THE $n$-DIMENSIONAL ANALOGUE OF THE CATENARY:
EXISTENCE AND NON-EXISTENCE

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We study "heavy" $n$-dimensional surfaces suspended from some prescribed $(n-1)$-dimensional boundary data. This leads to a mean curvature type equation with a non-monotone right hand side. We show that the equation has no solution if the boundary data are too small, and, using a fixed point argument, that the problem always has a smooth solution for sufficiently large boundary data.

Consider a material surface $M$ of constant mass density which is suspended from an $(n-1)$-dimensional surface $\Gamma$ in $\mathbb{R}^n \times \mathbb{R}^+$, $\mathbb{R}^+ = \{t > 0\}$, and hangs under its own weight. If $M$ is given as graph of a regular function $u: \Omega \to \mathbb{R}^+$ on a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, then $u$ provides an equilibrium for the potential energy $\mathcal{E}$ under gravitational forces,

$$\mathcal{E}(u) = \int_{\Omega} u \sqrt{1 + |Du|^2}.$$

Thus $u$ solves the Dirichlet problem

$$\begin{align*}
\text{div} \left\{ \frac{u \cdot Du}{\sqrt{1 + |Du|^2}} \right\} &= \sqrt{1 + |Du|^2} \quad \text{in } \Omega, \\
(1) \quad u &= \phi \quad \text{on } \partial \Omega
\end{align*}$$

The corresponding variational problem

$$\int_{\Omega} u \sqrt{1 + |Du|^2} + \frac{1}{2} \int_{\partial \Omega} |u^2 - \varphi^2| d\mathcal{H}_{n-1} \to \min$$

in the class

$$BV_2^+(\Omega) := \{u \in L^2(\Omega): u \geq 0, \ u^2 \in BV(\Omega)\}$$

has been solved by Bemelmans and Dierkes in [BD]. It was shown in [BD, Theorem 7] that the coincidence set $\{u = 0\}$ of a minimizer $u$ is non-empty provided that

$$\begin{align*}
(3) \quad |\varphi|_{\infty, \partial \Omega} < \frac{|\Omega|}{\mathcal{H}_{n-1}(\partial \Omega)},
\end{align*}$$

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where $|\Omega|$ denotes the Lebesgue measure of $\Omega$ and $\mathcal{H}_n$ denotes $n$-dimensional Hausdorff measure.

We want to show here that (1) has no solution in case (3) holds, whereas (1) has always a solution for sufficiently large boundary data. More precisely we prove the following existence-non-existence result.

**Theorem.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain of class $C^{2,\alpha}$, $\alpha > 0$, with non-negative (inward) mean curvature. Suppose $\varphi \in C^{2,\alpha}(\overline{\Omega})$ satisfies

\[ k_0 := \inf_{\partial \Omega} \varphi \geq c(n) \left(1 + \sqrt{2^{n+1}}\right)^2 |\Omega|^{1/n}, \]

where $c(n) = n^{-1} \omega_n^{1/n}$ is the isoperimetric constant. Then the Dirichlet problem (1) has a global regular solution $u \in C^{2,\alpha}(\overline{\Omega})$. Moreover, if $u \in C^{0,1}(\overline{\Omega})$ is a weak positive solution of (1) with Lipschitz constant $L$, then we have

\[ h := \sup_{\partial \Omega} \varphi \geq (1 + L^{-2})^{1/2} \frac{|A|}{\mathcal{H}_{n-1}(\partial A)} \]

for every Caccioppoli set $A \subset \Omega$.

Since $c(n)$ is the isoperimetric constant, we have

\[ c(n) |\Omega|^{1/n} \geq \frac{|\Omega|}{\mathcal{H}_{n-1}(\partial \Omega)} \]

and therefore it is an interesting question whether our existence result remains true if we replace (4) with an inequality of the form

\[ k_0 \geq \text{const.} \frac{|\Omega|}{\mathcal{H}_{n-1}(\partial \Omega)}. \]

The proof of the theorem is based on a priori bounds for solutions to the related problem

\[ \Delta u - \frac{D_i u D_j u}{1 + |Du|^2} D_i D_j u = f^{-1}, \]

which enable us to apply a fixed point argument. Notice that the operator

\[ \Delta - \frac{D_i u D_j u}{1 + |Du|^2} D_i D_j = (1 + |Du|^2) \cdot \Delta_M \]

where $\Delta_M$ is the Laplace-Beltrami operator on $M = \text{graph} \, u$.

Let us make some comments on the literature. For two dimensional parametric surfaces in $\mathbb{R}^3$ the existence problem has been investigated
by Böhme, Hildebrandt and Tausch [BHT]. To our knowledge the first existence result for the Dirichlet problem (1), in case $n = 2$, is due to Dierkes [D1]. The variational problem (2) is solved in [BD]. It is shown in [D2] that minima $u$ of (2) are regular up to the boundary provided only the boundary is mean curvature convex. A non-existence result of a different type has been proved by J. C. C. Nitsche in [N].

**Proof.** We consider regular solutions $u_f \in C^{2,\alpha}(\mathring{\Omega})$ of the related problem

$$
\begin{align*}
\sqrt{1 + |Du|^2} \text{div} \frac{Du}{\sqrt{1 + |Du|^2}} &= f^{-1} \quad \text{in } \Omega, \\
u &= \varphi \quad \text{on } \partial \Omega,
\end{align*}
$$

where $f \in C^{1,\alpha}(\mathring{\Omega})$ and $0 < d \leq f$. As a first step we establish a priori bounds for $\sup_{\Omega} u$ and $\inf_{\Omega} u$.

**Lemma.** Let $u_f \in C^{2,\alpha}(\mathring{\Omega})$ be a solution to the Dirichlet problem (6). If

$$f \geq d \geq \left(1 + \sqrt{2^{n+1}}\right) c(n)|\Omega|^{1/n},$$

and

$$k_0 = \inf_{\partial \Omega} \varphi \geq \left(1 + \sqrt{2^{n+1}}\right)^2 c(n)|\Omega|^{1/n},$$

then we have $h \geq u_f \geq d$.

**Proof of the Lemma.** The first inequality follows immediately from the maximum principle since $f$ is positive. To prove the second relation we chose $\delta \geq -k_0$ and put $w = \min(u + \delta, 0)$, $A(\delta) = \{x \in \Omega : u < -\delta\}$. Multiplying (6) with $w$, integrating by parts and using $w|_{\partial \Omega} = 0$, we obtain

$$\int_{\Omega} \frac{|Dw|^2}{\sqrt{1 + |Dw|^2}} \leq \int_{A(\delta)} f \sqrt{1 + |Du|^2},$$

hence

$$\int_{\Omega} |Dw| \leq |A(\delta)| + d^{-1} \int_{A(\delta)} |w|.$$

We use Sobolev's inequality on the left and Hölder's inequality on the right hand side and get with $c(n) = n^{-1} \omega_n^{1/n}$

$$|w|_{n/n-1} \cdot \{c^{-1}(n) - d^{-1}|\Omega|^{1/n}\} \leq |A(\delta)|,$$
where $|w|_{n/n-1}$ stands for the $L_{n/n-1}$-norm of $w$. Another application of Hölder's inequality yields
\[
(\delta_1 - \delta_2)|A(\delta_1)| \leq \left\{ \frac{c(n)d}{d - c(n)|\Omega|^{1/n}} \right\} |A(\delta_2)|^{1+1/n}
\]
for all $\delta_1 \geq \delta_2 \geq -k_0$. In view of a well-known lemma due to Stampacchia, [St, Lemma 4.1], this is easily seen to imply
\[
|A(-k_0 + 2^{n+1} \cdot c_1 |A(-k_0)|^{1/n})| = 0,
\]
where
\[
c_1 = \frac{c(n)d}{d - c(n)|\Omega|^{1/n}}.
\]
Clearly this means that
\[
u \geq k_0 - \frac{2^{n+1}dc(n)|\Omega|^{1/n}}{d - c(n)|\Omega|^{1/n}}.
\]
Since $k_0 \geq (1 + \sqrt{2^{n+1}})d$ and $d \geq (1 + \sqrt{2^{n+1}})c(n)|\Omega|^{1/n}$ we finally obtain $u \geq d$. \hfill \Box

To derive a gradient estimate at the boundary, we rewrite (6) into
\[
(1 + |Du|^2)\Delta u - D_i u D_j u D_i D_j u = f^{-1}(1 + |Du|^2).
\]
We can then apply the results of Serrin [Se1], see also [GT, Chapter 14.3]. Equation (7) satisfies the structure condition (14.41) in [GT] and the RHS is $\mathcal{O}(|Du|^2)$. So we obtain a gradient estimate on the boundary which is independent of $|D\phi|$
\[
\sup_{\partial \Omega} |Du| \leq c_2 = c_2(n, \Omega, h, |\phi|_{2, \Omega}),
\]
provided only that $\partial \Omega$ has non-negative (inward) mean curvature.

It is not possible to derive interior gradient estimates independent of $|D\phi|$, but we can prove
\[
\sup_{\Omega} |Du| \leq \max \left\{ 2, \frac{1}{4} \sup_{\Omega} |D\phi|, 2e^{4(hd^{-1}-1)} \sup_{\partial \Omega} |Du| \right\},
\]
which will be sufficient for our fixed point argument. Estimate (8) can be obtained from a careful analysis of the structure conditions in [GT, Chapter 15]. Here we present a self contained proof, using the geometric nature of equation (6). For a similar procedure we refer to [K].
In the following let \( v = (1 + |Dv|)^{1/2} \) and denote by \( H \) and \( \Delta \) the mean curvature and the Laplace-Beltrami operator on \( M = \text{graph} u \) respectively. Then equation (6) takes the form

(9) \[ v^2 \Delta u = f^{-1} \Leftrightarrow H = f^{-1}v^{-1}. \]

Let \( \tau_1, \tau_2, \ldots, \tau_n, \nu \) be an adapted local orthonormal frame on \( M \), such that \( \nu \) is the upper unit normal and

\[
\nabla_i \nu = -h_{il} \tau_l, \quad \nabla_i \tau_j = h_{ij} \nu,
\]

where \( \nabla_i \) is the tangential derivative with respect to \( \tau_i \) and \( h_{il} \) is the second fundamental form. Then we get for \( v = (1 + |Du|)^{1/2} = \langle \nu, e_{n+1} \rangle^{-1} \) the Jacobi-Codazzi equation

\[
\Delta v = \nabla_i v \nabla_i \nu_{n+1}^{-1} = \nabla_i (v^2 \langle h_{il} \tau_l, e_{n+1} \rangle)
= |A|^2 v + 2v^{-1} |\nabla v|^2 + v^2 \langle \nabla H, e_{n+1} \rangle,
\]

where \( |A|^2 = h_{il} h_{il} \). Now (9) implies

(10) \[ \Delta v = |A|^2 v + 2v^{-1} |\nabla v|^2 - f^{-2} v \langle \nabla f, e_{n+1} \rangle - f^{-1} \langle \nabla v, e_{n+1} \rangle. \]

If we now extend all functions from \( M \) to \( \mathbb{R}^{n+1} \) by

\[
f(\hat{x}, x_{n+1}) = f(\hat{x})
\]

such that

(11) \[
\nabla f = Df - \nu \langle Df, \nu \rangle, \quad D_{n+1} f = 0 \quad \text{and} \quad \langle \nabla f, e_{n+1} \rangle = -v^{-1} \langle Df, \nu \rangle
\]

then we derive from (10) and (11)

(12) \[ \Delta v \geq 2v^{-1} |\nabla v|^2 - f^{-1} \langle \nabla v, e_{n+1} \rangle - f^{-2} |Df|. \]

Next we compute for \( \alpha > 0 \) and \( g = e^{\alpha u} \cdot v \) the inequality

\[
\Delta g \geq e^{\alpha u} \{ 2v^{-1} |\nabla v|^2 - f^{-1} \langle \nabla v, e_{n+1} \rangle - f^{-2} |Df| \\
+ 2\alpha \nabla_i v \nabla_i u + \alpha v \Delta u + v \alpha^2 |\nabla u|^2 \}.
\]

Using again the equation (9) and

\[
\nabla i g = \nabla_i v e^{\alpha u} + \alpha v e^{\alpha u} \nabla_i u
\]
we obtain

\[
\Delta g \geq 2v^{-1} \nabla_i v \nabla_i g - f^{-1} \langle \nabla g, e_{n+1} \rangle + \alpha f^{-1} e^{\alpha u} v \langle \nabla u, e_{n+1} \rangle \\
- f^{-2} |Df| e^{\alpha u} + \{ v^{-1} \alpha f^{-1} + v \alpha^2 |\nabla u|^2 \} e^{\alpha u}.
\]
In view of relation (11) we finally conclude
\[ \Delta g \geq 2v^{-1}(\nabla v \cdot \nabla g - f^{-1}(\nabla g, e_{n+1}) + \{\alpha^2|\nabla u|^2 - \alpha f^{-1} - v^{-1}f^{-2}|Df|^2\})g \]
Now let again \( d < f < h \) and choose \( a = 4d^{-1} \). Then, since
\[ |\nabla u|^2 = \frac{|Du|^2}{1 + |Du|^2} \geq \frac{1}{2} \quad \text{for } |Du| \geq 1, \]
we see that \( g \) cannot have an interior maximum if
\[ v \geq \max \left\{ 2, \frac{1}{4} \sup_{\Omega} |Df| \right\}. \]
Therefore we get the estimate
\[ \sup_{\Omega} v \leq \max \left\{ 2, \frac{1}{4} \sup_{\Omega} |Df|, e^{4(hd^{-1} - 1)} \sup_{\partial \Omega} v \right\} \]
yielding (8).

To prove existence of a solution to equation (1) we now define the set
\[ \mathcal{M} := \left\{ f \in C^{1,\alpha}(\Omega) : d < f < h, \sup_{\Omega} |Df| \leq M \right\} \]
for \( M > 0 \) large and consider the operator
\[ T : \mathcal{M} \rightarrow C^{1,\alpha}(\Omega), \]
\[ f \rightarrow u_f. \]
In view of our estimates for \( u_f \) and \( |Du_f| \) we may then choose \( M \) so large that
\[ T(\mathcal{M}) \subset \mathcal{M}. \]
Moreover, standard theory ensures that \( T \) is continuous and \( T(\mathcal{M}) \) is precompact. So we can apply Schauder's fixed point theorem, see e.g. ([GT], Cor. 11.2) to obtain the existence of a regular \( u \in C^{2,\alpha}(\Omega) \) satisfying (1).

To prove the necessary conditions (5) we proceed similarly as in [G]. To this end let \( A \Subset \Omega \) have finite perimeter \( M(\partial A) \). There exists a sequence of positive functions \( \varphi_k \in C^1(\Omega) \) such that \( \varphi_k \rightarrow \varphi_A \) in \( L^1(\Omega) \), and
\[ \int_{\Omega} |D\varphi_k| \rightarrow M(\partial A), \]
where \( \varphi_A \) denotes the characteristic function of the set \( A \).

We test (1) with \( \varphi_k \) and integrate,
\begin{equation}
\int_{\Omega} \left\{ \frac{uDuD\varphi_k}{\sqrt{1 + |Du|^2}} + \varphi_k \sqrt{1 + |Du|^2} \right\} \, dx = 0.
\end{equation}
Now, since \( u \in \text{Lip}(\Omega) \) it follows from standard regularity theory that \( u \in C^\infty(\Omega) \) and therefore
\[
\text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) \geq 0 \quad \text{on } \Omega, \text{ whence } u \leq h.
\]

Using this in (13) we get
\[
\int_\Omega \phi_k \, dx \leq \frac{h \cdot L}{\sqrt{1 + L^2}} \int_\Omega |D\phi_k|
\]
and, letting \( k \to \infty \),
\[
|A| \leq \frac{h \cdot L}{\sqrt{1 + L^2}} M(\partial A), \quad \text{or}
\]
\[
h \geq \{1 + L^{-2}\}^{1/2} \frac{|A|}{M(\partial A)}.
\]

The general case follows by an approximation argument, using the fact that
\[
M(\partial [A \cap \Omega_\varepsilon]) \to M(\partial A) \quad \text{as } \varepsilon \to 0,
\]
where
\[
\Omega_\varepsilon := \{x \in \Omega: \text{dist}(x, \partial \Omega) > \varepsilon\}.
\]

This completes the proof of the theorem.

**Remark.** With the same method we could as well deal with the integral
\[
\int_\Omega u^\gamma \sqrt{1 + |Du|^2}, \quad \gamma > 0,
\]
the Euler equation of which is given by
\[
\text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = \frac{\gamma}{u\sqrt{1 + |Du|^2}}.
\]

Clearly, in this case the constants appearing in the theorem would depend on \( \gamma \) too, however we shall not dwell on this.

**References**


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