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**FANO BUNDLES OVER  $P^3$  AND  $Q_3$**

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## FANO BUNDLES OVER $P^3$ AND $Q_3$

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A vector bundle  $\mathcal{E}$  is called Fano if its projectivization  $P(\mathcal{E})$  is a Fano manifold. In this article we prove that Fano bundles exist only on Fano manifolds and discuss rank-2 Fano bundles over the projective space  $P^3$  and a 3-dimensional smooth quadric  $Q_3$ .

Fano bundles appear naturally as we strive to construct examples of Fano manifolds of dimension  $\geq 3$ ; they form interesting yet accessible class of Fano  $n$ -folds. For example: among 87 types of Fano 3-folds with  $b_2 \geq 2$  listed in [13] 22 types are ruled (i.e. obtained by projectivization of Fano bundles). Moreover some of the non-ruled manifolds listed there can be easily expressed as either finite covers of ruled 3-folds or divisors (or, more generally, complete intersections) in ruled Fano manifolds of higher dimension.

Let us mention another aspect of dealing with Fano bundles: it is how to determine whether or not a vector bundle is ample. This very fine property of a vector bundle cannot be determined by its numerical invariants, see [7]. Assuming the bundle to be stable helps to establish a sufficient condition for ampleness: [10], [17], which however is far from being necessary. In the present paper we take advantage of some already known facts about stable bundles with small Chern classes and determine that a bundle  $\mathcal{E}$  is not ample by finding its jumping lines or sections of  $\mathcal{E}(-k)$ .

Let us note that some results of this paper have already been published, see remarks after the proofs of Theorems (1.6) and (2.1).

**1. Fano bundles; preliminaries.** Let  $\mathcal{E}$  be a vector bundle of rank  $r \geq 2$  on a smooth complex projective variety  $M$ . Let us recall that the tautological line bundle  $\xi = \xi_{\mathcal{E}}$  on  $V = P(\mathcal{E})$  is uniquely determined by the conditions  $\xi_{\mathcal{E}}|_F \approx \mathcal{O}_F(1)$  and  $p_*\xi_{\mathcal{E}} = \mathcal{E}$ . By  $p$  we have denoted the projection morphism of  $V = P(\mathcal{E})$  onto  $M$  and by  $F$ —the fibre of  $p$ . Obviously,  $F \cong P^{r-1}$  and  $p: V \rightarrow M$  is a  $P^{r-1}$ -bundle. The Picard group of  $V$  can be expressed as a direct sum:  $\text{Pic}V \cong \mathbb{Z} \cdot \xi_{\mathcal{E}} \oplus p^*(\text{Pic}M)$ . Replacing  $\mathcal{E}$  by its twist with a line bundle  $\mathcal{L}$  on  $M$  does not affect

the projectivization and

$$\xi_{\mathcal{E} \otimes \mathcal{L}} = \xi_{\mathcal{E}} \otimes p^*(\mathcal{L}).$$

Moreover,  $\mathcal{E}$  is generated by global sections iff  $\xi_{\mathcal{E}}$  is. We have the following relative Euler sequence on  $V = P(\mathcal{E})$ :

$$(1.1) \quad 0 \rightarrow \mathcal{O}_V \rightarrow p^*(\mathcal{E})^\vee \otimes \xi_{\mathcal{E}} \rightarrow T_{V|M} \rightarrow 0$$

where the latter bundle is the relative tangent bundle of  $p$  and fits in the exact sequence

$$(1.2) \quad 0 \rightarrow T_{V|M} \rightarrow TV \rightarrow p^*TM \rightarrow 0.$$

We then obtain

$$(1.3) \quad c_1V = p^*(c_1M - c_1\mathcal{E}) + r\xi_{\mathcal{E}}$$

The theorem of Leray and Hirsch yields that in the cohomology ring of  $V$  the following holds

$$(1.4) \quad \xi_{\mathcal{E}}^r - p^*(c_1\mathcal{E})\xi_{\mathcal{E}}^{r-1} + p^*(c_2\mathcal{E})\xi_{\mathcal{E}}^{r-2} - \dots \pm p^*(c_r\mathcal{E}) = 0.$$

From now on we assume in this section that  $\mathcal{E}$  is a rank- $r$  Fano bundle on an  $n$ -fold  $M$ , i.e., that  $P(\mathcal{E})$  is a Fano manifold. We prove that such  $M$  must be Fano, as well.

(1.5) LEMMA. *Let  $C \subset M$  be a rational curve with a normalization  $\nu: P^1 \rightarrow C$ . Assume that  $\nu^*(\mathcal{E}) \cong \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \dots \oplus \mathcal{O}(a_r)$ , where  $a_1 \leq a_2 \leq \dots \leq a_r$ . Then*

$$(c_1M) \cdot C > \sum_{i=2}^r (a_i - a_1) \geq 0.$$

*Proof.* The right hand side inequality is obvious. To prove the left hand side inequality let us assume that  $W = P(\nu^*\mathcal{E})$ . The manifold  $W$  is then a  $P^{r-1}$ -bundle over  $P^1$ , with a projection  $\pi: W \rightarrow P^1$ . We have a section  $C_0$  of  $\pi$  associated to the epimorphism  $\nu^*\mathcal{E} \rightarrow \mathcal{O}(a_1) \rightarrow 0$ , such that

$$\xi_{\nu^*\mathcal{E}}|_{C_0} \cong \mathcal{O}_{P^1}(a_1).$$

The normalization map  $\nu: P^1 \rightarrow M$  lifts to a map  $\bar{\nu}: W \rightarrow V$ , making the following diagram commute

$$\begin{array}{ccc} W & \xrightarrow{\bar{\nu}} & V \\ \pi \downarrow & & \downarrow p \\ P^1 & \xrightarrow{\nu} & M \end{array}$$

By the choice of  $C_0$  we have

$$\bar{\nu}^*(\xi_{\mathcal{E}}) \cdot C_0 = a_1$$

and, since  $c_1V$  is ample, we obtain by (1.3)

$$\begin{aligned} 0 < c_1V \cdot \bar{\nu}(C_0) &= \bar{\nu}^*(c_1V) \cdot C_0 \\ &= r \cdot \bar{\nu}^*(\xi_{\mathcal{E}}) \cdot C_0 + (\pi \circ \nu)^*(c_1M) \cdot C_0 - (\pi \circ \nu)^*(c_1\mathcal{E}) \cdot C_0 \\ &= r \cdot a_1 + c_1M \cdot C - \sum_{i=1}^r a_i \end{aligned}$$

which yields the desired inequality.

(1.6) **THEOREM.** *If  $\mathcal{E}$  is a Fano bundle on a manifold  $M$  then  $M$  is a Fano  $n$ -fold.*

*Proof.* As  $c_1V = -K_V$  is ample, the cone of curves on  $V$  is spanned by the classes of extremal curves (see [12] for definitions and Theorem 1.2 on the cone of curves of a Fano manifold). Let us denote these curves by  $l_0, l_1, \dots, l_v$  with  $l_0$  contained in  $F$ , a fibre of the projection  $p: P(\mathcal{E}) \rightarrow M$ . We see that  $p^*(c_1M) \cdot l_0 = 0$  and for  $i > 0$ ,  $p(l_i)$  is a rational curve on  $M$ . Therefore from (1.5) it follows that

$$(1.7) \quad 0 < c_1M \cdot p(l_i) = p^*(c_1M) \cdot l_i$$

which means that  $p^*(c_1M)$  is numerically effective. Recall now (a conclusion from) the Kawamata-Shokurov contraction theorem, see (2.6) in [11]:

If  $D$  is nef and  $aD - K$  is ample for some  $a > 0$ , then  $D$  is semiample, i.e., some power of  $D$  is generated by global sections.

It follows that  $D := p^*(c_1M)$  is semiample. Since  $p: V \rightarrow M$  is a  $P^{r-1}$ -bundle, we have, for any integer  $k$ ,  $p_*p^*(\mathcal{O}(kc_1M)) \cong \mathcal{O}(kc_1M)$  and the images (under  $p_*$ ) of global sections of  $p^*(\mathcal{O}(kc_1M))$  are global sections of  $\mathcal{O}(kc_1M)$ . Therefore  $c_1M$  is semiample, hence to prove that it is ample it is enough to show that  $c_1M \cdot C > 0$  for any curve  $C$  in  $M$ .

Let  $C$  be an irreducible curve in  $M$ . Taking an appropriate component from an intersection of the inverse image  $p^{-1}(C)$  with general  $r - 1$  divisors from a very ample linear system, we can produce an irreducible curve  $C_1 \subset V$ , such that  $p(C_1) = C$ . Then  $C_1$  is numerically equivalent to a linear combination  $\sum a_i l_i$  with at least one  $a_i$  different from zero for  $i > 0$ . Let  $d$  be the degree of the map  $p|_{C_1}: C_1 \rightarrow C$ .

Now the inequality (1.7) gives

$$c_1 M \cdot C = \frac{1}{d} \cdot p^*(c_1 M) \cdot C_1 = \frac{1}{d} \cdot \left( \sum a_i \cdot p^*(c_1 M) \cdot l_i \right) > 0,$$

which concludes the proof of the theorem.

**REMARK.** Theorem (1.6) has already been known for bundles of rank 2 on surfaces [4] and 3-folds [1].

**2. Rank-2 Fano bundles on  $P^3$ .** The results stated below (Theorem (2.1)) can be understood as one more example of an exceptional character of the null-correlation bundle (see e.g. [3] or [15] for the definition of the null-correlation bundle).

**(2.1) THEOREM.** *The only rank-2 Fano bundles with  $c_1 = 0, -1$ , on  $\mathbb{P}^3$  are*

- (1)  $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}$ ,
- (2)  $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(-1)$ ,
- (3)  $\mathcal{E} = \mathcal{O}(-1) \oplus \mathcal{O}(+1)$ ,
- (4)  $\mathcal{E} = \mathcal{O}(-2) \oplus \mathcal{O}(+1)$ ,
- (5) *the null-correlation bundle  $\mathcal{N}$ .*

*Proof.* Let  $V = P(\mathcal{E})$ . We then have

$$(2.2) \quad \begin{aligned} -K_V &= 2\xi + (4 - c_1 \mathcal{E})H \\ &= \begin{cases} 2\xi + 4H = 2\xi_{\mathcal{E}(1)} + 2H = 2\xi_{\mathcal{E}(2)} & \text{if } c_1 = 0, \\ 2\xi + 5H = 2\xi_{\mathcal{E}(2)} + H = 2\xi_{\mathcal{E}(3)} - H & \text{if } c_1 = -1, \end{cases} \end{aligned}$$

and we see that any of the bundles listed above is Fano. Indeed, if  $\mathcal{E}$  is one of those listed as (1), (3) or (5) (respectively: (2) or (4)) then  $c_1 \mathcal{E} = 0$  (resp.  $c_1 \mathcal{E} = -1$ ) and  $\mathcal{E}(1)$  (resp.  $\mathcal{E}(2)$ ) is generated by its global sections. Now, since  $\rho(V) = 2$ , it follows from (2.2) that  $c_1 V$  is ample as the sum of two non-proportional nef divisors.

An easy corollary follows.

**(2.3)** For a normalized Fano bundle  $\mathcal{E}$  of rank 2 on  $P^3$ :

if  $c_1 \mathcal{E} = 0$ , then  $\mathcal{E}(2)$  is ample,

if  $c_1 \mathcal{E} = -1$ , then  $\mathcal{E}(3)$  is ample.

We shall discuss the two cases separately.

*Case  $c_1 = 0$ .* A straightforward consequence of the theorem of Leray and Hirsch (1.4) yields that in the cohomology ring of  $V$  the following holds

$$\xi^2 + c_2 H^2 = 0.$$

Since  $H^4 = 0$  and  $H^3\xi = 1$ , the above formula then gives

$$(2.4) \quad H^2\xi^2 = 0, \quad H\xi^3 = -c_2, \quad \xi^4 = 0,$$

so that  $(-K_V)^4 = (2\xi + 4H)^4 = 128(4 - c_2)$  and we see that  $c_2 < 4$ .

Assume first  $\mathcal{E}$  is not semistable, i.e.,  $H^0(\mathcal{E}(-1)) \neq 0$ . Let  $s$  be a non-zero section of  $\mathcal{E}(-1)$ . We claim that  $s$  does not vanish anywhere. Indeed, if  $Z = \{s = 0\}$  were not empty, then for a line  $L$  meeting  $Z$  in a finite number of points we would have

$$\mathcal{E}(-1)|_L = \mathcal{O}(d) \oplus \mathcal{O}(e) \quad \text{with } d \geq 1, d + e = -2,$$

contradicting (2.3). Therefore  $s$  does not vanish and thus  $\mathcal{E}(-1) = \mathcal{O} \oplus \mathcal{O}(-2)$ , hence  $\mathcal{E}$  is as in (3) of the theorem.

Let now  $\mathcal{E}$  be semistable but not stable:  $H^0(\mathcal{E}(-1)) = 0, H^0(\mathcal{E}) \neq 0$ . If a non-zero section of  $\mathcal{E}$  does not vanish anywhere,  $\mathcal{E}$  must then be  $\mathcal{O} \oplus \mathcal{O}$ . Otherwise a section vanishes on a curve. If the curve is not a single line then cutting it by a line leads to a contradiction, as above. But if a single line  $L$  was a zero set of a section of  $\mathcal{E}$  then, by the adjunction formula, the degree of the canonical divisor of  $L$  would be

$$\text{deg}(K_L) = (K_{P^3} + c_1\mathcal{E}) \cdot L = -4,$$

which is impossible. Because of Bogomolov's inequality  $c_1^2 < 4c_2$  for stable bundles, [15], it remains then to study stable bundles with  $c_1 = 0$  and  $c_2 = 1, 2, 3$ . In the first case  $\mathcal{E}$  is the null-correlation bundle  $\mathcal{N}$ , for which  $\mathcal{N}(2)$  is ample;  $\mathcal{N}$  is then Fano.

In the remaining cases we know that  $\mathcal{E}$  has multiple jumping lines, i.e. such lines  $L$  for which  $\mathcal{E}|_L = \mathcal{O}_L(-2) \oplus \mathcal{O}_L(2)$ , see [8], Proposition 9.11, and [18], respectively. In virtue of (2.3), such bundles cannot be Fano.

Case  $c_1 = -1$ . The multiplication table is now:

$$(2.5) \quad H^4 = 0, \quad H^3\xi = 1, \quad H^2\xi^2 = -1, \\ H\xi^3 = -c_2 + 1, \quad \xi^4 = 2c_2 - 1$$

and from

$$(-K_V)^4 = (2\xi + 5H)^4 = 32(-4c_2 + 17) > 0$$

we obtain that the only possible non-negative values for  $c_2$  are 0, 2 or 4 (recall that Schwarzenberger's condition says  $c_1c_2 \equiv 0 \pmod{2}$ ). Assume  $H^0(\mathcal{E}(-1)) \neq 0$ . As above, we show that no section  $s \neq 0$

vanishes: if  $Z = \{\text{zero}(s)\}$  were not empty, for a line  $L$  meeting  $Z$  at finitely many points we would have

$$\mathcal{E}(-1)|L = \mathcal{O}_L(d) \oplus \mathcal{O}_L(e) \quad \text{with } d \geq 1, d + e = -3,$$

contradicting (2.3). Therefore the sections  $\mathcal{E}(-1)$  do not vanish anywhere, so that  $\mathcal{E}$  is as in (4) of Theorem (2.1).

Let then  $H^0(\mathcal{E}(-1)) = 0$ ,  $H^0(\mathcal{E}) \neq 0$ . The zero set  $Z$  of a non-zero section is then a curve (if not empty). Again, if  $Z$  were anything different from a single line, for a line  $L$  that cuts  $Z$  at a finite number  $\geq 2$  of points we would have

$$\mathcal{E}|L = \mathcal{O}_L(d) \oplus \mathcal{O}_L(e), \quad d \geq 2, d + e = -1,$$

contradicting the ampleness of  $\mathcal{E}(3)$ . But  $c_1c_2$  is even so that the case  $c_1 = -1$ ,  $c_2 = 1$  does not hold, hence  $Z$  is not a line. The non-zero sections of  $\mathcal{E}$  do not vanish, hence  $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(1)$ .

It remains to exclude the cases of stable vector bundles with  $c_1 = -1$  and  $c_2 = 2$  or  $4$ . In the former case  $\mathcal{E}$  has multiple jumping lines, [9], Proposition 4.1, i.e., those for which  $\mathcal{E}|L = \mathcal{O}_L(-3) \oplus \mathcal{O}_L(2)$ , hence  $\mathcal{E}$  cannot be Fano in view of (2.3). In the latter one  $\mathcal{E}(2)$  has a section, see [2], Lemma 1, and  $2H + \xi$  is effective with

$$(2H + \xi)(c_1P(\xi))^3 = (2H + \xi)(2\xi + 5H)^3 = -17.$$

These bundles are then not Fano.

**REMARK.** Theorem (2.1) (in a somewhat weaker form) was first announced by Artiushkin, [1]. His proof was, however, incorrect: in line 36 on page 14 if  $E$  is a normalized bundle on  $P^3$ , then the tautological divisor  $\xi_E = L$  in op. cit. need not to be effective, therefore  $(-K)^3 \cdot L$  need not to be positive. Our actual proof is more complicated.

Let us conclude this section by proving that  $P(\mathcal{N})$  has a  $P^1$ -bundle structure over a 3-dimensional quadric  $Q_3$ . To see this, first let us recall that  $\mathcal{N}(1)$  can be defined as the bundle fitting in the following exact sequence on  $P^3$

$$0 \rightarrow \mathcal{O} \rightarrow \Omega P^3(2) \rightarrow \mathcal{N}(1) \rightarrow 0.$$

Note that  $P(\Omega P^3(2))$  is the incidence variety

$$I = \{(x, l) \in P^3 \times \text{Grass}(1, 3) : x \in l\}$$

and  $\text{Grass}(1, 3)$  is isomorphic to a 4-dimensional quadratic. Now, from the above exact sequence it follows that  $P(\mathcal{N}(1))$  is a divisor in  $I$  which is an inverse image of a hyperplane section of  $\text{Grass}(1, 3)$ .

Therefore:

(2.6) **PROPOSITION.** *The Fano 4-fold  $P(\mathcal{N}(1))$  is a projectivization of a rank-2 vector bundle on smooth quadratic  $Q_3 \subset \text{Grass}(1, 3)$ , obtained by restricting to  $Q_3$  the universal quotient bundle from  $\text{Grass}(1, 3)$ .*

**3. Bundles over  $Q_3$ .** Let us recall that the cohomology ring of  $Q_3$  is generated by the classes of  $[H] \in H^2(Q_3, \mathbb{Z})$ ,  $[L] \in H^4(Q_3, \mathbb{Z})$ , and  $[P] \in H^6(Q_3, \mathbb{Z})$  where  $H, L$  and  $P$  are a quadratic surface, a line and a point, respectively. There are the following relationships:  $[H]^2 = 2L$ ,  $[H][L] = [P]$  and hence  $[H]^3 = 2[P]$ . If  $\mathcal{F}$  is a coherent sheaf on  $Q_3$  with the Chern polynomial

$$1 + c_1(\mathcal{F})[H]t + c_2(\mathcal{F})[L]t^2 + c_3(\mathcal{F})[P]t^3,$$

then the numbers  $c_i$  are called the Chern classes of  $\mathcal{F}$ .

Recall the Riemann-Roch formula for  $\mathcal{F}$ , [5]

$$\chi(\mathcal{F}) = \frac{1}{6}(2c_1^3 - 3c_1c_2 + 3c_3) + \frac{3}{2}(c_1^2 - c_2) + \frac{13}{6}c_1 + \text{rank } \mathcal{F}.$$

Let now  $\mathcal{E}$  be a rank-2 vector bundle on  $Q_3$ . The theorem of Leray and Hirsch (1.4) gives the following relations between the generators of  $\text{Pic}(P(\mathcal{E})) \cong \mathbb{Z} \oplus \mathbb{Z}$

$$\begin{cases} \text{if } c_1 = 0, \text{ then } \xi^2 + \frac{1}{2}c_2(\mathcal{E})H^2 = 0; \\ \text{if } c_1 = -1, \text{ then } \xi^2 + \xi H + \frac{1}{2}c_2(\mathcal{E})H^2 = 0. \end{cases}$$

Because  $H^4 = 0$  and  $H^3\xi = 2$ , we obtain:

$$\text{if } c_1 = 0, \text{ then } H^2\xi^2 = 0, \quad H\xi^3 = -c_2, \quad \xi^4 = 0;$$

$$\text{if } c_1 = -1, \text{ then } H^2\xi^2 = -2, \quad H\xi^3 = 2 - c_2, \quad \xi^4 = 2c_2 - 2.$$

Let  $\mathcal{E}$  be a normalized rank-2 vector bundle on  $Q_3$  and  $V = P(\mathcal{E})$  its projectivization. We then have

$$(3.1) \quad c_1V = -K_V = \begin{cases} 2\xi + 3H & \text{when } c_1 = 0, \\ 2\xi + 4H & \text{for } c_1 = -1. \end{cases}$$

*Case of non-stable bundles.* Assume  $\mathcal{E}$  is non-stable with  $c_1(\mathcal{E}) = -1$ . If a non-zero section from  $H^0(\mathcal{E}(-1))$  vanishes at some point, let us consider a line  $L$  passing through this point and not contained in the zero set entirely. Then  $\mathcal{E}(-1)|L = \mathcal{O}(d) \oplus \mathcal{O}(e)$  with  $d \geq 1$ ,  $d + e = -3$  that contradicts the ampleness of  $\mathcal{E}(2)$ , (3.1).

Assume  $H^0(\mathcal{E}(-1)) = 0$ ,  $H^0(\mathcal{E}) \neq 0$ . Then a non-zero section of  $\mathcal{E}$  either does not vanish anywhere or it vanishes on a set of pure dimension 1. The divisor  $\xi_{\mathcal{E}}$  is effective on  $P(\mathcal{E})$  and

$$\xi \cdot (-K_V)^3 = 8\xi(\xi + 2H)^3 = 16(-2c_2 + 1),$$

and we see that  $c_2 \leq 0$ . But then sections of  $\mathcal{E}$  do not have zeros, hence  $\mathcal{E} = \mathcal{O}(-1) \oplus \mathcal{O}$ . Finally, we easily check that  $\mathcal{O}(-1) \oplus \mathcal{O}$  is a Fano bundle (because  $\mathcal{O}(1) \oplus \mathcal{O}(2)$  is ample).

In case  $c_1 = 0$  we exclude non-semistable bundles in a very similar way. Finally, if  $\mathcal{E}$  is semistable but not stable, that is  $H^0(\mathcal{E}) \neq 0 = H^0(\mathcal{E}(-1))$ , the divisor  $\xi_{\mathcal{E}}$  is effective and

$$0 < \xi(2\xi + 3H)^3 = 18(-2c_2 + 3)$$

so that  $c_2 \leq 0$  (recall that  $c_2 \equiv 0 \pmod 2$ , see [5], §1). If so, a non-zero section of  $\mathcal{E}$  does not vanish anywhere and  $\mathcal{E}$  must then be  $\mathcal{O} \oplus \mathcal{O}$ .

*Case of stable bundles with  $c_1 = 0$ .* From the condition  $K^4 > 0$  we easily obtain that if  $V = P(\mathcal{E})$  is Fano, then  $c_2 \leq 4$ , and since  $c_2 \equiv 0 \pmod 2$  it follows that either  $c_2 = 2$  or  $4$ . We believe that there is no Fano bundle on  $Q_3$  with  $c_1 = 0$ ,  $c_2 = 4$ , however we do not have enough information on these bundles to prove it.

In case of  $c_2 = 2$ , one can easily check that the pull-back  $\pi^*(\mathcal{N})$  of the null-correlation bundle, under a double covering  $\pi: Q_3 \rightarrow P^3$ , is Fano. Indeed,  $\pi^*(\mathcal{N})(1)$  is then spanned on  $Q_3$ , therefore  $-K_{P(\pi^*(\mathcal{N}))} = 2\xi_{\pi^*(\mathcal{N})(1)} + H$  is ample. On the other hand we have

**(3.2) PROPOSITION.** *If  $\mathcal{E}$  is a stable bundle on  $Q_3$  with  $c_1 = 0$ ,  $c_2 = 2$  such that  $\mathcal{E}(1)$  is spanned by global sections then  $\mathcal{E}$  is a pull-back  $\pi^*(\mathcal{N})$  of a null-correlation bundle  $\mathcal{N}$ , under a double covering  $\pi: Q_3 \rightarrow P^3$ .*

*Proof.* The argument is based on the following fact: for any two disjoint lines on  $P^3$  there exists a section of a twisted null-correlation bundle  $\mathcal{N}(1)$  vanishing exactly on these lines. Therefore, if we prove that a section of  $\mathcal{E}(1)$  vanishes on a set being a pullback, via a double covering  $\pi: Q_3 \rightarrow P^3$ , of two disjoint lines on  $P^3$ , then in view of Theorem 1.1 and Remark 1.1.1 from [8],  $\mathcal{E}(1)$  is a pullback of  $\mathcal{N}(1)$ ; if  $Z$  is the union of two disjoint lines and  $Y$  its pullback then it is easy to check that every isomorphism between  $\omega_Q(-2)|_Y$  and  $\omega_Y$  comes from  $\omega_P(-2)|_Z \simeq \omega_Z$ .

Assume  $\mathcal{E}$  is stable with  $c_1(\mathcal{E}) = 0$ ,  $c_2(\mathcal{E}) = 2$  on  $Q_3$ . We easily compute the following cohomology table of  $h^i(\mathcal{E}(-m))$

0	0	0	0	↑ $h^3$
1	1	0	0	↑ $h^2$
0	0	1	1	↑ $h^1$
0	0	0	0	↑ $h^0$
$m = 3$ $m = 2$ $m = 1$ $m = 0$				→

Indeed, vanishing of the lower and upper row is a consequence of the stability (plus Serre’s duality) and the “spectrum” technique, namely Corollary 2.4 in [5], gives

$$h^1(\mathcal{E}(-2)) = h^1(\mathcal{E}(-3)) = h^2(\mathcal{E}) = h^2(\mathcal{E}(-1)) = 0$$

and the remaining part of the table follows from computing the Euler-Poincaré characteristic.

Since  $\chi(\mathcal{E}(1)) = 5$  and  $h^2(\mathcal{E}(1)) = h^1(\mathcal{E}(-4)) = 0$  by Corollary 2.4 in [5], we see  $h^0(\mathcal{E}(1)) \geq 5$ . Let  $Y$  be the zero of a generic section.

Since  $H^0(\mathcal{E}) = 0$  and  $\mathcal{E}(1)$  is assumed to be globally generated,  $Y$  is a smooth (not necessarily connected) curve. From the diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 0 & \rightarrow & \mathcal{O}(-2) & \rightarrow & \mathcal{E}(-1) & \rightarrow & J_Y \rightarrow 0 \\
 & & & & \downarrow & & \\
 & & & & \mathcal{O}_{Q_3} & & \\
 & & & & \downarrow & & \\
 & & & & \mathcal{O}_Y & & \\
 & & & & \downarrow & & \\
 & & & & 0 & & 
 \end{array}$$

we calculate, with the aid of the cohomology table above, that  $h^0(\mathcal{O}_Y) = 2$ , i.e.,  $Y$  consists of two connected components, say  $Y_1$  and  $Y_2$ .

*Claim.*  $Y_1$  and  $Y_2$  are conics.

*Proof of claim.* Since  $c_2\mathcal{E}(1) = 4$  and both  $Y_i$  are smooth (therefore reduced) we have only to exclude the possibility that one of them is a line  $L$ . But then by the adjunction formula we would obtain

$$\deg(K_L) = (K_{Q_3} + c_1\mathcal{E}(1)) \cdot L = -1$$

which is impossible.

Let now  $H_i$  be the plane containing  $Y_i$ ,  $i = 1, 2$ ; clearly  $Q_3 \cap H_i = Y_i$  and  $H_1, H_2$  meet at one point in  $P^4$  off  $Q_3$ . Projecting  $Q_3 \subset P^4$  from this point onto a hyperplane  $H$  in  $P^4$  is a double covering of  $H$  and the images of  $Y_1$  and  $Y_2$  are two skew lines, say  $L_1$  and  $L_2$ . It then follows that  $\mathcal{E}(1)$  is the pull-back of the null-correlation bundle  $\mathcal{N}(1)$  corresponding to  $L_1$  and  $L_2$ .

**REMARK.** It is not entirely clear whether or not any stable bundle on  $Q_3$  with  $c_1 = 0$  and  $c_2 = 2$  enjoys the property stated in (3.2).

*Case of stable bundles with  $c_1 = -1$ ,  $c_2 = 1$ .* Here a more detailed description of Fano bundles can be given. Let  $\mathcal{E}$  be a stable bundle on  $Q_3$  with  $c_1 = -1$ ,  $c_2 = 1$ .

(3.3) The cohomology of such a bundle are the following:

- (1)  $h^0(\mathcal{E}(m)) = 0$  for  $m \leq 0$ ,
- (2)  $h^0(\mathcal{E}(1)) = 4$ ,
- (3)  $h^1(\mathcal{E}(m)) = h^2(\mathcal{E}(m)) = 0$  for all  $m$ ,
- (4)  $h^3(\mathcal{E}(m)) = 0$  for  $m \geq -2$ .

*Proof.* (1) is a criterion of stability, (4) is dual to (1), (2) will follow from (3), (4) and the Riemann-Roch formula. Corollary 2.4 in [5] gives  $h^1(\mathcal{E}(m)) = 0$  for  $m \leq -1$ . By duality,  $h^2(\mathcal{E}(m)) = 0$  for  $m \geq -1$  so that  $h^1(\mathcal{E}) = \chi(\mathcal{E}) = 0$ . The Castelnuovo criterion (see e.g. Lecture 14 in [14]) now yields that  $\mathcal{E}(m)$  are generated by global sections if  $m \geq 1$  and that all cohomology  $H^i(\mathcal{E}(m))$  vanish for  $i \geq 1$ ,  $i + m \geq 1$ . Now by duality (3) follows for any integer  $m$ .

Note that from the Castelnuovo criterion it follows that  $\mathcal{E}(1)$  is spanned; therefore  $\mathcal{E}(2)$  is ample and  $\mathcal{E}$  is Fano.

Now we prove that such  $\mathcal{E}$  is the one from (2.6). Since the bundle  $\mathcal{E}(1)$  is spanned and  $h^0(\mathcal{E}(1)) = 4$  it follows that the linear system  $|H + \xi|$  is base point free and of dimension 3. Let  $\varphi: P(\mathcal{E}) \rightarrow P^3$  be the map associated with this system.

(3.4). **PROPOSITION.**  $\varphi: P(\mathcal{E}) \rightarrow P^3$  is a  $P^1$ -bundle which is the projectivization of a null-correlation bundle.

*Proof.* First note that a general divisor  $D$  in the linear system  $|2H + \xi|$  is a Fano 3-fold listed as  $n^0$  17 in Table 2 [13]. The map  $\varphi|_D$  is a blow-down morphism from  $D$  onto  $P^3$ .

We claim that  $\varphi$  has no fibre of dimension  $\geq 2$ . Assume that  $S$  is such a fibre. Then  $f := D \cap S$  is isomorphic to  $P^1$  and  $\mathcal{O}_f(H) \cong \mathcal{O}_{P^1}(1)$ . In view of Theorem 2.1b', [6] we see that  $S \cong P^2$  and  $\mathcal{O}_S(H) \cong \mathcal{O}_{P^2}(1)$ . But in this case  $p: S \rightarrow Q_3$  is a plane embedding of  $P^2$  in  $Q_3$ , which is impossible.

Now any fibre of  $\varphi$  is numerically equivalent to  $(H + \xi)^3$  and, since  $H \cdot (H + \xi)^3 = 1$ , it follows that it must be isomorphic to  $P^1$ . The push-forward  $\varphi_*(\mathcal{O}(H))$  is a rank-2 Fano bundle on  $P^3$ . From the results of §2 we see that it is a null-correlation bundle.

**COROLLARY.** Any stable rank-2 bundle on  $Q_3$  with Chern classes  $c_1 = -1$ ,  $c_2 = 1$  is a pull-back of the universal quotient bundle on

## Grass(1, 3) via some hyperplane embedding

$$Q_3 \rightarrow \text{Grass}(1, 3) = Q_4 \subset P^5.$$

REMARK 1. The above example shows that the Horrocks splitting principle, as it stands on  $P^n$  (see e.g. [15]), cannot be applied literally to bundles on  $Q_3$  (see [16] for an analogue of the Horrocks splitting principle on  $Q_n$ ). Let us also notice that the bundle discussed above is uniform: its decomposition type is the same on all lines and smooth conics in  $Q_3$ .

REMARK 2. It is proved in [19] that  $V = P(\mathcal{N}) = P(\mathcal{E})$  (where  $\mathcal{E}$  is the bundle discussed above and  $\mathcal{N}$  is the null-correlation bundle on  $P^3$ ) is the only ruled Fano 4-fold of index 2 obtained from a non-decomposable bundle.

*Added in the proof.* Together with Ignacio Sols we have concluded the case of rank-2 Fano bundles on  $Q_3$ . Firstly, we have proved that the first twist of a stable bundle with  $c_1 = 0$ ,  $c_2 = 2$  is spanned by global sections (see Proposition (3.2) and the subsequent remark). Secondly, we have decided that bundles with  $c_1 = 0$ ,  $c_2 = 4$  are not Fano (see the discussion preceding (3.2)).

## REFERENCES

- [1] I. Yu. Artiushkin, *Four-dimensional Fano manifolds representable as  $P^1$ -bundles*, Vest. Mosk. Un-ta, ser 1, No. 1, (1987), 12–16, in Russian.
- [2] C. Banica and N. Manolache, *Rank 2 stable vector bundles on  $P^3(C)$  with Chern classes  $c_1 = -1$ ,  $c_2 = 4$* . Math. Z., **190** (1985), 315–339.
- [3] W. Barth, *Some properties of stable rank-2 vector bundles on  $P_n$* , Math. Ann., **226** (1977), 125–159.
- [4] I. V. Demin, *Three-dimensional Fano manifolds representable as line fiberings*, Izv. Akad. Nauk SSSR, **44**, n°4, (1980), 464–921, English translation in Math USSR Izv. 17 Addendum to this paper in Izv. Akad. Nauk SSSR 46 n°3, English translation in Math. USSR Izv. 20.
- [5] L. Ein and I. Sols, *Stable vector bundles on quadrics*, Nagoya Math. J., **96** (1984), 11–22.
- [6] T. Fujita, *On the structure of polarized varieties with  $\Delta$ -genera zero*, J. Fac. Sci. Univ. Tokyo, **22** (1975), 103–115.
- [7] W. Fulton, *Ample vector bundles, Chern classes, numerical criteria*. Inv. Math., **32** (1978), 171–178.
- [8] R. Hartshorne, *Stable Vector Bundles of Rank 2 on  $P^3$* , Math. Ann., **238** (1978), 229–280.
- [9] R. Hartshorne and I. Sols, *Stable rank 2 vector bundles on  $P^3$  with  $c_1 = -1$ ,  $c_2 = 2$* . J. Reine und Angew. Math., **325** (1981), 145–152.
- [10] T. Hosoh, *Ample vector bundles on a rational surface*, Nagoya Math. J., **59** (1975), 135–148.

- [11] Y. Kawamata, *The cone of curves of algebraic varieties*, Ann. Math., **119** (1984), 603f–633.
- [12] Sh. Mori, *Threefolds whose canonical bundles are not numerically effective*, Ann. Math., **116** (1982), 133–176.
- [13] Sh. Mori and Sh. Mukai, *Classification of Fano 3-folds with  $B_2 \geq 2$* , Manuscripta Math., **36** (1981), 147–162.
- [14] D. Mumford, *Lectures on curves on an algebraic surface*, Ann. of Math. Studies 59, Princeton Univ. Press 1966.
- [15] Ch. Okonek, M. Schneider and H. Spindler, *Vector Bundles on Complex Projective Spaces*, Birkhäuser 1981.
- [16] G. Ottaviani, *Critères de scindage pour les fibrés vectoriels sur les grassmanniens et les quadriques*, Comptes Rendus Acad. Sci. Paris, **305** (1987), 257–260.
- [17] M. Schneider, *Stabile Vektorraumbündel vom rang 2 auf der projektiven Ebene*. Nachr. Akad. Wiss. Göttingen 1976, 83–86.
- [18] S. S. Tikhomirov, *On stable two-dimensional vector bundles on  $P^3$  with Chern classes  $c_1 = 0$ ,  $c_2 = 3$  and 4*, Sbornik Nauch. Tr. Yarosl. Pedag. Inst. im. K. D. Ushinskogo, **180** (1979), 106–114, in Russian.
- [19] J. A. Wisniewski, *Ruled Fano 4-folds of index 2*, Proc. Amer. Math. Soc., **105** (1983), 55–61

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# Pacific Journal of Mathematics

Vol. 141, No. 1      November, 1990

<b>Yusuf Abu-Muhanna and Abdallah Khalil Lyzzaik</b> , The boundary behaviour of harmonic univalent maps .....	1
<b>Lawrence Jay Corwin, Allen Moy and Paul J. Sally, Jr.</b> , Degrees and formal degrees for division algebras and $GL_n$ over a $p$ -adic field .....	21
<b>Ulrich Dierkes and Gerhard Huisken</b> , The $n$ -dimensional analogue of the catenary: existence and nonexistence .....	47
<b>Peter Gerard Dodds, C. B. Huijsmans and Bernardus de Pagter</b> , Characterizations of conditional expectation-type operators .....	55
<b>Douglas Dokken and Robert Ellis</b> , The Poisson flow associated with a measure .....	79
<b>Larry J. Santoni</b> , Horrocks' question for monomially graded modules .....	105
<b>Zbigniew Slodkowski</b> , Pseudoconvex classes of functions. II. Affine pseudoconvex classes on $R^N$ .....	125
<b>Dean Ellis Smith</b> , On the Cohen-Macaulay property in commutative algebra and simplicial topology .....	165
<b>Michał Szurek and Jarosław Wisniewski</b> , Fano bundles over $P^3$ and $Q_3$ ...	197