_k(B^n)} |A_k(z)| d\lambda_{2n-2}(z)$$

$$= (1 - |a_k|^2) \int_{V_k} |\alpha_k(z)| d\lambda_{2n-2}(z).$$

Furthermore, $\tilde{\alpha}_k = A_k \circ Q_k \circ \phi_k$. With this new notation, it is enough to bound

$$\sum_k \sum_{j: j \neq k} (1 - |a_j|^2)(1 - |a_k|^2)^{2n} \int_{V_j} \frac{|A_k \circ Q_k \circ \phi_k(z)|}{|1 - z \cdot \overline{a_k}|^{2n}} d\lambda_{2n-2}(z).$$

The integral in question is equal to

$$\int_{\phi_k(V_j)} \frac{|A_k \circ Q_k(w)|}{|1 - z \cdot \overline{a_k}|^{2n}} |J_{\phi_k|V_j}(z)|^{-1} d\lambda_{2n-2}(w),$$

where $J_{\phi_k|V_j}(z)$ is the Jacobian of the map $\phi_k$ restricted to $V_j$. 
LEMMA 3.  

\[ |J_{\phi_k} V_j(z)| = \frac{(1 - |a_k|^2)^{n-1}}{|1 - z \cdot \bar{a}_k|^{2n}} \left[ |(a_k - a_j) \cdot a_j^*|^2 + (1 - |a_j|^2)(1 - |a_k|^2) \right] \]

\[ = \frac{(1 - |a_k|^2)^{n-1}}{|1 - z \cdot \bar{a}_k|^{2n}} |c_{jk}|^2, \]

with the notations from Lemma 1.

Thus the terms in the sum reduce to:

\[
\frac{(1 - |a_j|^2)(1 - |a_k|^2)^{n+1}}{|l_{jk}|^2 + (1 - |a_j|^2)(1 - |a_k|^2)} \int_{\phi_k(V_j)} |A_k \circ Q_k(w)| d\lambda_{2n-2}(w) 
\]

\[ = \frac{(1 - |a_j|^2)(1 - |a_k|^2)^{n+1}}{|c_{jk}|^2} \int_{Q_k \circ \phi_k(V_j)} |A_k(u)||J_{Q_k \circ \phi_k(V_j)}(w)|^{-1} d\lambda_{2n-2}(u). \]

LEMMA 4. Given \( a \in B^n \), let \( V = \{ z \in B^n : z \cdot \bar{a} = |a|^2 \} \). Then

(1) \[ |J_{Q_k|V}| = \left( \frac{|a \cdot \bar{a}_k|}{|a||a_k|} \right)^2 =: \cos^2 \theta. \]

(2) In the case where \( a \cdot \bar{a}_k \neq 0 \), \( Q_k(V) \) is the subset of \( Q_k(B^n) \) given by the equation

\[
\left( \frac{|a \cdot \bar{a}_k|}{|a||a_k|} \right)^{-2} |w_1 - Q_k(a)|^2 + |w_2|^2 < 1 - |a|^2,
\]

where \( w_1 \) is the coordinate in the \( Q_k(a) \) complex direction, and \( w_2 \) represents the \( n - 2 \) complex coordinates in the orthogonal directions within \( Q_k(B^n) \). \( Q_k(V) \) is thus an ellipsoid of radii \((\cos \theta)(1 - |a|^2)^{1/2}\) in the \( w_1 \) direction, and \((1 - |a|^2)^{1/2}\) in each of the \( w_2 \) directions. In the case where \( a \cdot \bar{a}_k = 0 \), we get simply \( Q_k(B^n) \cap V \) as the projection.

(3) \[ \max_{Q_k(V)} |z| = |a| \sin \theta + (1 - |a|^2)^{1/2} \cos \theta. \]

We apply this lemma with \( a = c_{jk}^0 \) and \( \theta = \theta_{jk} \). Since, under the separatedness condition, \( V_j \cap V_k = \emptyset \), we always have \( |c_{jk}^0 \cdot \bar{a}_k| = |a_k||c_{jk}^0| \cos \theta_{jk} > |a_k||c_{jk}^0|(1 - |c_{jk}^0|^2)^{1/2} > 0 \), i.e. \( c_{jk}^0 \cdot \bar{a}_k \neq 0 \). Replacing the Jacobian by its value (see Lemma 1(2)), we get for each term of the sum:

\[
= \frac{(1 - |a_j|^2)(1 - |a_k|^2)^{n+1}}{|a_k \cdot \bar{c}_{jk}|^2} \int_{Q_k \circ \phi_k(V_j)} |A_k(u)| d\lambda_{2n-2}(u).
\]
We now make use of (ii):

**Lemma 5.** If $T_\delta(V_j) \cap T_\delta(V_k) = \emptyset$, then there exists $\delta_1 = \delta_1(\delta) > 0$ such that

$$\max\{|z|: z \in Q_k \circ \phi_k(V_j)\} \leq \sqrt{1 - \delta_1^2} < 1.$$ 

Thus the distance to $\partial B^n$ from $Q_k \circ \phi_k(V_j)$ is at least $\delta_2 = 1 - \sqrt{1 - \delta_1^2}$. By the classical theory of Bergman spaces, this implies that $A_k$ satisfies a uniform estimate on $Q_k \circ \phi_k(V_j)$:

$$|A_k(u)| \leq \frac{C}{\delta_2^{2n-2}} \int_{Q_k(B^n)} |A_k(u)| \lambda_{2n-2}(u).$$

It follows from Lemma 4(2), applied with $a = c_{jk}^0$, that

$$\lambda_{2n-2}(Q_k \circ \phi_k(V_j)) = \cos^2 \theta_{jk}(1 - |c_{jk}^0|^2)^{n-1}.$$ 

Thus each term in our sum is bounded by

$$C(\delta) \frac{(1 - |a_j|^2)(1 - |a_k|^2)^{n+1}(1 - |c_{jk}^0|^2)^{n-1}}{|c_{jk}|^2} \int_{Q_k(B^n)} |A_k(u)| \lambda_{2n-2}(u),$$

which Lemma 1(3) and some arithmetic reduces to:

$$= C(\delta) \frac{(1 - |a_j|^2)^n(1 - |a_k|^2)^{2n}}{|c_{jk}|^{2n}} \int_{Q_k(B^n)} |A_k(u)| \lambda_{2n-2}(u).$$

We must estimate $|c_{jk}|^2 = |l_{jk}|^2 + (1 - |a_j|^2)(1 - |a_k|^2)$ from below. Simply writing that $a_k \notin T_\delta(V_j)$, condition (ii) implies $|l_{jk}| > \delta(1 - |a_j|^2)$.

**Case 1.** $(1 - \delta)|1 - a_j \cdot \bar{a}_k| \leq 2(1 - |a_j|)$. Then

$$|l_{jk}| > \delta(1 - |a_j|^2) \geq \frac{\delta(1 - \delta)}{2} |1 - a_j \cdot \bar{a}_k|.$$ 

**Case 2.** $(1 - \delta)|1 - a_j \cdot \bar{a}_k| > 2(1 - |a_j|)$. Then

$$|l_{jk}| = |a_k \cdot \bar{a}_j^* - a_j|$$

$$= |1 - a_k \cdot a_j - (1 - |a_j|)(1 + a_k \cdot \bar{a}_j^*)| \geq \delta|1 - a_k \cdot a_j|. $$
In either case, $|c_{jk}|^{2n} > |l_{jk}|^{2n} \geq C(\delta)|1 - a_j \cdot \bar{a}_k|^{2n}$, and our whole sum is majorized by

$$C(\delta) \sum_k (1 - |a_k|^2)^n \times \left( \sum_{j: j \neq k} \left[ \frac{(1 - |a_j|^2)(1 - |a_k|^2)}{|1 - a_j \cdot \bar{a}_k|^2} \right]^n \right) \int_{Q_k(B^n)} |A_k(u)| d\lambda_{2n-2}(u) \leq c_0 C(\delta) \sum_k (1 - |a_k|^2) \int_{V_k} |\alpha_k(z)| d\lambda_{2n-2}(z).$$

It will now be enough to pick

$$c_0 < \frac{1}{C(\delta)} \approx \delta^{2(n-1)} (\delta(1 - \delta))^{2n} \approx \delta^{6n-4},$$

which concludes the proof of the Main Lemma.


Proof of Lemma 1. Since $\phi_k = \phi_k^{-1}$,

$$\phi_k(V_j) = \phi_k^{-1}(V_j) = \{ z \in B^n : \phi_k(z) \cdot \bar{a}_j = |a_j|^2 \}.$$

This equation becomes:

$$a_k \cdot \bar{a}_j - \frac{z \cdot \bar{a}_k}{|a_k|^2} a_k \cdot \bar{a}_j (1 - s_k) - s_k z \cdot \bar{a}_j = |a_j|^2 (1 - z \cdot \bar{a}_k),$$

$$z \cdot \left( \left( |a_j|^2 - \frac{a_k \cdot \bar{a}_j}{|a_k|^2} (1 - s_k) \right) \bar{a}_k - s_k \bar{a}_j \right) = |a_j|^2 - a_k \cdot \bar{a}_j.$$

Let $|a_j| c_{jk} := ((1 - s_k)(a_j \cdot \bar{a}_k/|a_k|^2) - |a_j|^2) a_k + s_k a_j$, $l_{jk} := a_k \cdot \bar{a}_j - |a_j|$. The equation now reads $z \cdot \bar{c}_{jk} = l_{jk}$, or equivalently

$$z \cdot \frac{l_{jk} \bar{c}_{jk}}{|c_{jk}|^2} = \frac{|l_{jk}|^2}{|c_{jk}|^2} = \left| \frac{l_{jk} \bar{c}_{jk}}{|c_{jk}|^2} \right|^2.$$

We need to compute $|c_{jk}|^2$. Note first that

$$|a_j| c_{jk} \cdot \bar{a}_k = (1 - s_k) a_j \cdot \bar{a}_k - |a_j|^2 |a_k|^2 + s_k a_j \cdot \bar{a}_k = a_j \cdot \bar{a}_k - |a_j|^2 |a_k|^2;$$

and

$$|a_j| c_{jk} \cdot \bar{a}_j = (1 - s_k) \frac{|a_j \cdot \bar{a}_k|^2}{|a_k|^2} - |a_j|^2 a_k \cdot \bar{a}_j + s_k |a_j|^2.$$
Thus
\[ |a_j|^2 c_{jk}^2 = |a_j| c_{jk} \cdot |a_j| c_{jk} \]
\[ = |a_j| c_{jk} \cdot \bar{a}_k \left( (1 - s_k) \frac{a_j \cdot \bar{a}_k}{|a_k|^2} - |a_j|^2 \right) + (|a_j| c_{jk} \cdot \bar{a}_j)s_k \]
\[ = (1 - s_k) \frac{|a_j \cdot \bar{a}_k|^2}{|a_k|^2} - (1 - s_k) a_k \cdot \bar{a}_j |a_j|^2 - |a_j|^2 a_j \cdot \bar{a}_k + |a_j|^4 |a_k|^2 \]
\[ + s_k (1 - s_k) \frac{|a_j \cdot \bar{a}_k|^2}{|a_k|^2} - s_k |a_j|^2 a_k \cdot a_j + s_k^2 |a_j|^2 \]
\[ = |a_j \cdot \bar{a}_k|^2 - |a_j|^2 (a_k \cdot \bar{a}_j + a_j \cdot \bar{a}_k) + |a_j|^2 (1 - |a_k|^2) + |a_j|^4 |a_k|^2 \]
\[ = |a_j \cdot \bar{a}_k|^2 - |a_j|^2 (|a_j|^2 - |a_j|^4) (1 - |a_k|^2) \]
\[ = |a_j|^2 (|a_k \cdot \bar{a}_j^* - |a_j|^2) + (1 - |a_j|^2) (1 - |a_k|^2)). \]

This proves (1).

We get from the above
\[ \cos^2 \theta_{jk} = \frac{|a_j \cdot \bar{a}_k - |a_j|^2 |a_k|^2}{|a_k|^2 |a_j|^2 (|l_{jk}|^2 + |a_j|^2 (1 - |a_j|^2) (1 - |a_k|^2))}, \]
which proves (2) after cancelling \(|a_k|^2 |a_j|^2\) from top and bottom. Finally,
\[ |c_{jk}^0|^2 = \left| \frac{l_{jk}}{c_{jk}} \right|^2 = \frac{|l_{jk}|^2}{|l_{jk}|^2 + (1 - |a_j|^2) (1 - |a_k|^2)}, \]
from which (3) follows.

**Proof of Lemma 2.** Since \(d_G\) is automorphism-invariant, we can compute \(d_G(\phi_k(V_k), z)\) first. But \(P_k(z) = a_k\) for \(z \in V_k\), so \(\phi_k(V_k) = Q_k(B^n)\). Now fix \(z \in B^n\). We need to find
\[ \inf_{w \in Q_k(B^n)} \left( 1 - \frac{(1 - |z|^2)(1 - |w|^2)}{1 - z \cdot \bar{w}} \right) \]
\[ = 1 - (1 - |z|^2) \sup_{w \in Q_k(B^n)} \frac{1 - |w|^2}{1 - z \cdot \bar{w}}. \]

If \(z \cdot \bar{w} = Q_k(z) \cdot \bar{w}\) remains fixed, the largest value is obtained for \(|w|\) minimal, i.e. \(w\) parallel to \(Q_k(z)\). Set \(w = \alpha Q_k(z)^*\), with \(\alpha \in \Delta = \Delta^1 \subset C\). We have to study
\[ \max_{\alpha \in \Delta} \frac{1 - |\alpha|^2}{|A + B\alpha|^2}, \]
with \(A = 1, B = Q_k(z)^* \cdot z = |Q_k(z)| < 1\). This function is always differentiable and the gradient vanishes for \(\alpha = -\bar{B}/A\). The maximum
equals \((|A|^2 - |B|^2)^{-1} = 1/(1 - |Q_k(z)|^2)\).

\[
1 - \frac{1 - |z|^2}{1 - |Q_k(z)|^2} = \frac{|z|^2 - |Q_k(z)|^2}{1 - |Q_k(z)|^2} = \frac{|P_k(z)|^2}{1 - |z|^2 + |P_k(z)|^2}.
\]

That gives the distance from \(z\) to \(\phi_k(V_k)\). By invariance under automorphisms, \(d_G(V_k, z) = d_G(\phi_k(V_k), \phi_k(z))\), and we get (1) by substituting \(\phi_k(z)\) into the above formula.

Now we want to minimize \(d_G(\phi_k(V_k), z)\) over \(z \in \phi_k(V_j)\), i.e. for \(z \cdot c_{jk}^0 = |c_{jk}^0|^2\). Recall that \(P_k(z) = z \cdot a_k^*\). Let

\[
\Psi(z) := \frac{|z \cdot a_k^*|^2}{|z \cdot a_k^*|^2 + 1 - |z|^2} = \frac{1}{1 + (1 - |z|^2)/|z \cdot a_k^*|^2},
\]

so to minimize \(\Psi\) we have to maximize \(1 - |z|^2/|z \cdot a_k^*|^2\). We can reduce ourselves to the case where \(z \in \text{Span}(a_k, c_{jk}^0)\); otherwise, projecting \(z\) onto it will not change \(z \cdot a_k^*\) and will increase \(1 - |z|^2\). If \(z \in \phi_k(V_j) \cap \text{Span}(a_k, c_{jk}^0)\), we can write

\[
z = c_{jk}^0 + (1 - |c_{jk}^0|^2)^{1/2} \tilde{c}_{jk}\alpha,
\]

where \(\alpha\) is a complex number, \(\alpha \in \Delta\), and \(|\tilde{c}_{jk}| = 1\), \(\tilde{c}_{jk}^0 \in \text{Span}(a_k, c_{jk}^0)\), and \(\tilde{c}_{jk}^0 \cdot c_{jk}^0 = 0\). With this notation,

\[
1 - |z|^2 = (1 - |c_{jk}^0|^2)(1 - |\alpha|^2),
\]

\[
z \cdot a_k^* = c_{jk}^0 \cdot a_k^* + (1 - |c_{jk}^0|^2)^{1/2} \alpha \tilde{c}_{jk}^0 \cdot a_k^* =: A + B\alpha.
\]

Note that

\[
\frac{|c_{jk}^0 \cdot a_k^*|^2}{|c_{jk}^0|^2} + |\tilde{c}_{jk}^0 \cdot a_k^*|^2 = 1,
\]

so that

\[
|A|^2 = |c_{jk}^0 \cdot a_k^*|^2 = |c_{jk}^0|^2 \cos^2 \theta_{jk},
\]

\[
|B|^2 = (1 - |c_{jk}^0|^2) \left(1 - \frac{|c_{jk}^0 \cdot a_k^*|^2}{|c_{jk}^0|^2}\right) = (1 - |c_{jk}^0|^2)(1 - \cos^2 \theta_{jk}).
\]

As above, the maximum of \((1 - |\alpha|^2)/|A + B\alpha|^2\) is \(|A|^2 - |B|^2\)^{-1}, provided that \(|A| > |B|\). This last condition simply means that \(\phi_k(\overline{V}_k) \cap \phi_k(\overline{V}_j) = \emptyset\), i.e. \(\overline{V}_k \cap \overline{V}_j = \emptyset\). This is equivalent to \(|A|^2 > |B|^2\), which is easily rewritten into (2).
Getting back to $1 - \inf \{ d_G^2(z, w), z \in V_j, w \in V_k \}$, we find

$$
\frac{1}{1 + (1 - |c_{jk}^0|^2)/(|A|^2 - |B|^2)} = \frac{1 - |c_{jk}^0|^2}{|A|^2 - |B|^2 + (1 - |c_{jk}^0|^2)} = \frac{1 - |c_{jk}^0|^2}{\cos^2 \theta_{jk}}.
$$

Writing $d_G^2(V_j, V_k) \geq \delta_i^2$ gives (4) immediately. (3) follows from substituting the values given by Lemma 1 (2) and (3).

**Proof of Lemma 3.** Recall from [4] that the global Jacobian of $\phi_k$ is

$$
J_{\phi_k} = \left( \frac{1 - |a_k|^2}{|1 - z \cdot \overline{a_k}|^2} \right)^{n+1}.
$$

To restrict to $V_j$, we must divide out the dilation corresponding to the directions orthogonal to the source set, $a_j^* \perp V_j$, and to the target set, $c_{jk} \perp \phi_k(V_j)$. This will be $|D_{a_j^*}(\phi_k(z) \cdot \overline{c_{jk}})/|c_{jk}||^2$, where $D_{a_j^*}$ denotes the derivative in the complex direction of $a_j^*$.

$$
\phi_k(z) \cdot \overline{c_{jk}}
= a_k \left( 1 - (1 - s_k)z \cdot \overline{a_k}/|a_k|^2 \right) - s_k z
\cdot \left[ \left( (1 - s_k) \frac{a_j^* \cdot \overline{a_k}}{|a_k|^2} - |a_j| \right) a_k + s_k a_j^* \right]
= \frac{1}{1 - z \cdot \overline{a_k}} \left[ (1 - s_k) a_k \cdot \overline{a_j^*} - |a_j||a_k|^2 + s_k a_k \cdot \overline{a_j^*} \right. \\
+ \left[ (1 - s_k)^2 \frac{a_j^* \cdot \overline{a_k}}{|a_k|^2} - (1 - s_k)|a_j| - s_k (1 - s_k) \frac{a_j^* \cdot \overline{a_k}}{|a_k|^2} \right. \\
\left. + s_k|a_j| - s_k (1 - s_k) \frac{a_j^* \cdot \overline{a_k}}{|a_k|^2} \right] z \cdot \overline{a_k} - s_k z \cdot a_j^*
= \frac{1}{1 - z \cdot \overline{a_k}} \left[ a_k \cdot a_j^* - |a_j||a_k|^2 \right. \\
+ (a_j - a_k) \cdot \overline{a_j^*}(z \cdot \overline{a_k}) - (1 - |a_k|^2)z \cdot \overline{a_j^*}].
$$

Since $z \cdot \overline{a_k}$ and $z \cdot \overline{a_j^*}$ are linear forms,

$$
D_{a_j^*}(z \cdot \overline{a_j^*}) = a_j^* \cdot \overline{a_j^*} = 1 \quad \text{and} \quad D_{a_j^*}(z \cdot \overline{a_k}) = a_j^* \cdot \overline{a_k}.
$$
Thus
\[
D_{a_j}^*(\phi_k(z) \cdot \bar{c}_{jk}) = \frac{\phi_k(z) \cdot \bar{c}_{jk}}{1 - z \cdot \bar{a}_k} a_j \cdot a_k + \frac{1}{1 - z \cdot \bar{a}_k} [-l_{jk} a_j^* \cdot a_k - (1 - |a_k|^2)].
\]

For \( z \in \mathbb{C} \), \( z \cdot \bar{a}_j = |a_j|^2 \) and \( \phi_k(z) \cdot \bar{c}_{jk} = l_{jk} \), so that all that remains is the second term inside the square brackets:
\[
\left| D_{a_j}^* \left( \frac{\phi_k(z) \cdot \bar{c}_{jk}}{|c_{jk}|} \right) \right|^2 = \frac{(1 - |a_k|^2)^2}{|c_{jk}|^2 |1 - z \cdot \bar{a}_k|^2}.
\]

Dividing the global Jacobian by this quantity yields the result.

**Proof of Lemma 4.** (1) At any point of \( \mathbb{V} \), split the tangent space \( \mathbb{V}' \) into an orthogonal direct sum:
\[
\mathbb{V} = \mathbb{V}' \cap \text{Span}(a, a_k) \oplus \mathbb{V}'.
\]
The projection \( Q_k \) induces the identity on \( \mathbb{V}' \), so it is enough to consider the situation on the complex line \( \mathbb{V} \cap \text{Span}(a, a_k) = \text{Span}(\bar{u}) \), where \( \bar{u} := a_k - (a_k \cdot a/|a|^2) a \).

Thus
\[
|J_{Q_k}| = \frac{|Q_k(\bar{u})|^2}{|\bar{u}|^2},
\]
and an easy computation gives (1).

(2) If \( a \cdot a_k \neq 0 \), then \( Q_k|_\mathbb{V} \) is one-to-one. Let \( (Q_k|_\mathbb{V})^{-1}(w) = w + \lambda a_k \), where \( \lambda \in \mathbb{C} \).

\[
(w + \lambda a_k) \cdot \bar{a} = |a|^2 \Rightarrow \lambda = \frac{|a|^2 - w \cdot \bar{a}}{a_k \cdot \bar{a}}.
\]

Since we want the image under the projection of those points inside the ball,
\[
Q_k(\mathbb{V}) = \left\{ w \in Q_k(B^n) : |w|^2 + \frac{|a|^2 - w \cdot \bar{a}|^2}{|a_k \cdot \bar{a}|^2} |a_k|^2 < 1 \right\}.
\]

Using the \( w_1, w_2 \) notation, the above equation is written
\[
|w_1|^2 + |w_2|^2 + \frac{|a|^2 - w_1 \cdot \bar{a}|^2}{|a_k \cdot \bar{a}|^2} |a_k|^2 < 1.
\]

Notice that \( w_1 \cdot \bar{a} = w_1 \cdot \overline{Q_k(a)}, |w_1 \cdot \overline{Q_k(a)}|^2 = |w_1|^2 |Q_k(a)|^2 \), and
\[
|a|^2 = |Q_k(a)|^2 + \frac{|a \cdot a_k|^2}{|a_k|^2}.
\]
The equation becomes:
\[
|w_1|^2 \left(1 + \frac{|a_k|^2}{|a_k \cdot a|^2}\right) - \frac{|a_k|^2 |a|^2}{|a_k \cdot a|^2} (w_1 \cdot Q_k(a) + w_1 \cdot Q_k(a)) \\
+ \frac{|a_k|^2 |a|^4}{|a_k \cdot a|^2} + |w_2|^2 < 1
\]
which simplifies to
\[
\frac{|a_k|^2 |a|^2}{|a_k \cdot a|^2} |w_1 - Q_k(a)|^2 + |w_2|^2 < 1 - |a|^2.
\]

(3) In the above ellipsoid, the minimum distance to the boundary is attained when \(w_2 = 0\), and equals
\[
1 - |Q_k(a)| - (1 - |a|^2)^{1/2} \cos \theta = 1 - |a| \sin \theta - (1 - |a|^2)^{1/2} \cos \theta.
\]

**Proof of Lemma 5.** First, since \(V_j \cap T_\delta(V_k) = \emptyset\), \(\phi_k(V_j) \cap \phi_k(T_\delta(V_k)) = \emptyset\). Although tubes have no reason to be invariant under automorphisms, \(\phi_k(T_\delta(V_k))\) is not far from being a tube around \(Q_k(B^n) = \phi_k(V_k)\). More precisely, if \(|P_k(z)| < \delta/(1 + \delta)\), then \(\phi_k^{-1}(z) = \phi_k(z) \in T_\delta(V_k)\). Indeed,
\[
(\phi_k(z) - a_k) \cdot \bar{a}_k^* = \frac{-(1 - |a_k|^2)P_k(z)}{1 - z \cdot a_k},
\]
\[
|(\phi_k(z) - a_k) \cdot \bar{a}_k^*| \leq (1 - |a_k|^2) \frac{|P_k(z)|}{1 - |a_k|^2 |P_k(z)|} \\
\leq (1 - |a_k|^2) \frac{|P_k(z)|}{1 - |P_k(z)|} < \delta (1 - |a_k|^2)
\]
under the above hypothesis. It follows that for \(z \in \phi_k(V_j)\), since \(z \notin \phi_k(T_\delta(V_k))\), \(|P_k(z)| \geq \delta/(1 + \delta) =: \delta_1\), and consequently \(|Q_k(z)| = (1 - |P_k(z)|^2)^{1/2} \leq \sqrt{1 - \delta_1^2}\). \(\square\)

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