DEGREES AND FORMAL DEGREES FOR DIVISION ALGEBRAS AND GLₙ OVER A p-ADIC FIELD

LAWRENCE JAY CORWIN, ALLEN MOY AND PAUL J. SALLY, JR.
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We compute in the tame case, the degrees of the irreducible representations of a division algebra and the formal degrees of the discrete series of GL\((n)\) over a \(p\)-adic field and compare them.

1. Introduction. Let \(F\) be a \(p\)-adic field of characteristic zero, and let \(G = \text{GL}_n(F)\). Throughout this paper, we assume that \((n, p) = 1\) (the tame case). The discrete series of \(G\) consists of (equivalence classes of) irreducible, unitary representations of \(G\) whose matrix coefficients are square integrable \((\text{mod } Z)\), where \(Z\) is the center of \(G\). The discrete series splits into two distinct classes ([HC2], [J]):

1. Supercuspidal representations: irreducible unitary representations whose matrix coefficients are compactly supported \((\text{mod } Z)\);
2. Generalized special representations: irreducible unitary representations whose matrix coefficients are square integrable \((\text{mod } Z)\), and which are subrepresentations of representations induced from a proper parabolic subgroup of \(G\).

The supercuspidal representations of \(G\) were constructed by Howe [H2]. The first proof of the fact that all supercuspidal representations of \(G\) are contained in Howe’s construction was given by Moy [M]. The generalized special representations of \(G\) were characterized by Bernstein-Zelevinsky ([BZ], [Z]). We note that the Bernstein-Zelevinsky construction uses the supercuspidal representations of \(\text{GL}_m(F)\) where \(m\mid n \ (m < n)\). Since \((m, p) = 1\) in the present case, the requisite supercuspidal representations can be obtained from Howe’s construction.

The key to the study of the supercuspidal representations of \(G\) is the notion, due to Howe [H2], of an admissible character of an extension of degree \(n\) over \(F\). In fact, the supercuspidal representations of \(G\) are parametrized by (conjugacy classes of) admissible characters of extensions of degree \(n\) over \(F\), and generalized special representations are parametrized by (conjugacy classes of) admissible characters of
extensions of degree \( m \) over \( F \) where \( m|n, m < n \). (See [M] for additional details.)

Now, let \( D_n \) be a division algebra of dimension \( n^2 \) over \( F \), and let \( D_x \) be the multiplicative group of \( D_n \). The irreducible representations of \( D_x \) were constructed as induced representations by Corwin [Co] and Howe [HI]. In these constructions, the inducing representations are obtained from (conjugacy classes of) admissible characters of extensions of degree \( m \) over \( F \) where \( m|n \) (including \( m = n \)).

The proof by Moy [M] that Howe’s representations exhaust the supercuspidal representations of \( G \) uses the abstract matching theorem. The abstract matching theorem was proved by Deligne-Kazhdan-Vigneras [DKV] and Rogawski [R]. Recall that, if \( E/F \) is an extension of degree \( n \), then \( E \) can be embedded in both \( G \) and \( D_x \). In fact, any compact (mod center) Cartan subgroup of \( G \) (and \( D_x \)) is isomorphic to \( E \) for some extension of degree \( n \).

**Theorem 1.1 (Abstract Matching Theorem, [DKV], [R]).** There is a bijection \( \pi' \leftrightarrow \pi \) between irreducible representations of \( D_x \) and the discrete series of representations of \( G \) with the following properties:

1. If \( \theta_{\pi'} \) and \( \theta_{\pi} \) are the characters of \( \pi' \) and \( \pi \) respectively, and \( \gamma \) is a regular element in a compact (mod center) Cartan subgroup \( E^\times \), then
   \[
   \theta_{\pi'}(\gamma) = (-1)^{n-1} \theta_{\pi}(\gamma).
   \]

2. If the formal degree of the Steinberg representation [B] is normalized to be equal to one, then
   \[
   d(\pi') = d(\pi),
   \]
   where \( d(\pi') \) is the ordinary degree of the finite-dimensional representation \( \pi' \) and \( d(\pi) \) is the formal degree of the infinite-dimensional representation \( \pi \);

3. If \( \varepsilon(\pi', \psi) \), \( \varepsilon(\pi, \psi) \) are the \( \varepsilon \)-factors of \( \pi' \) and \( \pi \) respectively, then
   \[
   \varepsilon(\pi', \psi) = (-1)^{n-1} \varepsilon(\pi, \psi).
   \]

Here, \( \psi \) is a suitably chosen additive character on \( F \).

**Remarks 1.2.** (1) Moy’s proof [M] that the supercuspidal representations constructed by Howe and the generalized special representations constructed by Bernstein-Zelevinsky exhaust the discrete series of \( GL_n(F) \) uses the abstract matching theorem in an essential way. Thus, it is only after we use the abstract matching theorem that we
can assert that the concrete matching by admissible characters is actually bijection.

(2) The abstract matching theorem gives no indication as to which representations of \( D^\times \) correspond to the two distinct types of discrete series representations of \( G \).

(3) Recently, Howe-Moy [HM2] have given a proof of the completeness of Howe's construction without the use of Theorem 1.1.

To sharpen our focus, we introduce the following distinction. If \( E/F \) is an extension of degree \( m \), \( m|n \), \( m < n \), and \( \theta \) is an admissible character of \( E^\times \), we say that \( \theta \) is \textit{subadmissible} (for \( n \)). Thus, the term \textit{admissible character} will be used only for extensions \( E/F \) of degree \( n \). The conjugacy classes of admissible and subadmissible characters parametrize the irreducible representations of \( D^\times \). As indicated above, the supercuspidal representations of \( G \) correspond to admissible characters, and the generalized special representations of \( G \) correspond to subadmissible characters. Thus, it is natural to conjecture that, if \( \pi'_{\theta} \) is the irreducible representation of \( D^\times \) corresponding to an admissible (resp. subadmissible) character, then the discrete series representation \( \pi \) of \( G \) which corresponds to \( \pi'_{\theta} \) by the abstract matching theorem is supercuspidal (resp. generalized special).

This last assertion is indeed the case, and it is the purpose of this paper to give a proof using the degrees of the representations. To this end, we consider the following sets:

\begin{align*}
A'_{1} &= \{ \pi'_{\theta} \in (D^\times)^{\sim} | \theta \text{ is admissible} \} ; \\
A'_{2} &= \{ \pi'_{\theta} \in (D^\times)^{\sim} | \theta \text{ is subadmissible} \} .
\end{align*}

Here \( \pi'_{\theta} \) is the representation of \( D^\times \) constructed from \( \theta \) by Corwin and Howe, and \( (D^\times)^{\sim} \) is the unitary dual of \( D^\times \). In a similar fashion, we define

\begin{align*}
A_{1} &= \{ \pi_{\theta} \in \hat{G}_{d} | \theta \text{ is admissible} \} ; \\
A_{2} &= \{ \pi_{\theta} \in \hat{G}_{d} | \theta \text{ is subadmissible} \} .
\end{align*}

In this case, we have the supercuspidal representations (resp. generalized special representations) constructed by Howe (resp. Bernstein-Zelevinsky). \( \hat{G}_{d} \) denotes the discrete series in the unitary dual of \( G \).

Now, letting \( d(\pi) \) denote the ordinary or formal degree of a representation, we set

\begin{align*}
\Delta'_{1} &= \{ d(\pi'_{\theta}) | \pi'_{\theta} \in A'_{1} \} ; \\
\Delta'_{2} &= \{ d(\pi'_{\theta}) | \pi'_{\theta} \in A'_{2} \} ; \\
\Delta_{1} &= \{ d(\pi_{\theta}) | \pi_{\theta} \in A_{1} \} ; \\
\Delta_{2} &= \{ d(\pi_{\theta}) | \pi_{\theta} \in A_{2} \} .
\end{align*}
If we assume that $d(\text{Steinberg}) = 1$, then (2) in the abstract matching theorem implies that $\Delta'_1 \cup \Delta'_2 = \Delta_1 \cup \Delta_2$. We show in Theorem 4.1 that
\[
\Delta'_1 \cap \Delta'_2 = \Delta_1 \cap \Delta_2 = \emptyset, \quad \Delta'_1 = \Delta_1, \quad \text{and} \quad \Delta'_2 = \Delta_2.
\]
Since the trivial representation of $D^\times$ is in $A'_2$, it follows that, under the abstract matching, representations in $A'_1$ correspond to supercuspidal representations of $G$ and representations in $A'_2$ correspond to generalized special representations of $G$. It is interesting to note that the conductors of the representations $\pi_\theta$ and $\pi'_\theta$ appear naturally in the expressions for the formal degrees. This will be discussed in §4.

One of the more important consequences of (1.7) is worth observing here. Using the standard Frobenius formula for induced characters, we are able to give explicit formulas for the characters of the representations $\pi'_\theta \in (D^\times)^\wedge$. It follows from (1) of the abstract matching theorem that these are (up to a sign) explicit formulas for the characters of the discrete series of $G$ on the elliptic set. The distinction provided by the formal degrees tells us which of these are supercuspidal characters and which are generalized special characters. In turn, this allows us to analyze the differences between the two different classes of characters. This analysis is carried out in [CS].

In the case $n = p$, Carayol [C] has determined the formal degrees of the supercuspidal representations of $G$ and the degrees of the corresponding representations of $D^\times$. He has also observed the relationship between the formal degree and the conductor of a representation. Waldspurger [W] has computed the formal degrees of the discrete series of $G$ with a normalization which differs from ours. His techniques for obtaining these formulas are also different, but there are significant points of contact between some aspects of our computations and those of Waldspurger. In §4, we will give more detail about the relationship between our work and that of Carayol and Waldspurger.

In §2, we compute the formal degrees of the supercuspidal and generalized special representations of $G$. While the formal degrees of the supercuspidal representations are computed directly from their construction as induced representations in §2.1 and §2.2, the formal degrees of the generalized special representations are derived in §2.3 and §2.4 using the Hecke algebra isomorphisms proved in Howe-Moy [HM2]. This requires a discussion of the minimal $K$-types associated to generalized special representations.

Section 3 contains the calculation of the degrees of the irreducible representations of $D^\times$. Again, the degrees are computed from the inducing construction.
Finally, in §4, we prove the statement of (1.7). In addition, we make several observations concerning the relationship between degrees and characters, the appearance of the conductor in the expression for the degree of a representation, and the comparison of the formal degree of a generalized special representation with the formal degree of the associated supercuspidal representation. It is worth noting here, that our development hinges to a great extent on the fact that \((n,p) = 1\). However, if the formal degree of a generalized special representation is divided by an appropriate power of the associated supercuspidal representation, the resulting expression does not depend on the admissible character which parametrizes these representations. There is hope that such an expression pertains in the case when \(p|n\).

Some of the results in this paper were announced in [S1]. We adopt the usual notation: \(\mathcal{O}_F\) is the ring of integers in \(F\), \(\mathcal{P}_F\) the maximal ideal in \(\mathcal{O}_F\), and \(\omega_F\) a prime element in \(\mathcal{P}_F\). The \(F\)-conductor of a multiplicative character \(\phi\) on \(F^\times\) will be denoted by \(f_F(\phi)\).

2. Formal degrees for the discrete series of \(\text{GL}_n\). In this section, we compute the formal degrees of the supercuspidal and generalized special representations of \(G = \text{GL}_n(F)\). As mentioned in the Introduction, the formal degrees of the supercuspidal representations are computed directly from their construction as induced representations, while the formal degrees of the generalized special representations are computed by using isomorphisms of certain Hecke algebras. It turns out that the actual computations are remarkably similar for the two cases.

2.1. Degrees of the inducing representations. Let \(E/F\) be an extension of degree \(n\) \(((n,p) = 1)\), and let \(\theta\) be an admissible character of \(E^\times/F\) ([H2], [M]). The irreducible supercuspidal representations of \(G\) may be parametrized by (conjugacy classes of) admissible characters of extensions of degree \(n\) over \(F\). In fact, given \(\theta\), one constructs a compact (mod center) open subgroup \(K_\theta\) of \(G\) and an irreducible representation \(\sigma_\theta\) of \(K_\theta\) such that

\[
\pi_\theta = \text{Ind}_{K_\theta}^G \sigma_\theta
\]

is an irreducible supercuspidal representation of \(G\). Moreover, all irreducible supercuspidal representations can be constructed in this way ([H2], [M]).

Given an admissible character \(\theta\) of \(E^\times/F\), the construction of \(K_\theta\) and \(\sigma_\theta\) proceeds as follows. According to Howe [H2], there is a unique
tower of fields

\[(2.1.2)\quad E = E_t \supset E_{t-1} \supset \cdots \supset E_1 \supset E_0 = F,\]

and characters \(\chi, \phi_1, \ldots, \phi_t\) of \(F^\times, E_1^\times, \ldots, E_t^\times\) respectively such that \(\theta = (\chi \circ N_{E/F})(\phi_1 \circ N_{E/E_1}) \cdots (\phi_t)\). Each character \(\phi_k\) is generic over \(E_{k-1}\) (see [H2], [M]). For simplicity, we abuse the notation and write \(\phi_k = \phi_k \circ N_{E/E_k}\), so that

\[(2.1.3)\quad \theta = \chi \cdot \phi_1 \cdot \cdots \cdot \phi_t.\]

This is the Howe factorization of \(\theta\). It is unique in the sense that the conductorial exponents of the characters are unique, and \(f_E(\phi_1) > f_E(\phi_2) > \cdots > f_E(\phi_t) \geq 1\).

We set

\[(2.1.4)\quad n_k = [E: E_k], \quad e_k = e(E/E_k), \quad f_k = f(E/E_k),\]

\[k = 0, \ldots, t.\]

In particular, \(n_0 = n, n_t = 1, e_0 = e(E/F) = e,\) and \(f_0 = f(E/F) = f\).

If \(j_k = f_E(\phi_k)\), the \(E\)-conductor of \(\phi_k\), we define integers \(i_k, k = 1, 2, \ldots, t-1,\) as follows. For \(k = 1, 2, \ldots, t-1,\)

\[(2.1.5)\quad i_k = \begin{cases} 
  j_k/2, & \text{if } j_k \text{ even}, \\
  (j_k - 1)/2, & \text{if } j_k \text{ odd}.
\end{cases}\]

If \(f_E(\phi_t) = j_t > 1\), define \(i_t\) as above, and, if \(f_E(\phi_t) = j_t = 1\), set \(i_t = 1\).

**Remark 2.1.6.** (1) When \(j_t = 1, E/E_{t-1}\) is unramified [H2].

(2) The relationship between the \(E\)-conductor of \(\phi_k (= \phi_k \circ N_{E/E_k})\) and the \(E_k\)-conductor of \(\phi_k\) is

\[f_E(\phi_k) - 1 = e_k(f_E(\phi_k) - 1).\]

Now, writing \(\mathcal{O}_{E_k} = \mathcal{O}_k\), and \(\mathcal{P}_{E_k} = \mathcal{P}_k\), we define

\[(2.1.7)\quad \mathcal{O}_k = \begin{bmatrix} 
  M_{f_k}(\mathcal{P}_k) & M_{f_k}(\mathcal{O}_k) & \cdots & M_{f_k}(\mathcal{O}_k) \\
  M_{f_k}(\mathcal{P}_k) & M_{f_k}(\mathcal{P}_k) & \cdots & M_{f_k}(\mathcal{O}_k) \\
  \vdots & \vdots & \ddots & \vdots \\
  M_{f_k}(\mathcal{P}_k) & M_{f_k}(\mathcal{P}_k) & \cdots & M_{f_k}(\mathcal{P}_k)
\end{bmatrix},

where there are \(e_k\) blocks in each row and column. We regard \(1 + \mathcal{O}_k^h\) as a subgroup of \(G\) for any positive integer \(h\) (see [M]).

The inducing subgroup for \(\pi_\theta\) is then defined as

\[(2.1.8)\quad K_\theta = E^\times(1 + \mathcal{O}_k^{i_t})(1 + \mathcal{O}_k^{i_{t-1}})(1 + \mathcal{O}_k^{i_{t-2}}) \cdots (1 + \mathcal{O}_k^1)(1 + \mathcal{O}_0^{i_t}),\]
if $E(\phi_t) = j_t > 1$, and
\begin{equation}
K_\theta = E^\times K_{t-1}(1 + \delta_{i-1}^t)(1 + \delta_{i-2}^t) \cdots (1 + \delta_{i}^t),
\end{equation}
if $E(\phi_t) = j_t = 1$, where $K_{t-1} = GL_{n_{t-1}}(M_{t-1}).$

The inducing representation may be written as a tensor product
\begin{equation}
\sigma_\theta = \kappa_t \otimes \kappa_{t-1} \otimes \cdots \otimes \kappa_1 \otimes \chi,
\end{equation}
where $\chi$ is a character of $F^\times$ which can be removed by a twist for the purpose of computing formal degrees. From the construction of $\kappa_k$ ([H2], [M]), we have, for $1 \leq k < t$,
\begin{equation}
\deg(\kappa_k) = 1, \quad j_k \text{ even},
\end{equation}
\begin{equation}
\deg(\kappa_k) = [(1 + \delta_{k-1}^i):(1 + \delta_{k}^i)(1 + \delta_{k+1}^{i+1})]^{1/2}, \quad j_k \text{ odd}.
\end{equation}
If $j_t > 1$, the above formulas are still valid for $\deg(\kappa_t)$, and, if $j_t = 1$,
\begin{equation}
\det(\kappa_t) = \prod_{i=1}^{f_{t-1}} (q_{t-1}^i - 1),
\end{equation}
where $q_{t-1} = q^{f/f_{t-1}}.$

We now compute $\deg(\sigma_\theta)$ from the above data. We set $q_k = q^{f/f_k}, 1 \leq k \leq t$, so that $(q_k^{f_k})^{e_k} = (q^{f/f_k})^{e_k} = q^{f_n k}.$

**Lemma 2.1.13.** $[(1 + \delta_{k-1}^i):(1 + \delta_{k}^i)(1 + \delta_{k+1}^{i+1})] = q^{f_{n_k}/f_k} k.$

**Proof.**
\begin{align*}
[(1 + \delta_{k-1}^i):(1 + \delta_{k}^i)(1 + \delta_{k+1}^{i+1})] \\
= \left[ \frac{[(1 + \delta_{k-1}^i):(1 + \delta_{k+1}^{i+1})]}{[(1 + \delta_{k}^i):(1 + \delta_{k+1}^{i+1})]} \right] = \left[ \frac{q_{k-1}^{e_{k-1}}}{q_k^{e_k}} \right]^{e_k}.
\end{align*}

**Lemma 2.1.14.** If $\theta$ is an admissible character for $E^\times/F([E:F] = n)$, and $\sigma_\theta$ is the representation of $K_\theta$ given by (2.1.10), then
\begin{enumerate}
\item $\deg(\sigma_\theta) = q^{\beta(\theta)}$, where
$$\beta(\theta) = \left( f/2 \right) \sum_{k: j_k \text{ odd}} (n_{k-1} - n_k), \quad E(\phi_t) > 1,$$
\item $\deg(\sigma_\theta) = q^{\beta(\theta)} [\prod_{j=1}^{f_{t-1}} (q_{t-1}^j - 1)],$ where
$$\beta(\theta) = \sum_{k: j_k \text{ odd}} (n_{k-1} - n_k), \quad E(\phi_t) = 1.$$
\end{enumerate}
Proof. This is an immediate consequence of (2.1.11), (2.1.12) and Lemma 2.1.13.

2.2. Normalization of volumes and formal degrees of supercuspidal representations. In order to use the abstract matching theorem for purposes of comparison between representation of $D^\chi$ and $G$, we must normalize measures so that the formal degree of the Steinberg representation is equal to one (Theorem 1.1). We begin by recalling the basic formula for formal degrees ([HC1], p. 5). If $\pi$ is a representation of $G$ which is square integrable (mod Z), and $Z_0$ is a cocompact subgroup of $Z$ (i.e. $Z/Z_0$ is compact), then

$$\int_{G/Z_0} |(v|\pi(x)v)|^2 \, dx = \deg(\pi, G/Z_0)^{-1},$$

where $v$ is a unit vector in the space of $\pi$, and $dx$ is a Haar measure on $G/Z_0$. The formal degree $\deg(\pi, G/Z_0)$, in fact, depends on the normalization of $dx$.

In the case of the Steinberg representation, it is well known ([R]) that

$$\deg(St, G/Z) \cdot \text{vol}(KZ/Z) = \frac{1}{n} \prod_{k=1}^{n-1} (q^k - 1).$$

It should be observed that, in imposing this normalization, we are simultaneously normalizing Haar measures on $G$ and $Z$ so that $\text{vol}_G(K)/\text{vol}_Z(K \cap Z) = \text{vol}_{G/Z}(KZ/Z)$.

For our purposes, it is convenient to get an analogue of (2.2.2) for any cocompact subgroup $Z_0$ of $Z$. To this end, we impose the normalizations

$$\text{vol}_G(K) = \frac{1}{n} \prod_{k=1}^{n-1} (q^k - 1);$$

(iii) $\text{vol}_{Z_0}(K \cap Z_0) = 1$.

We then have

$$\deg(St, G/Z_0) \cdot \text{vol}(KZ_0/Z_0) = \frac{1}{n} \prod_{k=1}^{n-1} (q^k - 1).$$

In particular, if $Z_0$ is the discrete subgroup of $G$ given by $Z_0 = \langle \omega_F I_{n \times n} \rangle$, then (2.2.3), (iii), gives counting measure on $Z_0$. This will
be used below in determining the formal degrees of the generalized special representations.

We now turn to the formal degrees of the supercuspidal representations of \( G \). If \( K_\theta \) and \( \sigma_\theta \) are defined as in (2.1.8) (or (2.1.9)) and (2.1.10), and \( \pi_\theta \) is given as the irreducible supercuspidal representation induced from \( \sigma_\theta \), then it is easy to see ([S2]) that a non-trivial matrix coefficient of \( \sigma_\theta \) may be extended to \( G \) by defining it to be zero on the complement of \( K_\theta \), thus yielding a matrix coefficient of \( \pi_\theta \). It follows that \( \deg(\pi_\theta, G/Z) = \deg(\sigma_\theta) / \vol(ZK_\theta/Z) \). So, to complete our calculation for \( \deg(\pi_\theta, G/Z) \), we must determine \( \vol(ZK_\theta/Z) \) relative to the normalization (2.2.2).

Define

\[
K^{(e)} = \left\{ \begin{bmatrix} \GL_f(\mathcal{O}_F) & \mathcal{O}_F \\ \mathcal{O}_F & \GL_f(\mathcal{O}_F) \end{bmatrix} \right\},
\]

and let \( Z^{(e)} \) be the subgroup generated by

\[
z^{(e)} = \begin{bmatrix}
0 & I_f & 0 & 0 \\
0 & 0 & I_f & \cdots \\
\omega_f I_f & 0 & 0 & \cdots \\
\end{bmatrix},
\]

where \( f = f(E/F) \) and there are \( e = e(E/F) \) blocks in each row and column. Then \( Z^{(e)} \) normalizes \( K^{(e)} \) and \( K_\theta \) is a subgroup of \( Z^{(e)} K^{(e)} \) for any admissible character \( \theta \).

We now have

\begin{equation}
\vol(ZK_\theta/Z) = \vol(Z^{(e)} K^{(e)}/Z)[Z^{(e)} K^{(e)} : K_\theta]^{-1}.
\end{equation}

Moreover, \( Z^{(e)} K^{(e)} \cap ZK = ZK^{(e)} \), and

\begin{equation}
\vol(Z^{(e)} K^{(e)}/Z) = [Z^{(e)} K^{(e)} : ZK^{(e)}] [ZK^{(e)} : ZK]^{-1} \vol(ZK/Z).
\end{equation}

Thus, we must compute the three indices in (2.2.4) and (2.2.5).

**Lemma 2.2.6.** (1) If \( f_E(\phi_i) > 1 \), then

\[
[Z^{(e)} K^{(e)} : K_\theta] = |\GL_f(q)|^e(q^f - 1)^{-1} q^{\alpha f},
\]

where \( \alpha = \sum_{k=1}^i i_k(n_{k-1} - n_k) - n + 1 \).
(2) If \( f_E(\phi_t) = 1 \), then
\[
[Z^{(e)}K^{(e)} : K_\theta] = |GL_f(q)|^e |GL_{f_{t-1}}(q_{t-1})|^{-1} q^\alpha f,
\]
where \( \alpha = \sum_{k=1}^{t-1} i_k(n_{k-1} - n_k) - n + 1. \)

**Proof.** (1) We have \([Z^{(e)}K^{(e)} : K_\theta] = [K^{(e)} : \varepsilon_E(1 + \varepsilon_{i_1}^0) \cdots (1 + \varepsilon_{i_t}^0)].\)

Since \( 1 + \varepsilon_{i_1}^0 \) is normal in \( K^{(e)} \), this last index is equal to
\[
[K^{(e)} : (1 + \varepsilon_{i_1}^0)][\varepsilon_E^X : 1 + \varepsilon_{i_1}^1]^{-1} \left( \prod_{k=1}^{t-1} [\varepsilon_{i_{k+1}}^k : \varepsilon_{i_k}^k] \right)^{-1}.
\]

Now, the following facts lead to the stated formula for \([Z^{(e)}K^{(e)} : K_\theta].\)

First,
\[
[K^{(e)} : (1 + \varepsilon_{i_1}^0)] = [K^{(e)} : (1 + \varepsilon_0)][\varepsilon_{i_1}^0] = |GL_f(q)|^e q^{f(n_{i_1} - 1)}.
\]

Second,
\[
[\varepsilon_E^X : 1 + \varepsilon_{i_1}^1] = [\varepsilon_E^X : 1 + \varepsilon_{i_1}^0][\varepsilon_{i_1}^1] = (q^f - 1)(q^f)^{i_1 - 1}.
\]

And, finally,
\[
[\varepsilon_{i_{k+1}}^k : \varepsilon_{i_k}^k] = q^{f n_k(i_k - i_{k+1})}, \quad k = 1, \ldots, t - 1.
\]

(2) When \( f_E(\phi_t) = 1 \), we have \( E_t/E_{t-1} \) unramified, \( n_{t-1} = f_{t-1} \),
and \( i_t = 1. \) Also, \([\varepsilon_E^X : 1 + \varepsilon_{i_1}^1]\) is replaced by \([K_{t-1} : 1 + \varepsilon_{t-1}] = |GL_{f_{t-1}}(q_{t-1})|.\)

In both cases, the transformation from
\[
\sum n_k(i_k - i_{k+1}) \quad \text{to} \quad \sum i_k(n_{k-1} - n_k)
\]
should be noted.

We now turn to (2.2.5).

**Lemma 2.2.7.** (1) \([Z^{(e)}K^{(e)} : ZK^{(e)}] = e.\)
(2) \([ZK : ZK^{(e)}] = |GL_n(q)|/|GL_f(q)|^e q^{(n^2 - nf)/2}.\)

**Proof.** (1) \( Z^{(e)} \) has order \( e \) mod \( Z.\)

(2) Let \( K_1 \) be the first congruence subgroup of \( K. \) Then \( K_1 \) is a subgroup of \( K^{(e)} \) and
\[
[ZK : ZK^{(e)}] = [K : K_1]^{-1} = |GL_n(q)|/|GL_f(q)|^e (\varepsilon_F/\varepsilon_F)^{f^2} e^{(e-1)/2}.
\]
But \( f^2 e(e - 1)/2 = (n^2 - nf)/2. \)

We are now in a position to give an explicit formula for \( \deg(\pi_\theta, G/Z). \)
Theorem 2.2.8. Let $\pi_\theta$ be the irreducible supercuspidal representation induced from $\sigma_\theta$ where $\theta$ is an admissible character of $E^\times/F([E : F] = n)$. Let $\{n_k, i_k\}$ be the data from the Howe factorization of $\theta$ given in (2.1.4) and (2.1.5). Then, if $\text{vol}(KZ/Z)$ is given by (2.2.2),

$$\deg(\pi_\theta, G/Z) = [f(q^n - 1)/(q^{n/e} - 1)]q^{(f/2)(\alpha(\theta) + 2 - n - e)},$$

where $\alpha(\theta) = \sum_{k=1}^i j_k(n_{k-1} - n_k)$.

Proof. As observed above, we have

$$\deg(\pi_\theta, G/Z) = \deg(\sigma_\theta)/\text{vol}(ZK_\theta/Z).$$

The result follows from (2.1.5), Lemma 2.1.14, Lemma 2.2.6, Lemma 2.2.7, and some elementary arithmetic.

2.3. Hecke algebra isomorphisms. We now consider the generalized special representations. Let $E/F$ be an extension of degree $m$, $m | n$, $m < n$, and let $\theta$ be a subadmissible character of $E^\times/F$. As in (2.1.2), there is a unique tower of fields

$$(2.3.1)\ E = E_t \supset E_{t-1} \supset \cdots \supset E_1 \supset E_0 = F,$$

and the associated Howe factorization

$$(2.3.2)\ \theta = \phi_t \phi_{t-1} \cdots \phi_1.$$

Remark 2.3.3. Here we use the same conventions as above, that is, $\phi_k$ is used to denote $\phi_k \circ N_{E/E_k}$, and the character $\chi$ of $F^\times$ which appears in the Howe factorization of $\theta$ is twisted away for purposes of computing the formal degrees.

Let $n = am$, and extend $E$ to an extension $E'/F$ such that

$$(2.3.4)\ [E' : E] = a \text{ and } E'/E \text{ is totally ramified}.$$

Thus, $[E' : F] = n = am$. Moreover, if $e = e(E/F)$, $f = f(E/F)$, and $n_k, e_k, f_k$ are defined as in (2.1.4), we set

$$(2.3.5)\ e' = e(E'/F) = ea, \ f' = f(E'/F) = f, \ n_k' = [E' : E_k] = an_k, \ e_k' = e(E'/E_k) = ae_k, \ f_k' = f(E'/E_k) = f_k.$$

Note that $n_0' = n$, and $n_1' = a$.

Define $\theta' = \theta \circ N_{E'/E}$. Then, we can write

$$(2.3.6)\ \theta' = \phi_t' \phi_{t-1}' \cdots \phi_1',$$
where \( \phi'_{k} = \phi_{k} \circ \text{NE}/E \) (see Remark 2.3.3). If \( J_{E}(\phi_{k}) = j_{k} \), then \( J_{E}(\phi'_{k}) = j'_{k} \), where \( J_{E}(\phi'_{k}) - 1 = a(J_{E}(\phi_{k}) - 1) \), that is,

\[
(2.3.7) \quad j'_{k} - 1 = a(j_{k} - 1).
\]

In analogy with (2.1.5), we set

\[
(2.3.8) \quad i'_{k} = \begin{cases} j'_{k}/2, & \text{if } j'_{k} \text{ even,} \\ (j'_{k} - 1)/2, & \text{if } j'_{k} \text{ odd, } k = 1, 2, \ldots, t - 1. \end{cases}
\]

If \( J_{E}(\phi'_{l}) = j'_{l} > 1 \), define \( i'_{l} \) as above. If \( J_{E}(\phi'_{l}) = j'_{l} = 1 \), set \( i'_{l} = 1 \). We note from (2.3.7) that \( j'_{l} = 1 \) if and only if \( j_{l} = 1 \).

For the Hecke algebra isomorphisms to which we referred at the beginning of §2, we must define subgroups of \( G \) which are analogous to those in §2.1. Thus, we define \( \mathcal{E}_{k} \) (in \( M_{n}(F) \)) as in (2.1.7), with \( e'_{k} \) and \( f'_{k} \) replacing \( e_{k} \) and \( f_{k} \) respectively. Let \( G_{a} = \text{GL}_{a}(E) \), and let \( B_{a} \) be the Iwahori subgroup of \( G_{a} \), considered as a subgroup of \( G \). If \( j'_{l} > 1 \), we set

\[
(2.3.9) \quad J_{\theta} = B_{a}(1 + \mathcal{E}_{l-1}^{i'_{l}}) \cdots (1 + \mathcal{E}_{1}^{i'_{1}})(1 + \mathcal{E}_{0}^{i'_{0}}).
\]

If \( j'_{l} = j_{l} = 1 \), we write \( h = n_{t-1} = [E: E_{l-1}] \), and let \( P_{t-1} \) be the \( (h, h, \ldots, h) \) (\( a \) times) parahoric subgroup of \( \text{GL}_{ah}(E_{t-1}) \). Then, if \( j'_{l} = j_{l} = 1 \), we define

\[
(2.3.10) \quad J_{\theta} = P_{t-1}(1 + \mathcal{E}_{t-1}^{i'_{t-1}}) \cdots (1 + \mathcal{E}_{1}^{i'_{1}})(1 + \mathcal{E}_{0}^{i'_{0}}).
\]

If \( \pi_{\theta} \) is the generalized special representation constructed from \( \theta \), we write \( (\Omega_{\theta}, J_{\theta}) \) for the minimal \( K \)-type associated to \( \pi_{\theta} \) ([HM2]). The representation \( \Omega_{\theta} \) is constructed in a manner which is very similar to the construction of the inducing representations \( \sigma_{\theta} \) for supercuspidal representations (see (2.1.10) ff). In particular, if \( j'_{l} > 1 \), \( \Omega_{\theta} = (\phi'_{l} \circ \text{det}) \otimes \kappa_{t-1} \otimes \cdots \otimes \kappa_{1} \), where \( \phi'_{l} \circ \text{det} \) is a one dimensional representation on \( B_{a} \), and \( \kappa_{k} \) is defined as in [M]. If \( j'_{l} = 1 \), we consider \( \phi_{l} \) as a character on the anisotropic Cartan subgroup of \( \text{GL}_{ah}(q_{l-1}) \) where \( q_{t-1} = q^{i'_{l}/j'_{l}-1} \). Let \( \kappa_{t-1} \) be the cuspidal representation of \( \text{GL}_{ah}(q_{t-1}) \) associated to \( \phi_{l}([G]) \). We then let \( \kappa_{t-1} \) be \( \otimes \kappa_{t-1} \) (\( a \) times) inflated to \( P_{t-1} \), and set \( \Omega_{\theta} = \kappa_{t-1} \otimes \kappa_{t-2} \otimes \cdots \otimes \kappa_{1} \), where \( \kappa_{k}, 1 \leq k \leq t - 2 \), is defined as in [M]. Note that \( \text{deg}(\kappa_{t-1}) = \prod_{k=1}^{h_{t-1}-1}(q_{k}^{h_{t-1}-1} - 1)^{a} \).

The following lemma is the analogue of Lemma 2.1.14 for the case of generalized special representations.
**Lemma 2.3.11.** If \( \theta \) is a subadmissible character for \( E^\times / F \), \( [E: F] = m, m \mid n, m < n \), and \((\Omega_\theta, J_\theta)\) is the minimal \( K \)-type associated to \( \theta \), then

\[
\begin{align*}
(1) \quad \deg(\Omega_\theta) &= q^{\gamma(\theta)} \sum_{k: j'_k \text{ odd}, k < \ell} (n'_{k-1} - n_k'), j'_\ell > 1, \\
(2) \quad \deg(\Omega_\theta) &= q^{\gamma(\theta)} \left[ \prod_{j=1}^{h-1} (q_{t-1}^j - 1) \right]^a,
\end{align*}
\]

where \( \gamma(\theta) = f(\ell/2) \sum_{k: j'_k \text{ odd}, k < \ell} (n'_{k-1} - n_k'), j'_\ell = 1. \) (Here, as in \S 2.1, \( q_{t-1} = q^{f(\ell/2)}. \))

Before stating the basic theorem on Hecke algebra isomorphisms, we make a simple observation. Set \( T = \langle \tilde{\omega}_F I_{n \times n} \rangle \) and \( T_a = \langle \tilde{\omega}_E I_{a \times a} \rangle \). We can choose the prime elements \( \tilde{\omega}_F \) and \( \tilde{\omega}_E \) so that, under the above embedding of \( G_a \) into \( G \), \( T \) is a subgroup of \( T_a \). It is clear that

\[
[T_a : T] = e.
\]

**Theorem 2.3.13 ([HM2]).** The Hecke algebras \( \mathcal{H}(G/T, J_\theta T/T, \Omega_\theta) \) and \( \mathcal{H}(G_a/T, B_a T/T, 1) \) are isomorphic. This isomorphism carries discrete series to discrete series and preserves Plancherel measure. In particular, the generalized special representation \( \pi_\theta \) of \( G \) corresponds to the Steinberg representation of \( G_a \), and ([HM2]), (5.2))

\[
\deg(\pi_\theta, G/T) \ vol(J_\theta T/T) = \deg(\Omega_\theta) \ deg(St, G_a/T) \ vol(B_a T/T).
\]

Our goal is to determine the values of the factors in the formula (2.3.14). First of all, the degree of the minimal \( K \)-type \( \Omega_\theta \) is given in Lemma 2.3.11. Second, from (2.2.1) and (2.3.12), it follows that

\[
\deg(St, G_a/T) \ vol(B_a T/T)
\]

\[
= e^{-1} \ deg(St, G_a/T_a) \ vol(K_a T_a/T_a)
\]

\[
\times [\ vol(B_a T/T) / \ vol(K_a T_a/T_a) ],
\]

where \( K_a = GL_a(\mathcal{O}_E) \). From the normalizations given by (2.2.3), we obtain

\[
\ vol(B_a T/T) / \ vol(K_a T_a/T_a) = \ vol(B_a) / \ vol(K_a).
\]
Thus, to determine $\deg(\pi_\theta, G/T)$ from (2.3.14), we must compute the volumes $\text{vol}(J_\theta T/T) = \text{vol}(J_\theta)$, and $\text{vol}(B_a)/\text{vol}(K_a)$ relative to the normalization of Haar measures given by (2.2.3) and (2.2.2').

2.4. Formal degree of generalized special representations. The computation of $\text{vol}(J_\theta)$ is similar to those contained in Lemma 2.2.6 and Lemma 2.2.7. In the present case, $J_\theta$ is compact, whereas, in §2.2, the subgroup $K_\theta$ is compact mod $Z$. Here, we define

$$K^{(ea)} = \left\{ \begin{bmatrix} GL_f(\mathcal{O}_F) \\
\mathcal{O}_F \\
\mathcal{P}_F \\
\mathcal{O}_F \\
GL_f(\mathcal{O}_F) \end{bmatrix} \right\},$$

where we have $ea$ copies $GL_f(\mathcal{O}_F)$ along the diagonal ($ae f = am = n$).

In analogy with (2.2.4) and (2.2.5), we have

(2.4.1) $$[K : J_\theta] = [K : K^{(ae)}][K^{(ae)} : J_\theta],$$

and

(2.4.2) $$\text{vol}(J_\theta) = \text{vol}(K)/[K : J_\theta].$$

**Lemma 2.4.3.** (1) If $j_t' > 1$, then

$$[K^{(ae)} : J_\theta] = |GL_f(q)|^{ae} (q^f - 1)^{-a} q^{nf},$$

where $\gamma = \sum_{k=1}^{t-1} i_k(n_k' - 1) - n + a$.

(2) If $j_t' = 1$, then

$$[K^{(ae)} : J_\theta] = |GL_f(q)|^{ae} |GL_h(q_{t-1})|^{-a} q^{nf},$$

where $\gamma = \sum_{k=1}^{t-1} i_k(n_k' - 1) - n + a$, and $h = n_{t-1} = [E : E_{t-1}] = f_{t-1}$.

**Proof.** As expected, the proof is similar to that of Lemma 2.2.6.

(1) If $j_t' > 1$, then, since $B_a$ and $(1 + \mathcal{O}_{k+1}^{i_k t})$ are normal in $J_\theta$, $k = 0, \ldots, t - 1$,

$$[K^{(ae)} : J_\theta] = [K^{(ae)} : 1 + \mathcal{O}_{t}^{i_t}] [B_a : 1 + \mathcal{O}_{t}^{i_t}]^{-1} \prod_{k=1}^{t-1} [\mathcal{O}_{k}^{i_k t} : \mathcal{O}_{k}^{i_k t-1}]^{-1}.$$

The proof now proceeds as in Lemma 2.2.6 with the observation that

$$[B_a : 1 + \mathcal{O}_{t}^{i_t}] = [B_a : 1 + \mathcal{O}_{t}][\mathcal{O}_{t} : \mathcal{O}_{t}^{i_t}] = (q^f - 1)^a q^{fa(i_t'-1)}.$$

(2) If $j_t' = 1$, $[B_a : 1 + \mathcal{O}_{t}^{i_t}]$ is replaced by

$$[P_{t-1} : 1 + \mathcal{O}_{b_t}^{i_t}] = [P_{t-1} : 1 + \mathcal{O}_{b_t-1}][\mathcal{O}_{b_t-1} : \mathcal{O}_{b_t}^{i_t}] = |GL_h(q_{t-1})|^{a} h^{2a(i_t'-1)}.$$
We need three more observations before giving the formula for $\deg(\pi_\theta, G/T)$. First, from (2.2.2'), we have

\begin{equation}
\deg(\text{St}, G_a/T_a) \vol(K_a T_a/T_a) = \frac{1}{a} \prod_{k=1}^{a-1} (q^{f^k} - 1).
\end{equation}

Second, an easy calculation yields

\begin{equation}
\vol(B_a) / \vol(K_a) = [K_a : B_a]^{-1}
= (q^f - 1)^a (q^{f^a} - 1)^{-1} \left\{ \frac{1}{\prod_{k=1}^{a-1} (q^{f^k} - 1)} \right\}^{-1}.
\end{equation}

Finally, from (2.2.3)

\begin{equation}
\vol(K) / [K : K^{(ae)}] = [\text{GL}_f(q)^{ae} q^{(n/2)(1-f)}] / [n(q^n - 1)]
\end{equation}

since $[K : K^{(ae)}] = |\text{GL}_n(q)| / |\text{GL}_f(q)|^{-a} q^{-f_2(\text{ae} - 1)/2}$ (see the proof of Lemma 2.2.7(2)).

**Theorem 2.4.7.** Let $E/F$ be an extension of degree $m$, $m|n$, $m < n$, and write $n = ma$. Let $\theta$ be a subadmissible character on $E^\times / G$, and let $\pi_\theta$ be the generalized special representation constructed from $\theta$. Let $e = e(E/F)$, $f = f(E/F)$, and $\{n_k, j_k\}$ be the data from the Howe factorization of $\theta$ given in (2.1.4) and (2.1.5). Then

$$
\deg(\pi_\theta, G/T) = \left[ f(q^n - 1) / (q^{n/e} - 1) \right] q^{(af/2)(\alpha(\theta) + a + 1 - a\alpha - e)}
$$

where $\alpha(\theta) = \sum_{k=1}^t j_k (n_{k-1} - n_k)$.

**Proof.** We have

$$
\deg(\pi_\theta, G/T) = e^{-1} [\vol(J_\theta)]^{-1} \deg(\Omega_\theta) \deg(\text{St}, G_a/T_a)
\times \vol(K_a T_a/T_a) / [\vol(B_a) / \vol(K_a)]
$$

from (2.3.14), (2.3.15) and (2.3.16). The values of the various factors in the above expression are given in Lemma 2.3.11, Lemma 2.4.3, (2.4.4), (2.4.5) and (2.4.6). Some elementary arithmetic gives us

$$
\deg(\pi_\theta, G/T) = [(n/ea)(q^n - 1) / (q^{fa} - 1)] q^{(f/2)(\alpha(\theta) + 2a - ea - n)}
$$

where $\alpha'(\theta) = \sum_{k=1}^t j_k' (n_{k-1}' - n_k')$.

The final formula is obtained by using (2.3.5) and (2.3.7).

**Remark 2.4.8.** When $a = 1$, then $m = n$ and the formula in Theorem 2.4.7 reduces to the formula in Theorem 2.2.8. Note that the normalizations of measures given in (2.2.3) shows that, in the calculations leading to the formal degrees of both the supercuspidal and generalized special representations, Haar measure on $G$ is given by (2.2.3), (i).
3. Degrees for the representations of $D^\times_n$. This section contains the calculation of the degrees of the irreducible representations of $D^\times_n = D^\times_n$. Since $D^\times$ is compact mod $Z_D$ (where $Z_D$ is the center of $D$, $Z_D \simeq F^\times$), these representations are finite dimensional. Moreover, as pointed out in the Introduction, the irreducible representations of $D^\times$ may be constructed as induced representations, and the inducing representations are determined by admissible or subadmissible characters of extensions $E/F$ of degree $m$, where $m \mid n$.

Let $\mathfrak{D}$ be the integers in $D$, $\mathfrak{D}$ a prime element in $D$, and set $\mathfrak{D}^r = \mathfrak{D}^r \mathfrak{D}$, $r \geq 1$. Let $F_n$ be an unramified extension of degree $n$ over $F$ which is embedded in $D$. The residue class field $\overline{F}_n$ is also the residue class field of $D$, and $|\overline{F}_n| = q^n$. We may choose $\mathfrak{D}_n$ so that $\mathfrak{D}_n^r$ is a prime element of $F$. We set $K_0 = \mathfrak{D}^\times$, and $K_h = 1 + \mathfrak{D}^h$, $h \geq 1$.

Now, let $E$ be an extension of degree $m$ over $F$, where $m \mid n$, and, as in §2, write $e = e(E/F)$ and $f = f(E/F)$. Contrary to the situation in §2, there is no need here to separate the cases $m = n$ and $m < n$. We write

$$ n = ma, $$

noting that we may have $a = 1$. Let $\theta$ be an admissible or subadmissible character of $E^\times/F$ with Howe factorization given by (2.1.2) and (2.1.3) (or (2.3.1) and (2.3.2)). As usual, we twist away the character $\chi$ for the purpose of computing degrees.

Using the notation in (2.3.5) without referring to an auxiliary extension $E'$, we set

$$ e'_k = ae(E/E_k), \quad f'_k = f(E/E_k), \quad n'_k = a[E : E_k] = an_k. $$

In particular $n_0 = m$.

For the $E$-conductor of $\phi_k$, we write $\mathcal{J}_E(\phi_k) = j_k$, and define $\mathcal{J}_D(\phi_k) = j'_k$, where

$$ j'_k - 1 = af(j_k - 1). $$

As in (2.3.8), we set, for $1 \leq k \leq t$,

$$ i'_k = \begin{cases} \frac{j'_k}{2}, & \text{if } j'_k \text{ even}, \\ \frac{(j'_k - 1)}{2}, & \text{if } j'_k \text{ odd}. \end{cases} $$

We set $j'_{t+1} = 1$ and $i'_{t+1} = 0$.

Let $\pi'_\theta$ be the representation of $D^\times$ corresponding to $\theta$. In order to compute $\deg(\pi'_\theta)$, we recall a few facts about its construction. We
embed $E$ (and hence $E_k$) in $D$, and let $D_k$ be the division algebra

$$\begin{align*}
(3.5) \quad D_k = \{ x \in D | xy = yx \text{ for all } y \in E_k \}, \quad 0 \leq k \leq t.
\end{align*}$$

Note that $D_0 = D$. Now, define

$$\begin{align*}
(3.6) \quad H_\theta = D^\times_t (K_{i_t} \cap D^\times_{i_{t-1}}) (K_{i_{t-1}} \cap D^\times_{i_{t-2}}) \cdots (K_{i_2} \cap D^\times_1) (K_{i_1}).
\end{align*}$$

Then ([Co], [H1], [M]), there is an irreducible representation $\sigma_\theta'$ of $H_\theta$ such that

$$\begin{align*}
(3.7) \quad \pi_\theta' = \operatorname{Ind}_{H_\theta}^{D^\times_\theta} \sigma_\theta'.
\end{align*}$$

From [Co], [M], we can write

$$\begin{align*}
(3.8) \quad \sigma_\theta' = \kappa'_{i_t} \otimes \kappa'_{i_{t-1}} \otimes \cdots \otimes \kappa'_{i_1}.
\end{align*}$$

For the remainder of this section, we write $f_k = f(E_k/F), \ k = 0, 1, \ldots, t$. This should not be confused with the notation $f_k = f(E/E_k)$ as used in §2. Note that $\overline{f}_{k-1} \mid \overline{f}_k$.

**Lemma 3.9** ([Co], [M]). Let $\sigma_\theta'$ be the representation of $H_\theta$ given in (3.8). Then, for $1 \leq k \leq t$,

$$\begin{align*}
\deg(\kappa'_{i_k}) = \begin{cases} 
1 & \text{if } f_k \text{ is even}, \\
q^{\alpha(k)/2} & \text{if } f_k \text{ is odd}, 
\end{cases}
\end{align*}$$

where

$$\begin{align*}
\alpha(k) = \begin{cases} 
0 & \text{if } \overline{f}_{k-1} \nmid i'_k, \\
f e'_{k-1} & \text{if } \overline{f}_{k-1} \mid i'_k, \\
f (e'_{k-1} - e'_k) & \text{if } \overline{f}_k \mid i'_k.
\end{cases}
\end{align*}$$

Now, for the degree of $\pi_\theta'$, we have

$$\begin{align*}
(3.10) \quad \deg(\pi_\theta') = [D^\times : H_\theta] \deg(\sigma_\theta'), \quad \text{and}
(3.11) \quad [D^\times : H_\theta] = [D^\times : H_\theta K_0][H_\theta K_0 : H_\theta K_1][H_\theta K_1 : H_\theta].
\end{align*}$$

We calculate each of the three indices in (3.11) separately.

**Lemma 3.12.** $[D^\times : H_\theta K_0] = f$.

**Proof.** First note that $[D^\times : H_\theta K_0] = [D^\times / K_0 : H_\theta K_0 / K_0]$. Let $\nu$ be the usual valuation on $D^\times$. Then $\nu : D^\times \to \mathbb{Z}$ has kernel $K_0$. Under this map, $\nu(H_\theta)$ is generated by $\nu(\tilde{\omega}_t)$, where $\tilde{\omega}_t$ is a prime in $D_t$. Since $\tilde{\omega}_t$ must commute with the unramified piece of $E_t$, $\nu(\tilde{\omega}_t) = f$. 
**Lemma 3.13.** \([H_0K_0 : H_0K_1] = (q^n - 1)/(q^{n/e} - 1)\).

*Proof.* We have \([H_0K_0 : H_0K_1] = [H_0K_0/\langle \omega_i \rangle K_1 : H_0K_1/\langle \omega_i \rangle K_1]\). Now \(H_0K_0/\langle \omega_i \rangle K_1\) is isomorphic to \(\overline{F}_n\), and \(H_0K_1/\langle \omega_i \rangle K_1\) is isomorphic to the multiplicative group of the residue class field of \(D_i\). But, \(D_i\) is a division algebra of index \(a\) over \(E_i\), and the residue class field of \(E_i = E\) has order \(q^f\).

Before calculating the remaining index, \([H_0K_1 : H_0]\), we establish some notation. Define, for \(j \geq 0\),

\[
(3.14) \quad \beta_j = \begin{cases} 
  n - fe' & \text{if } i_{k+1}' < j < i_k' \text{ and } \overline{f}_k \mid j; \\
  n & \text{if } i_{k+1}' \leq j < i_k' \text{ and } \overline{f}_k \nmid j.
\end{cases}
\]

Note that \(i_{k+1}' < i_k'\) from the properties of the Howe factorization; and that \(\beta_0 = n - af\).

**Lemma 3.15.** Let \(\beta_j\) be defined as above. Then \([H_0K_1 : H_0] = q \sum_{j=0}^{i_k' - 1} \beta_j\).

*Proof.* Since \(K_{i_k'} \subset H_0\), we have

\[
[H_0K_1 : H_0] = \prod_{j=1}^{i_k'} [H_0K_j : H_0K_{j+1}].
\]

Suppose \(i_{k+1}' \leq j < i_k'\). We have

\[
[H_0K_j : H_0K_{j+1}] = [K_j/K_{j+1}(H_0 \cap K_{j+1}) : 1].
\]

For \(j \geq 1\), \(K_j/K_{j+1} \cong \mathcal{D}_D^j/\mathcal{D}_D^{j+1} \cong \overline{F}_n\), and the elements of \(H_0 \cap K_j\) correspond, under this isomorphism, to

\[
\{\delta \in \overline{F}_n | \delta \omega_D \in D_k \text{ (mod } \mathcal{D}_D^{j+1})\}.
\]

The number of such elements is equal to \(q^{fe'}\) if \(\overline{f}_k \mid j\) and 1 if \(\overline{f}_k \nmid j\).

Hence, if \(i_{k+1}' \leq j < i_k'\), we get

\[
[H_0K_j : H_0K_{j+1}] = \begin{cases} 
  q^{n-fe'} & \text{if } \overline{f}_k \mid j, \\
  q^n & \text{if } \overline{f}_k \nmid j.
\end{cases}
\]

**Corollary 3.16.** Let \(E/F\) be an extension of degree \(m\), \(m|n\), and let \(\theta\) be an admissible or subadmissible character of \(E^\times/F\). Let \(\pi'_\theta\) be the irreducible representations of \(D^\times\) associated to \(\theta\). Then

\[
(3.17) \quad \deg(\pi'_\theta) = [f(q^n - 1)/(q^{n/e} - 1)]q^{(1/2) \sum_{k=1}^i \alpha(k)} q^{\sum_{j=1}^{i_k' - 1} \beta_j}.
\]
For the computations below, it is convenient to define

\[
\alpha_1(k) = \begin{cases} 
0 & \text{if } j_{k+1}' \text{ is even or } \overline{f}_k \nmid i_{k+1}', \\
fe'_k & \text{otherwise}; 
\end{cases} \\
\alpha_2(k) = \begin{cases} 
0 & \text{if } j_k' \text{ is even or } \overline{f}_k \nmid i_k'
\end{cases} \text{ otherwise.}
\]

(3.18)

Then, we have \(\alpha(k) = \alpha_1(k - 1) - \alpha_2(k), 1 \leq k \leq t.\) We observe that \(\alpha_1(t) = n/e,\) and that \(\alpha_1(0)\) is defined.

Set

\[
(3.19) \quad \gamma(k) = (1/2)(\alpha_1(k) - \alpha_2(k)) + \sum_{j=i_{k+1}'}^{i_k-1} \beta_j, \quad 1 \leq k \leq t.
\]

To compute \(\gamma(k),\) we consider four cases. Note that \(\overline{f}_k \mid j_k' - 1\) always.

(I) \(j_{k+1}' \text{ even, } j_k' \text{ even.}\) Here, \(\alpha_1(k) = \alpha_2(k) = 0,\) and \(\overline{f}_k \nmid j_{k+1}'\) is odd. The number of multiples of \(\overline{f}_k\) in \([i_{k+1}', i_k' - 1)\) is \(1/2 \overline{f}_k(j_{k+1}' - j_k').\) So

\[
\gamma(k) = n(i_k' - i_{k+1}') - (fe'_k)(1/2\overline{f}_k)(j_{k+1}' - j_k')
= (1/2)(j_{k+1}' - j_k')(n - n_{k+1}').
\]

(II) \(j_{k+1}' \text{ odd, } j_k' \text{ even.}\) Then \(\overline{f}_k\) is odd, so that \(\overline{f}_k \nmid j_{k+1}', \overline{f}_k \nmid i_k'.\) This gives \(\alpha_1(k) = fe'_k,\) \(\alpha_2(k) = 0.\) The number of multiples of \(\overline{f}_k\) in \([j_{k+1}' - 1, i_k' - 1)\) is \((1/2\overline{f}_k)(j_{k+1}' - j_k').\) Thus, the number of multiples of \(\overline{f}_k\) in \([i_{k+1}', i_k' - 1)\) is \((1/2\overline{f}_k)(j_{k+1}' - j_k') + 1/2.\) We then have

\[
\gamma(k) = n(i_k' - i_{k+1}') - (fe'_k)(1/2\overline{f}_k)(j_{k+1}' - j_k') - (1/2)fe'_k + (1/2)fe'_k
= (1/2)(j_{k+1}' - j_k')(n - n_{k+1}') + (n/2).
\]

(III) \(j_{k+1}' \text{ even, } j_k' \text{ odd.}\) Here \(\overline{f}_k\) is odd, so that \(\overline{f}_k\) is odd. Therefore, \(\overline{f}_k \mid i_k', \overline{f}_k \nmid i_k', \alpha_1(k) = 0, \alpha_2(k) = fe'_k.\) The reasoning is similar to that in (II) above, but here we lose a multiple of \(\overline{f}_k.\) We get

\[
\gamma(k) = (1/2)(j_{k+1}' - j_k')(n - n_{k+1}') - (n/2).
\]

(IV) \(j_{k+1}' \text{ odd, } j_k' \text{ odd.}\) We must consider subcases here.

If \(\overline{f}_k \mid i_k'\) and \(\overline{f}_k \mid i_{k+1}',\) then \(\alpha_1(k) = \alpha_2(k) = fe'_k,\) and the number of multiples of \(\overline{f}_k\) in the relevant interval is \((1/2\overline{f}_k)(j_{k+1}' - j_k').\) This gives \(\gamma(k) = (1/2)(j_{k+1}' - j_k')(n - n_{k+1}').\)
If $\overline{f}_k |i'_k$, but $\overline{f}_k \nmid i'_{k+1}$, then $\alpha_1(k) = 0$ and $\alpha_2(k) = fe'_k$, and the number of multiples of $\overline{f}_k$ in the relevant interval is $(1/2\overline{f}_k)(j'_k - j'_{k+1}) - (1/2)$. Thus,

$$
\gamma(k) = (1/2)(j'_k - j'_{k+1})(n - n'_k) + (1/2)fe'_k - (1/2)fe'_k
$$

$$
= (1/2)(j'_k - j'_{k+1})(n - n'_k).
$$

The cases $\overline{f}_k \nmid i'_k$, $\overline{f}_k |i'_{k+1}$, and $\overline{f}_k \nmid i'_k$, $\overline{f}_k \nmid i'_{k+1}$ are similar, and give $\gamma(k) = 1/2(j'_k - j'_{k+1})(n - n'_k)$.

We now sum the $\gamma(k)$.

**Lemma 3.20.** (a) If $j'_1$ is even, then

$$
\sum_{k=1}^{t} \gamma(k) = (1/2) \left[ \sum_{k=1}^{t} (j'_k - j'_{k+1})(n - n'_k) + n \right].
$$

(b) If $j'_1$ is odd, then

$$
\sum_{k=1}^{t} \gamma(k) = (1/2) \left[ \sum_{k=1}^{t} (j'_k - j'_{k+1})(n - n'_k) \right].
$$

**Proof.** (a) Here, case (II) occurs once more than case (III).

(b) If $j'_1$ is odd, case (II) and case (III) balance out.

We now have, from (3.17), (3.18) and Lemma 3.20,

(3.21) $\deg(\pi'_\theta) = [f(q^n - 1)/(q^{n/e} - 1)]q^{\sum_{k=1}^{t} \gamma(k)} + [1/2][\alpha_1(0) - \alpha_1(t)] - \beta_0$.

Note that $\alpha_1(0) = 0$ if $j'_1$ is even, and $\alpha_1(0) = n$ if $j'_1$ is odd. Also, $\alpha_1(t) = af$, and $\beta_0 = n - af$ in all cases. It follows that

(3.22) $\sum_{k=1}^{t} \gamma(k) + [1/2][\alpha_1(0) - \alpha_1(t)] - \beta_0$

$$
= \frac{1}{2} \left[ \sum_{k=1}^{t} (j'_k - j'_{k+1})(n - n'_k) - n + af \right],
$$

and that

(3.23) $\deg(\pi'_\theta) = [f(q^n - 1)/(q^{n/e} - 1)]q^{1/2}[\sum_{k=1}^{t} (j'_k - j'_{k+1})(n - n'_k) - n + af]$.
Now, from (3.2) and (3.3), we see that
\[
\sum_{k=1}^{t}(j'_k - j'_{k+1})(n - n'_k) = n(j'_1 - j'_{t+1}) - \sum_{k=1}^{t} n'_k(j'_k - j'_{k+1})
\]
\[
= nj'_1 - n - n'_0j'_1 + n'_t + \sum_{k=1}^{t} j'_k(n'_{k-1} - n'_k)
\]
\[
= -n + a + \sum_{k=1}^{t}[1 + af(j_k - 1)](an_{k-1} - an_k)
\]
\[
= -n + a(n_0 - n_t) - a^2f(n_0 - n_t) + (a^2f)\sum_{k=1}^{t} j_k(n_{k-1} - n_k)
\]
\[
= -a^2 fm + a^2 f + (a^2 f)\sum_{k=1}^{t} j_k(n_{k-1} - n_k).
\]
Thus, the exponent of \( q \) in (3.23) is
\[
(3.24) \quad (af/2) \left[ a \sum_{k=1}^{t} j_k(n_{k-1} - n_k) + a + 1 - am - e \right].
\]

**Theorem 3.25.** Let \( E/F \) be an extension of degree \( m \), \( m|n \), and write \( n = ma \) (here, we may have \( m = n \) and \( a = 1 \)). Let \( \theta \) be an admissible or subadmissible character of \( E^x/F \) and let \( \pi'_\theta \) be the irreducible representation of \( D^x \) constructed from \( \theta \). Let \( e = e(E/F) \), \( f = f(E/F) \), and let \( \{n_k, j_k\} \) be the data from the Howe factorization of \( \theta \). Then
\[
\deg(\pi_\theta) = [f(q^n - 1)/(q^{n/e} - 1)]q^{(af/2)[a0(\theta)+a+1-am-e]}
\]
where \( \alpha(\theta) = \sum_{k=1}^{t} j_k(n_{k-1} - n_k) \).

**4. Comparison of degrees.** We are now in a position to wrap things up in fine fashion. In particular, we prove the statements in (1.7) and derive some consequences. We begin with a simple lemma.

**Lemma 4.1.** If \( a, b, c, d \) are positive integers such that \( a|B \) and \( c|d \), and \( q \) is a power of a prime, then \( a/(q^b - 1) = c/(q^d - 1) \) if and only if \( a = c \) and \( b = d \).

**Proof.** If \( b = d \), then \( a = c \). So, assume \( b > d \). It is easy to see that \((q^b - 1, q^d - 1) = q^r - 1 \) where \( r = (b, d) \). Let \( q^b - 1 = (q^r - 1)b' \),
\[ q^d - 1 = (q^r - 1)d'. \] Then \( ad' = b'c \) and \((b', d') = 1\). Thus, \( b' \mid a \). Since \( r < b/2 \), \((q^r - 1)^2 < q^b - 1\). It follows that \( q^b - 1 < (b')^2 \leq a^2 \leq b^2\). But, if \( q \geq 3 \), \( q^b - 1 > b^2 \) for all positive integers \( b \). If \( q = 2 \), then \( q^b - 1 > b^2 \) when \( b > 4 \). We are left with \( q = 2, 4 \geq b \geq 1 \), and an enumeration of cases shows that there can be no solution with these restrictions.

**Disjunction of formal degrees.**

**Theorem 4.2.** Let \( d(\pi'_\theta) \) and \( d(\pi_\theta) \) be the degrees and formal degrees given in Theorem 2.2.8, Theorem 2.4.7 and Theorem 3.25. Let \( \Delta'_1, \Delta'_2, \Delta_1 \) and \( \Delta_2 \) be the sets defined in (1.5) and (1.6). Then \( \Delta'_1 \cap \Delta'_2 = \Delta_1 \cap \Delta_2 = \Delta'_1 \cap \Delta_1 = \Delta_2 \). Thus, \( \Delta'_1 = \Delta_1 \) and \( \Delta'_2 = \Delta_2 \).

**Proof.** This is an immediate consequence of Lemma 4.1 and the fact that \( \Delta'_1 \cup \Delta'_2 = \Delta_1 \cup \Delta_2 \).

**Remark 4.3.** The disjunction of formal degrees provided by Theorem 4.2 uses only the factor in the degrees which is prime to \( p \), that is, \( f(q^n - 1)/(q^{n/e} - 1) \). This factor depends only on the field \( E \) and not on the particular admissible or subadmissible character. We expect that Theorem 4.2 is true in an appropriate sense when \( p \mid n \).

**Comparison of characters.**

**Theorem 4.4.** (1) If \( \theta \) is an admissible character of \( E^\times /F([E:F]=n) \) and \( \pi'_\theta \) is the representation of \( D^\times \) parametrized by \( \theta \), then the representation \( \pi \) of \( G \) corresponding to \( \pi'_\theta \) under the abstract matching theorem is supercuspidal. Moreover, if \( \Theta_{\pi'_\theta} \) is the character of \( \pi'_\theta \), then \( \Theta_{\pi} = (-1)^{n-1} \Theta_{\pi'_\theta} \) is a supercuspidal character on the elliptic set in \( G \).

(2) If \( \theta \) is a subadmissible character (for \( n \)) of \( E^\times /F, [E:F]=m, m \mid n, m < n \), and \( \pi'_\theta \) is the representation of \( D^\times \) parametrized by \( \theta \), then the representation \( \pi \) of \( G \) corresponding to \( \pi'_\theta \) under the abstract matching theorem is a generalized special representation of \( G \). Moreover, if \( \Theta_{\pi'_\theta} \) is the character of \( \pi'_\theta \), then \( \Theta_{\pi} = (-1)^{n-1} \Theta_{\pi'_\theta} \) is a generalized special character on the elliptic set in \( G \).

**Proof.** This follows from Theorem (1.1) since the trivial representation of \( D^\times \) is in \( A_2' \) and the Steinberg representation of \( G \) is in \( A_2 \) (see (1.3) and (1.14)).
REMARK 4.5. (1) On the elliptic set in $G$, the character of a discrete series representation $\pi$ is equal to $\pm d(\pi)$ near 1. It follows from Theorem 4.4 that supercuspidal characters can be distinguished from generalized special characters by their values near 1 on the elliptic set.

(2) Theorem 4.4 tells us that representations of $D^\times$ in $A'_1$ correspond to supercuspidal representations of $G$, and representations in $A'_2$ correspond to generalized special representations of $G$. In fact, Remark 4.3 allows us to refine this correspondence. Thus, if $\pi'_\theta \in A'_1$ (resp. $A'_2$) and $\pi$ is the corresponding representation in $A_1$ (resp. $A_2$), then $\pi$ is parametrized by an admissible (resp. subadmissible) character of a field $E/F$ which has the same ramification index and residue class degree as the field associated to $\theta$.

(3) In general, the concrete matching by admissible characters is not the same as that given by Theorem 1.1 (1). It would be of some interest to determine the exact relation between these two matchings (see [M] for additional details).

Dependence on $\theta$.

THEOREM 4.6. Let $E/F$ be an extension of degree $m$, and let $\theta$ be an admissible character of $E^\times/F$. Let $\pi_\theta$ be the supercuspidal representation of $GL_m(F)$ determined by $\theta$, and let $\pi'^\theta$ be the generalized special representation of $GL_n(F)$ corresponding to $\pi_\theta$, where $n = ma$. Then

$$d(\pi_\theta)^a / d(\pi'^\theta)$$

$$= f^{a-1} [(q^m - 1)^2 (q^{fa} - 1)/(q^f - 1)^a (q^n - 1)] q^{(fa(a-1)(1-e)/2}}.$$
Theorem 2.4.7, we see that there is no direct relationship between the conductor of $\pi_\theta$ and the formal degree of $\pi_\theta$. We do note however that, if the data from the Howe factorization is known, in particular $j_1 = f_E(\theta)$, then the formula for the conductor (4.8) is an immediate consequence.

References


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**Rutgers University**  
New Brunswick, NJ 08903

**University of Washington**  
Seattle, WA 98195

AND

**University of Chicago**  
Chicago, IL 60637
Yusuf Abu-Muhanna and Abdallah Khalil Lyzzaik, The boundary behaviour of harmonic univalent maps ........................................ 1

Lawrence Jay Corwin, Allen Moy and Paul J. Sally, Jr., Degrees and formal degrees for division algebras and GL$_n$ over a $p$-adic field ............ 21

Ulrich Dierkes and Gerhard Huisken, The $n$-dimensional analogue of the catenary: existence and nonexistence ........................................ 47

Peter Gerard Dodds, C. B. Huijsmans and Bernardus de Pagter, Characterizations of conditional expectation-type operators ....................... 55

Douglas Dokken and Robert Ellis, The Poisson flow associated with a measure .................................................................................... 79

Larry J. Santoni, Horrocks’ question for monomially graded modules ........ 105

Zbigniew Slodkowski, Pseudoconvex classes of functions. II. Affine pseudoconvex classes on $R^N$ ......................................................... 125

Dean Ellis Smith, On the Cohen-Macaulay property in commutative algebra and simplicial topology ...................................................... 165

Michał Szurek and Jarosław Wisniewski, Fano bundles over $P^3$ and $Q_3$ ... 197