SOMMES EXPONENTIELLES DONT LA GÉOMÉTRIE EST TRÈS BELLE: $p$-ADIC ESTIMATES

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0. Introduction. Let $K = \mathbb{F}_q$ be the field with $q$ elements (char $K = p \neq 2$, $q = p^r$), $\mathcal{X} \in K^\times$, $g_1, \ldots, g_n$ positive integers relatively prime and prime to $p$ ($n \geq 2$) and let $\mathcal{Y}_\mathcal{X}$ be the variety defined over $K$ by $\prod_{i=1}^{n} t_i^{g_i} = \mathcal{X}$. Let $\Omega$ be a complete algebraically closed field containing $\mathbb{Q}_p$, $\Theta: K \to \Omega^\times$ an additive character and for each $i \in \{1, \ldots, n\}$ let $\chi_i: K^\times \to \Omega^\times$ be a multiplicative character. Let $\bar{c}_1, \ldots, \bar{c}_n$ be non-zero elements of $K$, and let $\bar{f}(i) = \sum_{i=1}^{n} \bar{c}_i t_i^{k_i}$, where $k_1, \ldots, k_n$ are positive integers prime to $p$. For each $m \in \mathbb{Z}_+$ let $K_m$ be the extension of $K$ of degree $m$. We consider the twisted exponential sums

\[
S_m(\bar{f}, \mathcal{Y}_\mathcal{X}) = \sum_{(i_1, \ldots, i_n) \in \mathcal{Y}_\mathcal{X}(K_n)} \prod_{i=1}^{n} \chi_i \circ N_{K_m/K}(i_i) \times \Theta \circ \text{Tr}_{K_m/K}(\bar{f}(i))
\]

and the associated $L$ function:

\[
L = L(\bar{f}, \mathcal{Y}_\mathcal{X}, T) = \exp \left( - \sum_{m=1}^{\infty} S_m(\bar{f}, \mathcal{Y}_\mathcal{X}) T^m / m \right).
\]

Our main results are the following:

A. We show that $L^{-1} h$ is a polynomial of degree

\[
h = \left( \sum_{i=1}^{n} g_i / k_i \right) \prod_{i=1}^{n} k_i.
\]

B. We compute explicitly a lower bound for the Newton polygon of $L^{-1} h$; this lower bound is independent of the prime number $p$ and its endpoints coincide with those of the Newton polygon (Theorem 5.1 and Corollary 5.1).

C. Provided $p$ lies in certain congruence classes, we show that our lower bound is in fact the exact Newton polygon of $L^{-1} h$ (Theorem 5.3).
D. As a consequence we obtain p-adic estimates for the sums (0.1), since they are related to the reciprocal roots \{\gamma_i\}_{i=1}^n of (0.2) by the equation

\[ S_m(J, \gamma) = (-1)^{n+1}(\gamma_1^m + \cdots + \gamma_n^m). \]

We emphasize that our lower bound for the Newton polygon can be computed explicitly: To fix notations, we assume that the multiplicative characters \( \chi_i \) are of the form \( \chi_i(t) = \omega(t)^{-(q-1)\rho_i/r} \), where \( r \) and \( \rho_i \) are natural integers, \( r|q-1 \), \( 0 \leq \rho_i < r \). For \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \), let \( \sigma(\alpha) = \text{Inf}_i \alpha_i/g_i \) and \( J(\alpha) = \frac{1}{r} \sum_{i=1}^n \alpha_i/k_i \). Let \( \tilde{\Delta}_p' \) be the finite subset of \( \mathbb{Z}^n \) defined by

\[ \alpha \in \tilde{\Delta}_p' \iff \begin{cases} 0 \leq \sigma(\alpha) < r \\
\alpha_i \equiv \rho_i \pmod{r}, & i = 1, \ldots, n \\
\sigma(\alpha) \leq \alpha_i/g_i \leq \sigma(\alpha) + rk_i/g_i, & i = 1, \ldots, n. \end{cases} \]

Whenever two elements \( \alpha \) and \( \beta \) of \( \tilde{\Delta}_p' \) satisfy \( J(\alpha) = J(\beta) \) and \( \alpha_i \equiv \beta_i \pmod{k_i} \) for all \( i \), we only keep the first of these two elements for the lexicographic order and eliminate the other: let \( \Delta_p \) be the resulting set. \( \Delta_p \) contains \( h = (\sum_{i=1}^n g_i/k_i) \prod_{i=1}^n k_i \) elements, and the slopes of our lower bound are the values on \( \Delta_p \) of the weight function \( w(\alpha) = J(\alpha) - \frac{1}{r} \sigma(\alpha) \sum_{i=1}^n g_i/k_i \). For example, if \( \gamma \) is the variety \( t_1t_2t_3 = 1 \) and \( J(t) = t_1^3 + t_2^2 + t_3 \), with trivial twisting characters \( \chi_i \), then \( L^{-1} \) is a polynomial of degree 26. When \( p \equiv 1 \pmod{18} \) its reciprocal roots have p-adic ordinal 0, 1/3, 7/18, 4/9, 1/2, 2/3 (twice), 13/18, 7/9, 5/6, 8/9, 17/18, 1 (twice), 19/18, 10/9, 7/6, 11/9, 23/18, 4/3 (twice), 3/2, 14/9, 29/18, 5/3, 2. When \( p \not\equiv 1 \pmod{18} \), the Newton polygon of \( L^{-1} \) lies above the Newton polygon whose sides have these slopes and their endpoints coincide.

If \( n = 2 \), \( k_1 = k_2 = 1 \), \( g_1 = g_2 = 1 \), and the twisting characters are trivial, the sum (0.1) is the Kloosterman sum, which was first investigated from a p-adic point of view by B. Dwork in [9]. More general situations have been studied by S. Sperber ([13], [14], [15]) and Adolphson-Sperber ([1], [2]). We have made extensive use of the work of these authors, especially from [15]. On the other hand, using l-adic cohomology, P. Deligne [6] has shown, in the case \( g_1 = \cdots = g_n = k_1 = \cdots = k_n = 1 \), that the reciprocal roots \( \{\gamma_i\}_{i=1}^n \) of \( L^{-1} \) have complex absolute value \( q^{n-1/2} \); this was later extended by N. Katz [10]—from whom we borrow the title of this article—to include the case \( k_1 = \cdots = k_n \) and general \( g_1, \ldots, g_n \). We complement
here this result, by obtaining $p$-adic estimates for the $\gamma_i$'s. Our approach departs from previous literature on the subject by the use of a new trace formula (Theorem 1.1) which provides a more balanced treatment and avoids the restriction $g_n = k_n = 1$ ([4], [15]).

Using Dwork's methods, we construct cohomology spaces $W_{x, \rho}$ on which a Frobenius map acts, $\mathcal{F}_x: W_{x, \rho} \to W_{x^t, \rho}$. These spaces have dimension $h$, and if $x = x^q$ is a Teichmüller point, the eigenvalues of $\mathcal{F}_x$ are the reciprocal zeros of (0.2). The choice of a good basis for the space $W_{x, \rho}$ is crucial in obtaining estimates for the Newton polygon of the $L$-function: its elements are those of the set $\{x^{-(\sigma(\alpha))/r} t^\alpha | \alpha \in \Delta_\rho \}$, chosen so as to minimize the weight function $w(\alpha)$.

Define $\rho^{(0)} = \rho, \rho^{(1)}, \ldots, \rho^{(r)} = \rho$ by the conditions

$$\begin{cases} \rho \rho_i^{(j+1)} - \rho_i^{(j)} \equiv 0 \pmod{r} \\ 0 \leq \rho_i^{(j)} < r \end{cases} \forall i, j$$

For each $\alpha^{(j)} \in \widetilde{\Delta}_\rho^{(j)}$, there exist (Lemma 2.8) unique elements $\alpha^{(j+1)} \in \widetilde{\Delta}_\rho^{(j+1)}$ and $\delta^{(j)} \in \mathbb{Z}^n$ satisfying

$$\begin{cases} p \left( \frac{\alpha^{(j+1)}}{rk_i} - \sigma(\alpha^{(j+1)}) \frac{g_i}{rk_i} \right) - \left( \frac{\alpha^{(j)}}{rk_i} - \sigma(\alpha^{(j)}) \frac{g_i}{rk_i} \right) = \delta^{(j)}_i \\ 0 \leq \delta^{(j)}_i < r \end{cases}$$

If $\alpha = \alpha^{(0)} \in \widetilde{\Delta}_\rho$, let $Z(\alpha) = \sum_{j=0}^{r-1} w(\alpha^{(j)})$. We show that the Newton polygon of $L^{-1}_n$ lies below that of $\mathcal{H}_\rho(T) = \prod_{\alpha \in \Delta_\rho} (1 - p^{Z(\alpha)} T)$, and their endpoints coincide (Theorem 5.2 and Corollary 5.1). On the other hand, if $p \equiv 1 \pmod{r}$, the Newton polygon of the $L$-function lies above that of $\mathcal{H}_\rho(T) = \prod_{\alpha \in \Delta_\rho} (1 - q^{w(\alpha)} T)$ (Theorem 5.1). If furthermore $pg_i \equiv g_i \pmod{k_ig_j}$ for all $i, j$, then $\mathcal{H}_\rho(T) = \mathcal{H}_\rho(T)$ and therefore their common Newton polygon is that of $L^{-1}_n$.

The precise determination of the Newton polygon in other congruence classes requires finer estimates for the Frobenius matrix. This question has been solved by Adolphson-Sperber ([2]) in the case $n = 2, g_1 = g_2 = 1, k_1 = k_2$. We expect to address this question more fully in a subsequent article.

In [5], we studied the deformation equation when $k_n = g_n = 1$. With only minor changes, this treatment can be reconciled with the point of view adopted here. Let us simply indicate that the deformation operator of [5, p. 9-04] should be replaced by

$$\eta_y = E_y + \pi Mc_n \frac{d_n}{a_n} t^{d_n},$$
1. Trace formula. Let $g_1, \ldots, g_n$ be positive integers ($n \geq 2$), $g = (g_1, \ldots, g_n)$. We assume that $\text{g.c.d.}(g_1, \ldots, g_n) = 1$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ we define:

\[
\begin{align*}
\omega_{i,j}(\alpha) &= \frac{\alpha_i}{g_i} - \frac{\alpha_j}{g_j}, \quad i, j = 1, \ldots, n; \\
\sigma(\alpha) &= \inf \left\{ \frac{\alpha_1}{g_1}, \ldots, \frac{\alpha_n}{g_n} \right\}.
\end{align*}
\]

Let $\mu$ be a fixed positive integer; for any $\alpha \in \mathbb{Z}^n$ let $\phi_\alpha : \mathbb{Z}^n \to \mathbb{Z}/\mu\mathbb{Z}$ be the group homomorphism defined by $\phi_\alpha(y_1, \ldots, y_n) = \sum_{i=1}^n \frac{y_i}{\mu} \alpha_i$.

**Lemma 1.1.** Let $\alpha \in \mathbb{Z}^n$; the following conditions are equivalent:

(i) There exists $\beta \in \mathbb{Z}^n$ such that $\omega_{i,j}(\alpha) = \mu \omega_{i,j}(\beta)$ for all $i, j = 1, \ldots, n$.

(ii) There exist $\beta \in \mathbb{Z}^n$ and $l \in \{1, \ldots, n\}$ such that $\omega_{i,l}(\alpha) = \mu \omega_{i,l}(\beta)$ for all $i = 1, \ldots, n$.

(iii) $\ker(\phi_\alpha) \subseteq \ker(\phi_\beta)$.

**Proof.** The equivalence of (i) and (ii) is obvious from the definitions. Suppose that $\alpha$ satisfies condition (ii) and let $\gamma = (\gamma_1, \ldots, \gamma_n) \in \ker(\phi_\beta)$. By assumption, $\alpha_ig_i = \alpha_ig_i + \mu(\beta_ig_i - \beta_ig_i)$ for all $i$, hence:

\[
g_i \sum_{i=1}^n \gamma_i \alpha_i = \left( \sum_{i=1}^n \gamma_i g_i \right) (\alpha_l - \mu \beta_l) + \mu g_l \sum_{i=1}^n \gamma_i \beta_i.
\]

Since $g_i(\alpha_l - \mu \beta_l) = g_l(\alpha_l - \mu \beta_l)$ for all $i$ and $\text{g.c.d.}(g_1, \ldots, g_n) = 1$, it follows that $g_l$ divides $\alpha_l - \mu \beta_l$. Hence $\sum_{i=1}^n \gamma_i \alpha_i \equiv 0 \pmod{\mu}$ i.e. $\gamma \in \ker(\phi_\alpha)$ and (ii)$\Rightarrow$(iii).

Suppose that $\ker(\phi_\beta) \subseteq \ker(\phi_\alpha)$ and, for $i = 1, \ldots, n - 1$, let $\tau_i = \text{g.c.d.}(g_i, g_n)$.

Since

\[
\frac{g_n}{\tau_i} g_i - \frac{g_i}{\tau_i} g_n = 0,
\]

our assumption implies the existence of integers $z_1, \ldots, z_{n-1}$ satisfying

\[
\frac{g_n}{\tau_i} \alpha_i - \frac{g_i}{\tau_i} \alpha_n = \mu z_i \quad \text{for all } i = 1, \ldots, n - 1.
\]

Furthermore, for each such $i$, there are integers $\beta_i$ and $\beta_n^{(i)}$ such that:

\[
(1.2(i)) \quad z_i = \beta_i \frac{g_n}{\tau_i} - \beta_n^{(i)} \frac{g_i}{\tau_i}.
\]
Thus
\[
\frac{\alpha_i}{g_i} - \frac{\alpha_n}{g_n} = \mu \left( \frac{\beta_i}{g_i} - \frac{\beta_n(i)}{g_n} \right) \quad \text{for all } i = 1, \ldots, n - 1.
\]

Observe that, if \((\beta_i, \beta_n(i))\) is a solution of equation (1.2(i)), then so is \((\beta_i + g_i/\tau_i, \beta_n(i) + g_n/\tau_i)\). We must show the existence of solutions satisfying \(\beta_n(i) = \ldots = \beta_n(n-1)\). Let \(i, j \in \{1, \ldots, n - 1\}\) with \(i \neq j\):

\[
\frac{\alpha_i}{g_i} - \frac{\alpha_j}{g_j} = \mu \left( \frac{\beta_n(j) - \beta_n(i)}{g_n} + \frac{\beta_i}{g_i} - \frac{\beta_j}{g_j} \right).
\]

On the other hand, just as above, we can find integers \(e_i\) and \(e_j\) such that:

\[
\frac{\alpha_i}{g_i} - \frac{\alpha_j}{g_j} = \mu \left( \frac{e_i}{g_i} - \frac{e_j}{g_j} \right).
\]

Hence, letting \(\delta_i = \beta_i - e_i, \delta_j = \beta_j - e_j\) and \(\tau_{i,j} = \gcd(\tau_i, \tau_j)\) we can write:

\[
(\beta_n(j) - \beta_n(i)) \frac{g_i g_j \tau_{i,j}}{\tau_i \tau_j} = \frac{g_n \tau_{i,j}}{\tau_i \tau_j} (\delta_j g_i - \delta_i g_j).
\]

Since \(g_n \tau_{i,j}/\tau_i \tau_j\) and \(g_i g_j \tau_{i,j}/\tau_i \tau_j\) are relatively prime, there exists \(Z \in \mathbb{Z}\) such that

\[
\beta_n(j) - \beta_n(i) = Z \frac{g_n \tau_{i,j}}{\tau_i \tau_j}.
\]

In turn, there exist \(\xi, \eta \in \mathbb{Z}\) such that \(Z \tau_{i,j} = \xi \tau_i + \eta \tau_j\) and therefore

\[
\beta_n(j) - \beta_n(i) = \xi \frac{g_n}{\tau_j} + \eta \frac{g_n}{\tau_i}.
\]

If we let \(r_k = g_n/\tau_k\) \((k = 1, \ldots, n - 1)\), we have just proved that, for all \(i, j \in \{1, \ldots, n - 1\}\):

\[
(1.3) \quad \beta_n(j) - \beta_n(i) \in r_i \mathbb{Z} + r_j \mathbb{Z}.
\]

We now proceed by induction. Let \(k < n - 1\) and suppose that we have found solutions \((\tilde{\beta}_i, \tilde{\beta}_n(i))\) of equations (1.2(i)) for all \(i\), with the property that \(\tilde{\beta}_n(1) = \ldots = \tilde{\beta}_n(k) =: \tilde{\beta}_n\).

Let \(m_k = \text{l.c.m.}(r_1, \ldots, r_k)\). By (1.3), \(\tilde{\beta}_n - \tilde{\beta}_n(k+1) \in m_k \mathbb{Z} + r_{k+1} \mathbb{Z}\) and therefore there are integers \(\lambda, \zeta\) such that \(\tilde{\beta}_n + \lambda m_k = \tilde{\beta}_n(k+1) + \zeta r_{k+1}\).
Let:
\[
\begin{align*}
\beta_n^{(i)} &= \tilde{\beta}_n^{(i)} + \lambda m_k & 1 \leq i \leq k \\
\beta_i &= \tilde{\beta}_i + \lambda \frac{g_i}{g_n} m_k & 1 \leq i \leq k \\
\beta_n^{(k+1)} &= \tilde{\beta}_n^{(k+1)} + \zeta r_{k+1} \\
\beta_{k+1} &= \tilde{\beta}_{k+1} + \zeta \frac{g_{k+1}}{\tau_{k+1}} \\
\beta_n^{(j)} &= \tilde{\beta}_n^{(j)} & j > k + 1 \\
\beta_j &= \tilde{\beta}_j & j > k + 1
\end{align*}
\]

For each \( i = 1, \ldots, n - 1 \), \((\beta_i, \beta_n^{(i)})\) is a solution of (1.2(i)) and we have \(\beta_n^{(1)} = \cdots = \beta_n^{(k+1)}\). Finally we obtain \(\beta = (\beta_1, \ldots, \beta_n)\) with \(\omega_{i,n}(\alpha) = \mu \omega_{i,n}(\beta) \forall i = 1, \ldots, n\).

Hence (iii)\(\Rightarrow\)(ii).

**Notation.** If \(\alpha, \beta \in \mathbb{Z}^n\) satisfy \(\omega_{i,j}(\alpha) = \mu \omega_{i,j}(\beta)\) for all \(i, j = 1, \ldots, n\) we shall write:
\[(1.4)\]
\[
\omega(\alpha) = \mu \omega(\beta).
\]

**Remark 1.1.** Let \(\alpha, \beta \in \mathbb{Z}^n\) satisfying (1.4) and let \(l \in \{1, \ldots, n\}\), then
\[(1.5)\]
\[
\sigma(\alpha) = \frac{\alpha_l}{g_l} \Leftrightarrow \sigma(\beta) = \frac{\beta_l}{g_l}.
\]

Let:
\[(1.6)\]
\[
S = \{\alpha \in \mathbb{Z}^n \mid 0 \leq \sigma(\alpha) < 1\}.
\]

**Lemma 1.2.** Let \(\alpha, \beta \in S\); then \(\alpha = \beta \Leftrightarrow \omega(\alpha) = \omega(\beta)\).

**Proof.** The first implication is obvious. Conversely, suppose that \(\omega(\alpha) = \omega(\beta)\) and let \(l\) be an index such that \(\sigma(\alpha) = \alpha_l/g_l\). By the remark above, \(\sigma(\beta) = \beta_l/g_l\).

By assumption, \(g_l(\alpha_l - \beta_l) = g_l(\alpha_i - \beta_i)\) for all \(i\). If \(\gamma_1, \ldots, \gamma_n\) are integers satisfying \(\sum_{i=1}^n \gamma_i g_i = 1\), then \(\alpha_l - \beta_l = g_l \sum_{i=1}^n \gamma_i (\alpha_i - \beta_i)\) and therefore \(g_l\) divides \(\alpha_l - \beta_l\).

Since \(\alpha\) and \(\beta\) are elements of \(S\), \(-g_l < \alpha_l - \beta_l < g_l\), hence \(\alpha_l = \beta_l\) and it follows that \(\alpha_i = \beta_i\) for all \(i\). \(\Box\)

We fix \(r\), a positive integer, and for each \(\alpha \in \mathbb{Z}^n\) we set
\[(1.7)\]
\[
\sigma(\alpha) = \frac{1}{r} \sigma(\alpha).
\]
Let:
\[(1.8) \quad E = \{a \in \mathbb{Z}^n | 0 \leq \sigma(a) < 1\} = \{a \in \mathbb{Z}^n | 0 \leq \sigma(a) < r\}.\]

If \(\rho \in \mathbb{Z}^n\), with \(0 \leq \rho_i < r\) we set
\[(1.9) \quad Z^{(\rho)} = \{\alpha \in \mathbb{Z}^n | \alpha_i \equiv \rho_i \pmod{r} \text{ for all } i\},\]
\[(1.10) \quad E^{(\rho)} = Z^{(\rho)} \cap E.\]

**Lemma 1.3.** Let \(\alpha, \beta \in E^{(\rho)}\); then \(\alpha = \beta \iff \omega(\alpha) = \omega(\beta)\).

**Proof.** Suppose that \(\omega(\alpha) = \omega(\beta)\) and assume that \(\alpha_i \geq \beta_i\) for some index \(i\). Then \(\alpha_i > \beta_i\) for all \(i\) and, letting \(\gamma_i = (\alpha_i - \beta_i)/r\), \(\gamma = (\gamma_1, \ldots, \gamma_n)\) is an element of \(S\), with \(\omega(\gamma) = 0\). Lemma 1.2 implies that \(\gamma = (0, \ldots, 0)\). \(\Box\)

We now fix \(p\), a prime number, with \((p, r) = 1\). If \(\rho \in \mathbb{Z}^n\), \(0 \leq \rho_i < r\), we let \(\rho' \in \mathbb{Z}^n\) be the unique element satisfying
\[(1.11) \quad \begin{cases} 0 \leq \rho'_i < r, \\ p \rho'_i - \rho_i \equiv 0 \pmod{r}. \end{cases}\]

**Lemma 1.4.** Let \(\alpha \in Z^{(\rho)}\) satisfying the equivalent conditions of Lemma 1.1 with \(\mu = p\). Then, in (i) and (ii), \(\beta\) can be chosen uniquely so that
\[(1) \quad \beta \in E^{(\rho')}; \quad (2) \quad \sigma(\alpha) - p\sigma(\beta) \in \mathbb{Z}.\]

**Proof.** Suppose that \(\omega(\alpha) = p \omega(\delta)\). Certainly, \(\delta\) may be chosen (uniquely) so that \(0 \leq \sigma(\delta) < 1\). By Remark 1.1, \(g_i(\sigma(\alpha) - p \sigma(\delta)) = \alpha_i - p\delta_i \forall i\). Let \(\gamma_1, \ldots, \gamma_n\) be integers satisfying \(\sum_{i=1}^n \gamma_i g_i = 1\):
\[
\sum_{i=1}^n g_i \gamma_i (\sigma(\alpha) - p \sigma(\beta)) = \sum_{i=1}^n \gamma_i (\alpha_i - p\delta_i),
\]
hence \(\sigma(\alpha) - p\sigma(\delta) \in \mathbb{Z}\). In particular, \(p\delta - \alpha\) belongs to the cyclic subgroup of \(\mathbb{Z}^n\) generated by \(g\). Since \(\gcd.g, (p, r) = 1 = \gcd.(g_1, \ldots, g_n)\), there is a unique integer \(\lambda, 0 \leq \lambda < r\), such that \(p(\delta + \lambda g) - \alpha \in r\mathbb{Z}^n\). Now set \(\beta = \delta + \lambda g\). \(\Box\)

Let \(\mathbb{Q}_p\) be the completion of the field of rational numbers for the \(p\)-adic valuation, and \(\Omega\) an algebraically closed field containing \(\mathbb{Q}_p\). We denote by “\(\text{ord}\)” the valuation on \(\Omega\) normalized so that \(\text{ord}p = 1\). Let \(\kappa\) be a positive integer such that \(r \mid p^{\kappa} - 1\), let \(q = p^{\kappa}\) and let
$x \in \Omega^x$ be a Teichmüller point: $x^q = x$. Let $K$ be an extension of $\mathbb{Q}_p$ in $\Omega$ containing $x$. Let $t_1, \ldots, t_n$ be indeterminates. We shall use multi-index notation: if $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $t^\alpha = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$.

Fix $k_1, \ldots, k_n$ positive integers. Given $b, c \in \mathbb{R}$ with $b \geq 0$, let:

$$(1.12) \mathcal{L}(b, c) = \left\{ \xi = \sum_{\alpha \in \mathbb{N}^n} B_\alpha t^\alpha \mid B_\alpha \in K \text{ and ord } B_\alpha \geq b \sum_{i=1}^n \frac{\alpha_i}{k_i} + c \right\};$$

$$(1.13) \mathcal{L}(b) = \bigcup_{c \in \mathbb{R}} \mathcal{L}(b, c).$$

For each $\rho = (\rho_1, \ldots, \rho_n) \in \mathbb{Z}^n$ with $0 \leq \rho_i < r$ we let

$$(1.14) \mathcal{L}_\rho(b, c) = \left\{ \xi = \sum_{\alpha \in \mathbb{N}^n} B_\alpha t^\alpha \in \mathcal{L}(b, c) \mid B_\alpha = 0 \text{ if } \alpha \notin Z(\rho) \right\};$$

$$(1.15) \mathcal{L}_\rho(b) = \bigcup_{c \in \mathbb{R}} \mathcal{L}_\rho(b, c).$$

$\mathcal{L}(b, c), \mathcal{L}(b), \mathcal{L}_\rho(b, c), \mathcal{L}_\rho(b)$ are $p$-adic Banach spaces with the norm

$$||\xi|| = \sup_{\alpha} p^{c_\alpha}, \quad c_\alpha = b \sum_{i=1}^n \frac{\alpha_i}{k_i} - \text{ord } B_\alpha.$$

Let $\mathcal{N} = \sum_{i=1}^n g_i/k_i$ and

$$(1.16) \overline{\mathcal{L}}(b, c) = \left\{ \eta = \sum_{\alpha \in \mathbb{E}} C_\alpha t^\alpha \mid C_\alpha \in K \text{ and ord } C_\alpha \geq b \left( \sum_{i=1}^n \frac{\alpha_i}{k_i} - \mathcal{N} \sigma(\alpha) \right) + c \right\};$$

$$(1.17) \overline{\mathcal{L}}(b) = \bigcup_{c \in \mathbb{R}} \overline{\mathcal{L}}(b, c);$$

$$(1.18) \overline{\mathcal{L}}_\rho(b, c) = \left\{ \eta = \sum_{\alpha \in \mathbb{E}} C_\alpha t^\alpha \in \overline{\mathcal{L}}(b, c) \mid C_\alpha = 0 \text{ if } \alpha \notin E(\rho) \right\};$$

$$(1.19) \overline{\mathcal{L}}_\rho(b) = \bigcup_{c \in \mathbb{R}} \overline{\mathcal{L}}_\rho(b, c).$$

$\mathcal{L}(b, c), \overline{\mathcal{L}}(b), \overline{\mathcal{L}}_\rho(b, c), \overline{\mathcal{L}}_\rho(b)$ are $p$-adic Banach spaces with the norm

$$||\eta|| = \sup_{\alpha} p^{c_\alpha}, \quad c_\alpha = b \left( \sum_{i=1}^n \frac{\alpha_i}{k_i} - \mathcal{N} \sigma(\alpha) \right) - \text{ord } B_\alpha.$$
If \( \alpha, \beta \in \mathbb{Z}^n \), there exist \( \tau \in \mathbb{Z} \) and \( \delta \in E \), uniquely defined, such that \( \alpha + \beta = \delta + \tau r \) and we set

\[
(1.20) \quad t^{\alpha} * t^{\beta} = x^{\tau} t^\delta.
\]

Since \( \sigma(\alpha + \beta) \geq \sigma(\alpha) + \sigma(\beta) \) and \( \sigma(\delta + \tau r) = \sigma(\delta) + \tau r \), this operation makes \( \mathcal{L}(b) \) (respectively \( \mathcal{L}_p(b) \)) into a \( K \)-algebra; if \( \zeta \) is an element of \( \mathcal{L}(b, c') \), then \( \eta \to \zeta * \eta \) maps \( \mathcal{L}(b, c) \) continuously into \( \mathcal{L}(b, c + c') \).

Let \( \phi \) be the \( K \)-linear map whose action on monomials is given by

\[
(1.21) \quad \phi(t^{\alpha}) = t_1^{\alpha_1} * t_2^{\alpha_2} * \cdots * t_n^{\alpha_n}.
\]

For each \( p \), \( \phi \) is a continuous algebra homomorphism from \( \mathcal{L}_p(b, c) \) into \( \mathcal{L}(b, c) \). If \( \alpha \in \mathbb{Z}^{(p)} \) we define

\[
(1.22) \quad \psi(t^{\alpha}) = \begin{cases} x^{\nu(\alpha) - p \nu(\beta)} t^{\beta} & \text{if } \exists \beta \in E(p') \text{ such that } \omega(\alpha) = p \omega(\beta), \\ 0 & \text{otherwise}. \end{cases}
\]

Note that if \( \alpha, \beta \in \mathbb{Z}^n \), then

\[
(1.23) \quad \psi(t^{\alpha} * t^{\beta}) = \psi(t^{\alpha + \beta}).
\]

It follows from Lemma 1.4 that \( \psi \) extends to a continuous linear map from \( \mathcal{L}_p(b, c) \) into \( \mathcal{L}_p(pb, c) \). Since \( r \mid q - 1 \), \( \psi' \) maps \( \mathcal{L}_p(b, c) \) into \( \mathcal{L}_p(qb, c) \). If \( b' > b \), then \( \mathcal{L}_p(b', c) \) is a subspace of \( \mathcal{L}_p(b, c) \) and the canonical injection \( i: \mathcal{L}_p(b', c) \to \mathcal{L}_p(b, c) \) is completely continuous \([12, \S 9]\).

We fix \( F(t) = \sum_{\alpha \in \mathbb{N}^n} B_\alpha t^{\alpha} \) an element of \( \mathcal{L}(rb) \) and we let \( \bar{F}(t) = \phi(F(t')) \) \( \in \mathcal{L}_\mathbb{Q}(b) \). We define \( \mathcal{F}_p \) to be the composition:

\[
\mathcal{L}_p(qb) \xrightarrow{i} \mathcal{L}_p(b) \xrightarrow{\bar{F}(t)} \mathcal{L}_p(b) \xrightarrow{\psi'} \mathcal{L}_p(qb).
\]

By \([12, \S 3]\), \( \mathcal{F}_p \) is a completely continuous endomorphism of \( \mathcal{L}(qb) \). Its trace and Fredholm determinant are well defined and

\[
\det(I - T\mathcal{F}_p) = \exp \left( - \sum_{m=1}^{\infty} \text{tr}(\mathcal{F}_p^m) \frac{T^m}{m} \right) \text{ is a } p \text{-adic entire function.}
\]

For \( m \in \mathbb{N}^* \) we let

\[
(1.24) \quad \forall_m = \{(t_1, \ldots, t_n) \in K^n \mid t_i^{m-1} = 1 \text{ and } t_1^{g_1} \times \cdots \times t_n^{g_n} = x\}.
\]

**Theorem 1.1.**

\[
(q - 1)^{n-1} \text{tr}(\mathcal{F}_p \mid \mathcal{L}_p(qb)) = \sum_{t \in \forall_1} \left( \prod_{i=1}^{n} t_i^{-(q-1)\rho_i / r} \right) F(t).
\]
Proof. Write $F(t) = \sum_{\alpha \in S} \sum_{\lambda \in \mathbb{N}} B_{\alpha + \lambda g} t^{\alpha + \lambda g}$. Let $G(t) = \sum_{\alpha \in S} C_{\alpha} t^{\alpha}$, with $C_{\alpha} = \sum_{\lambda \in \mathbb{N}} B_{\alpha + \lambda g} x^{\lambda}$. For each $i = 1, \ldots, n$ let $\delta_i = -\rho_i (q - 1)/r$ and set $X_\rho(t) = \prod_{i=1}^{n} t_i^{\delta_i}$. Then $\sum_{t \in \mathbb{F}_q} X_\rho(t) F(t) = \sum_{t \in \mathbb{F}_q} X_\rho(t) G(t)$.

On the other hand, $\hat{F}(t) = \phi(F(t)) = \sum_{\alpha \in S} C_{\alpha} t^{\alpha} = G(t')$.

Note that for each $\beta \in \mathbb{Z}^n$ we can find $\gamma \in \mathbb{Z}^n$ such that $\omega(\gamma) = (q - 1)\omega(\beta)$. Since $r \mid q - 1$, we can choose $\gamma$ so that $\gamma_i \equiv 0 \pmod{r}$ for all $i$. Furthermore, after adding or subtracting multiples of $rg$, we may assume that $\gamma \in E$. Accordingly, for each $\beta \in \mathbb{Z}^n$, we denote by $\tilde{\beta}$ the unique (by Lemma 1.3) element of $S$ satisfying $\omega(r \tilde{\beta}) = (q - 1)\omega(\beta)$.

For fixed $\beta \in E(\rho)$,

$$\mathcal{F}_\rho(t^\beta) = \sum_{\alpha \in S} C_{\alpha} \psi'(t^{\alpha + \beta}) = \sum_{\alpha \in S} C_{\alpha} x^{(r \tilde{\beta} - (q - 1)\epsilon(\beta))} t^\gamma,$$

where the last sum is indexed by the set of all $\alpha \in S$ such that $\omega(r \alpha + \beta) = q \omega(\gamma)$, $\gamma \in E(\rho)$. The coefficient of $t^\beta$ in this sum is $C_{\beta} x^{(r \tilde{\beta} - (q - 1)\epsilon(\beta))}$, and therefore,

$$\text{tr}(\mathcal{F}_\rho) = \sum_{\beta \in E(\rho)} C_{\beta} x^{(r \tilde{\beta} - (q - 1)\epsilon(\beta))}.$$ (1.25)

There remains to show that $(q - 1)^{n-1} \text{tr}(\mathcal{F}_\rho) = \sum_{t \in \mathbb{F}_q} X_\rho(t) G(t)$, and it is sufficient to check this when $G(t)$ is a single monomial, $G(t) = C_{\alpha} t^{\alpha}$. Let $G = \mathbb{Z}/(q - 1)\mathbb{Z}$; if $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_n)$ and $\bar{b} = (\bar{b}_1, \ldots, \bar{b}_n)$ are two elements of $G$, we let $\bar{a} \cdot \bar{b} = \sum_{i=1}^{n} \bar{a}_i \bar{b}_i$. Fix $\zeta$ a primitive $(q - 1)$-st root of unity. Since $\text{g.c.d.}(g_1, \ldots, g_n) = 1$, we can find $\overline{\gamma} \in G$ such that $x = \zeta^{\overline{\gamma} \cdot \overline{\epsilon}}$. Let $H = \{\overline{\eta} \in G \mid \overline{\eta} \cdot \overline{\epsilon} = 0\}$:

$$\sum_{t \in \mathbb{F}_q} X_\rho(t) t^{\alpha} = \zeta^{\overline{\gamma} \cdot (\overline{\delta} + \overline{\alpha})} \sum_{\eta \in H} \zeta^{\overline{\eta} \cdot (\overline{\delta} + \overline{\alpha})}.$$ 

The homomorphism from $G$ into $\mathbb{Z}/(q - 1)\mathbb{Z}$ sending $\overline{\eta} \in G$ into $\overline{\eta} \cdot \overline{\epsilon}$ is surjective, with kernel $H$; hence $|H| = (q - 1)^{n-1}$. Furthermore, $\overline{\eta} \rightarrow \zeta^{\overline{\eta} \cdot (\overline{\delta} + \overline{\alpha})}$ is a character of $H$. Therefore

$$\sum_{\overline{\eta} \in H} \zeta^{\overline{\eta} \cdot (\overline{\delta} + \overline{\alpha})} = \begin{cases} (q - 1)^{n-1} & \text{if } \overline{\eta} \cdot (\overline{\delta} + \overline{\alpha}) = \overline{0} \forall \overline{\eta} \in H; \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 1.1, $\overline{\eta} \cdot (\overline{\delta} + \overline{\alpha}) = \overline{0} \forall \overline{\eta} \in H$ if and only if there exists $\epsilon \in \mathbb{Z}^n$ such that $\omega(\delta + \alpha) = (q - 1)\omega(\epsilon)$ or equivalently $\omega(r \alpha) = (q - 1)\omega(r \epsilon + \rho)$. 


Thus $\eta \cdot (\delta + \alpha) = 0 \forall \eta \in H$ if and only if there exists $\beta \in E^\rho$ (necessarily unique) such that $\omega(r\alpha) = (q-1)\omega(\beta)$. If so,

$$\alpha_i - \rho_i \frac{(q-1)}{r} \equiv g_i[\omega(r\alpha) - (q-1)\omega(\beta)] \pmod{q-1}$$

for all $i$;

hence $\xi^{\gamma(\delta + \alpha)} = x^{\omega(r\alpha) - (q-1)\omega(\beta)}$.

\[ \square \]

**Lemma 1.5.** Let $F(t) \in \mathcal{L}(rb)$; then $\psi^/ \circ (\star F(t^q)) = \star F(t) \circ \psi^/$.

**Proof.** It is sufficient to check that, for a monomial $t^\beta$, $\beta \in \mathbb{Z}^n$:

$$\psi^/ (t^\alpha \star t^\beta) = t^\beta \star \psi^/ (t^\alpha) \quad \text{for all } \alpha \in E.$$

$$\psi^/ (t^\alpha \star t^\beta) = \begin{cases} 
\chi^\omega(\alpha + \beta - \gamma) t^\gamma & \text{if } \omega(\alpha + \beta) = q\omega(\gamma) \\
0 & \text{otherwise.} 
\end{cases}$$

Suppose that $\omega(q\beta + \alpha) = q\omega(\delta)$. Then $\omega(\alpha) = q\omega(\delta - \beta)$; let $\lambda \in \mathbb{Z}$ be such that $\delta - \beta + \lambda r = \gamma$ is an element of $E$:

$$\psi^/ (t^\alpha) = \chi^\omega(\alpha) t^\gamma,$$

$$t^\beta \star \psi^/ (t^\alpha) = \chi^\omega(\alpha + \beta) t^\gamma.$$

Suppose that $\sigma(\delta) = \delta_i / g_i$; Remark 1.1 shows that $\sigma(q\beta + \alpha) = (q\beta_i + \alpha_i) / g_i$. Thus,

$$\omega(q\beta + \alpha) - q\omega(\delta) = \frac{1}{rg_i} (q\beta_i + \alpha_i - q\delta_i) = \frac{1}{rg_i} (\alpha_i - q\gamma_i) + q\lambda.$$

Likewise, if $\sigma(\alpha) = \alpha_k / g_k$, then

$$\sigma(\gamma) = \frac{\gamma_k}{g_k} \quad \text{and} \quad \frac{1}{g_i} (\alpha_i - q\gamma_i) = \frac{1}{g_k} (\alpha_k - q\gamma_k).$$

\[ \square \]

**Corollary 1.1.**

$$(q^m - 1)^{n-1} \operatorname{tr}(\mathcal{F}_\rho^m | \mathcal{F}_\rho(qb))$$

$$= \sum_{t \in \mathcal{F}_m} \left( \prod_{i=1}^n t_i^{-(q^m - 1)p_i / r} \right) F(t) F(t^q) \cdots F(t^{q^{n-1}}).$$

\[ \square \]

2. Special subsets of $\mathbb{Z}^n$. Let $a = (a_1, \ldots, a_n)$ and $d = (d_1, \ldots, d_n)$ be two $n$-tuples of positive integers.
Let \( M = \text{l.c.m.}(a_1, \ldots, a_n) \) and \( D = \text{l.c.m.}(d_1, \ldots, d_n) \). If \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \) we let

\[(2.1) \quad s(\alpha) = \inf \left\{ \frac{\alpha_1}{a_1}, \ldots, \frac{\alpha_n}{a_n} \right\}.\]

Let \( J : \mathbb{Z}^n \to \frac{1}{D} \mathbb{Z} \) be the map defined by

\[(2.2) \quad J(\alpha) = \sum_{i=1}^{n} \frac{\alpha_i}{d_i}.\]

We define an equivalence relation on \( \mathbb{Z}^n \) by setting:

\[(2.3) \quad \alpha \sim \alpha' \text{ if and only if } \alpha_i \equiv \alpha'_i \pmod{d_i} \text{ for all } i = 1, \ldots, n.\]

There are \( \prod_{i=1}^{n} d_i \) equivalence classes, which we call "congruence classes"; if \( \alpha \in \mathbb{Z}^n \), we denote by \( \overline{\alpha} \) its congruence class.

Let

\[(2.4) \quad \Delta' = \left\{ \alpha \in \mathbb{Z}^n \mid s(\alpha) \leq \frac{\alpha_i}{a_i} \leq \frac{s(\alpha)}{a_i} + \frac{d_i}{a_i} \quad \forall i = 1, \ldots, n \right\}.\]

If \( \alpha \) and \( \beta \) are two elements of \( \Delta' \) we set

\[(2.5) \quad \begin{cases} \alpha \mathcal{R} \beta \text{ if and only if } \alpha \sim \beta \text{ and } J(\alpha) = J(\beta); \\ \Delta = \Delta' / \mathcal{R}. \end{cases}\]

We identify \( \Delta \) with the subset of \( \Delta' \) obtained by choosing, in each equivalence class for \( \mathcal{R} \), the first element in lexicographic order.

**Lemma 2.1.** Let \( \alpha \in \Delta \) and let \( \beta \in \mathbb{Z}^n \) be such that \( \beta \sim \alpha \) and \( J(\beta) = J(\alpha) \); then

\[s(\beta) \leq s(\alpha).\]

**Proof.** If \( \beta \neq \alpha \), there is an index \( i \) such that \( \beta_i < \alpha_i \). Since \( \beta \sim \alpha \), we have in fact \( \beta_i \leq \alpha_i - d_i \). Hence

\[\frac{\beta_i}{a_i} \leq \frac{\alpha_i}{a_i} - \frac{d_i}{a_i} \leq s(\alpha).\]

For each \( i \in \{1, \ldots, n\} \) we denote by \( U_i \) the element of \( \mathbb{Z}^n \) with 1 in the \( i \)-th position and 0 elsewhere.
Lemma 2.2. Let $K \in \frac{1}{\beta} \mathbb{Z}$ and let $\alpha$ be a congruence class in $\mathbb{Z}^n$ such that $\alpha \cap J^{-1}(K) \neq \emptyset$. Then there exists a unique element $\beta \in \Delta$ such that $\beta \in \alpha$ and $J(\beta) = K$.

Proof. Let $S(\alpha, K) = \max \{ s(\delta) | \delta \in \alpha \text{ and } J(\delta) = K \}$.

Pick $\delta \in \alpha$ with $J(\delta) = K$ and $s(\delta) = S(\alpha, K)$.

If $\delta_i/a_i \leq s(\delta) + d_i/a_i$ for all $i$, then $\delta \in \Delta'$ so $\Delta' \cap J^{-1}(K) \neq \emptyset$ and we are done.

Suppose now that $\delta_i/a_i > s(\delta) + d_i/a_i$ for some index $i$ and let $k$ be the index such that $\delta_k/a_k$ is maximum among those satisfying the last inequality. Let also $l$ be an index such that $s(\delta) = \delta_l/a_l$; note that necessarily $k \neq l$.

Let

$$
\gamma = \delta - d_k U_k + d_l U_l; \quad \frac{\gamma_k}{a_k} > s(\delta) \quad \text{and} \quad \frac{\gamma_l}{a_l} > s(\delta).
$$

Hence $s(\gamma) \geq s(\delta)$ and Lemma 2.1 implies $s(\gamma) = s(\delta)$.

Furthermore $\gamma_l/a_l = s(\gamma) + d_l/a_l$. Repeating the process if necessary, after a finite number of steps we obtain $\varepsilon \in \Delta' \cap \alpha$ with $J(\varepsilon) = K$. \qed

Notation. If $\beta$ satisfies the conditions of Lemma 2.2 we write

(2.6) \hspace{1cm} \beta = \tau(\alpha, K).

Let

(2.7) \hspace{1cm} N = J(a) = \sum_{i=1}^{n} \frac{a_i}{d_i}.

Observe that $\alpha \in \Delta \iff \alpha + a \in \Delta$. Thus, if $\alpha \cap J^{-1}(K) \neq \emptyset$:

(2.8) \hspace{1cm} \tau(\alpha, K) + a = \tau(\alpha + a, K + N).

Lemma 2.3. Let $K \in \frac{1}{\beta} \mathbb{Z}$ and let $\alpha$ be a congruence class in $\mathbb{Z}^n$ such that $\alpha \cap J^{-1}(K) \neq \emptyset$; let $\beta = \tau(\alpha, K), \delta = \tau(\alpha, K + 1)$; there exists an index $\lambda = \lambda(\alpha, K) \in \{1, \ldots, n\}$ such that $\beta = \delta - d_\lambda U_\lambda$. Furthermore $s(\beta) = \beta_\lambda/a_\lambda$.

Proof. Let

$$
s = \max \left\{ \frac{\delta_1 - d_1}{a_1}, \ldots, \frac{\delta_n - d_n}{a_n} \right\}
$$

(2.9) \hspace{1cm} \beta = \sum_{i=1}^{n} \frac{a_i}{d_i} \gamma_i.

Note that $\gamma_i \geq 0$ for all $i$ and $\gamma_\lambda < 0$ for some $\lambda$. Replacing $\gamma_i$ by $\gamma_i + m_i$ with $m_i$ so that $\gamma_\lambda = 0$, we have $s = \sum_{i=1}^{n} \frac{a_i}{d_i} m_i$.

Hence $\gamma_\lambda = -\sum_{i=1}^{n} \frac{a_i}{d_i} m_i > s$, or $\gamma_\lambda < s$. Repeating the process if necessary, after a finite number of steps we obtain $\varepsilon \in \Delta' \cap \alpha$ with $J(\varepsilon) = K$. \qed
and let \( l \) be the smallest index such that \( s = (\delta_l - d_l)/a_l \). Let \( \gamma = \delta - d_l U_j \): for all \( i \neq l \),
\[
\frac{\delta_i}{a_i} \geq s(\delta) \geq \frac{\delta_l - d_l}{a_l} = \frac{\gamma_l}{a_l}, \quad \text{hence } s(\gamma) = \frac{\gamma_l}{a_l} = s.
\]
Furthermore, for all \( i \neq l \), \( (\gamma_i - d_i)/a_i \leq s(\gamma) \) so \( \gamma \in \Delta' \). Suppose that there exists \( \varepsilon \in \Delta' \) such that \( \varepsilon \prec \gamma \) and \( \varepsilon \) precedes \( \gamma \) in the lexicographic ordering. Let \( j \) be the smallest index such that \( \varepsilon_j \neq \gamma_j \); then \( \varepsilon_j \leq \gamma_j - d_j \) and there exists \( k > j \) such that \( \varepsilon_k \geq \gamma_k + d_k \):
\[
s(\varepsilon) \leq \frac{\varepsilon_j}{a_j} \leq \frac{\gamma_j - d_j}{a_j} \leq s(\gamma),
\]
\[
s(\gamma) \leq \frac{\gamma_k}{a_k} \leq \frac{\varepsilon_k - d_k}{a_k} \leq s(\varepsilon).
\]
Hence \( s(\gamma) = s(\varepsilon) = s \), \( \varepsilon_j = \gamma_j - d_j \), \( \varepsilon_k = \gamma_k + d_k \); in particular \( s = (\gamma_j - d_j)/a_j \) so we must have \( j \neq l \); hence \( \varepsilon_j = \delta_j - d_j \) and therefore \( j > l \). Let now \( \delta' = \delta - d_j U_j + d_k U_k \):
\[
s \leq \frac{\varepsilon_j}{a_j} = \frac{\delta_j - d_j}{a_j} \leq s(\delta)
\]
\[
s(\delta) \leq \frac{\delta_k}{a_k} = \frac{\gamma_k}{a_k} = \frac{\varepsilon_k - d_k}{a_k} = s.
\]
Thus
\[
s = s(\delta') = s(\delta) = \frac{\delta_l}{a_l} = \frac{\delta_j - d_j}{a_j}.
\]
Furthermore,
\[
\frac{\delta_i}{a_i} = \frac{\delta_i}{a_i} \leq s(\delta') + \frac{d_i}{a_i} \quad \text{if } i \neq j, k, \quad \text{and } \frac{\delta_k}{a_k} = \frac{\delta_k + d_k}{a_k} = s(\delta') + \frac{d_k}{a_k}.
\]
Hence \( \delta' \in \Delta, \delta' \prec \delta \) and \( \delta' \) precedes \( \delta \) in the lexicographic ordering. This contradicts the choice of \( \delta \). Hence \( \gamma = \beta = \tau(\alpha, K) \) and \( l = \lambda(\alpha, K) \).
\[\square\]

We now let
\[
(2.9) \quad \tilde{\Delta} = \{ \alpha \in \Delta \mid 0 \leq s(\alpha) < 1 \}
\]
\[
(2.10) \quad \bar{\Delta} = \{ \alpha \in \Delta \mid 0 \leq J(\alpha) < N \}
\]

**Lemma 2.4.** \( |\tilde{\Delta}| = |\bar{\Delta}| \).

**Proof.** We construct two maps:
\[
i: \tilde{\Delta} \rightarrow \bar{\Delta}
\]
\[
i^*: \bar{\Delta} \rightarrow \tilde{\Delta}
\]
Let \( \alpha \in \tilde{\Delta} \): we can find \( \mu_{\alpha} \in \mathbb{N} \), \( r_{\alpha} \in \frac{1}{D}\mathbb{N} \), unique such that \( J(\alpha) = N\mu_{\alpha} + r_{\alpha} \) and we set:

\[
(2.11) \quad \iota(\alpha) = \alpha - \mu_{\alpha} a.
\]

Clearly, \( \iota(\alpha) \in \Delta \) with \( s(\iota(\alpha)) = s(\alpha) - \mu_{\alpha} \) and \( 0 \leq J(\iota(\alpha)) < N \); hence \( \iota(\alpha) \in \Delta \). If \( \beta \in \tilde{\Delta} \), there exist \( \nu_{\beta} \in \mathbb{N} \) and \( k_{\beta} < 1 \) unique such that \( s(\beta) = \nu_{\beta} + k_{\beta} \); we set:

\[
(2.12) \quad \iota^*(\beta) = \beta - \nu_{\beta} a.
\]

Clearly \( \iota^*(\beta) \in \Delta \) with \( 0 \leq s(\iota^*(\beta)) < 1 \), i.e. \( \iota^*(\beta) \in \tilde{\Delta} \).

It is now straightforward to check that \( \iota \) and \( \iota^* \) are inverse to each other. \( \square \)

**Lemma 2.5.** Let \( \delta = \frac{1}{D} \prod_{i=1}^{n} d_i \). If \( K \in \frac{1}{D}\mathbb{Z} \), then \( J^{-1}(K) \) meets exactly \( \delta \) congruence classes in \( \mathbb{Z}^n \).

*Proof.* Let \( G = \mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_n\mathbb{Z} \) and let \( H = \frac{1}{D}\mathbb{Z}/\mathbb{Z} \). \( J: \mathbb{Z}^n \rightarrow \frac{1}{D}\mathbb{Z} \) induces a group homomorphism:

\[
(2.13) \quad \overline{J}: G \rightarrow H.
\]

It is sufficient to prove that \( |\overline{J}^{-1}(h)| = \delta \) for any \( h \in H \). Let

\[
\delta_i = \prod_{\substack{1 \leq j \leq n \\backslash j \neq i}} d_j.
\]

Observe that \( \delta = \gcd(\delta_1, \ldots, \delta_n) \) and therefore there exist integers \( \alpha_1, \ldots, \alpha_n \) such that \( \delta = \sum_{i=1}^{n} \alpha_i \delta_i \). Dividing by \( \prod_{i=1}^{n} d_i \) we obtain \( \frac{1}{D} = \sum_{i=1}^{n} \alpha_i/d_i \), showing that \( \overline{J} \) is surjective. Hence, for \( h \in H \),

\[
|J^{-1}(h)| = \frac{|G|}{|H|} = \frac{\prod_{i=1}^{n} d_i}{D} = \delta. \quad \square
\]

**Lemma 2.6.** \( |\tilde{\Delta}| = N \prod_{i=1}^{n} d_i \).

*Proof.* By Lemma 2.5, \( J^{-1}(K) \cap \Delta \) has exactly \( \delta \) elements for each \( K \in \frac{1}{D}\mathbb{Z} \). Hence, using the definition of \( \tilde{\Delta} \), \( |\tilde{\Delta}| = N \prod_{i=1}^{n} d_i \). The conclusion follows from Lemma 2.4. \( \square \)

Let \( r \) be a fixed positive integer and let \( g = (g_1, \ldots, g_n) \), \( k = (k_1, \ldots, k_n) \) be \( n \)-tuples of positive integers, with \( \gcd(g_1, \ldots, g_n) = 1 \).
From now on we shall assume that $a_i = r g_i$ and $d_i = r k_i$ for all $i = 1, \ldots, n$. Thus, in (1.7) and (2.1):

\[(2.14)\quad s(\alpha) = s(\alpha) \quad \forall \alpha \in \mathbb{Z}^n.\]

If $\rho = (\rho_1, \ldots, \rho_n) \in \mathbb{Z}^n$, with $0 \leq \rho_i < r$ we let

\[\Delta_{\rho} = \{ \alpha \in \Delta \mid \alpha_i \equiv \rho_i \mod r \}; \]
\[\tilde{\Delta}_{\rho} = \tilde{\Delta} \cap \Delta_{\rho}; \]
\[\widehat{\Delta}_{\rho} = \widehat{\Delta} \cap \Delta_{\rho}.\]

**Lemma 2.7.** $|\widetilde{\Delta}_{\rho}| = |\widehat{\Delta}_{\rho}| = N \prod_{i=1}^n k_i$.

**Proof.** The map $i: \tilde{\Delta} \to \widehat{\Delta}$ of Lemma 2.4 restricts to a bijection between $\tilde{\Delta}_{\rho}$ and $\widehat{\Delta}_{\rho}$. Hence $|\tilde{\Delta}_{\rho}| = |\widehat{\Delta}_{\rho}|$. Let $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{Z}^n$, with $0 \leq \eta_i < r$. If $\alpha \in \tilde{\Delta}_{\rho}$ we let $\gamma = \alpha - \rho + \eta$. There is a unique integer $\lambda_{\alpha}$ such that $K_{\alpha} = J(\gamma) + \lambda_{\alpha} N$ satisfies $0 \leq K_{\alpha} < N$, and we set $F_{\rho, \eta}(\alpha) = \tau(\gamma + \lambda_{\alpha} a, K_{\alpha})$. $F_{\rho, \eta}$ maps $\tilde{\Delta}_{\rho}$ and $\tilde{\Delta}_{\rho}$ and is easily seen to be injective. Hence, the $r^n$ sets $\tilde{\Delta}_{\rho}$, $0 \leq \rho_i < r$, all have the same cardinality

\[|\tilde{\Delta}_{\rho}| = \frac{1}{r^n} |\Delta| = N \prod_{i=1}^n k_i. \quad \Box\]

**Lemma 2.8.** Let $p$ be a prime number, with $(p, a_i) = (p, d_i) = 1$ for all $i$; let $\rho \in \mathbb{Z}^n$, with $0 \leq \rho_i < r$ and let $\rho' \in \mathbb{Z}^n$ satisfying $0 \leq \rho'_i < r$ and $p \rho'_i - \rho_i \equiv 0 \pmod{r}$ $\forall i$. If $\alpha' \in \tilde{\Delta}_{\rho'}$, there exist $\alpha \in \tilde{\Delta}_{\rho}$ and integers $\delta_1, \ldots, \delta_n$ uniquely determined by the conditions:

\[\left\{ \begin{array}{l}
p \left( \frac{\alpha'_i}{d_i} - s(\alpha') \frac{a_i}{d_i} \right) - \left( \frac{\alpha_i}{d_i} - s(\alpha) \frac{a_i}{d_i} \right) = \delta_i, \\0 \leq \delta_i < p - 1.
\end{array} \right\}

Furthermore:

(i) Let $l \in \{1, \ldots, n\}$, then

\[s(\alpha) = \frac{\alpha_l}{a_l} \iff s(\alpha') = \frac{\alpha'_l}{a_l} \iff \delta_l = 0.\]

(ii) $\alpha' \mapsto \alpha$ is a bijection between $\tilde{\Delta}_{\rho'}$ and $\tilde{\Delta}_{\rho}$.

**Proof.** Certainly, using notation (1.4), there exists $\beta \in \mathbb{Z}^n$ such that $\omega(\beta) = p \omega(\alpha')$, and an argument similar to that of Lemma 1.4 shows
that \( \beta \) can be chosen uniquely in \( E^{(\rho)} \). Furthermore, if \( s(\alpha') = \alpha'_l/a_l \), then \( s(\beta) = \beta_l/a_l \). Since \( \alpha' \in \Delta \), we have
\[
0 \leq \frac{\alpha'_l}{a_l} - \frac{\alpha_l}{a_l} \leq \frac{d_i}{a_i},
\]
hence
\[
0 \leq \frac{\beta_l}{a_l} - \frac{\beta_l}{a_l} \leq p \frac{d_i}{a_i}
\]
for all \( i \).

If
\[
\frac{\beta_l}{a_l} - \frac{\beta_l}{a_l} < p \frac{d_i}{a_i},
\]
there is a unique integer \( \delta_i, 0 \leq \delta_i \leq p - 1 \), such that
\[
0 \leq \frac{\beta_l - \delta_i d_i}{a_i} - \frac{\beta_l}{a_l} < \frac{d_i}{a_i}.
\]
If
\[
\frac{\beta_l}{a_l} - \frac{\beta_l}{a_l} = p \frac{d_i}{a_i}
\]
we set \( \delta_i = p - 1 \).

Now let \( \alpha_i = \beta_i - \delta_i d_i \) for all \( i \). It is straightforward to check that \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( \delta = (\delta_1, \ldots, \delta_n) \) have the required properties. \( \square \)

**Lemma 2.9.** Let \( \rho = (\rho_1, \ldots, \rho_n) \in \mathbb{N}^n \), with \( 0 \leq \rho_i < r \). Then
\[
\sum_{\alpha \in \Delta_\rho} w(\alpha) = N \prod_{i=1}^n k_i \frac{(n-1)}{2}.
\]

**Proof.** Let \( G = \prod_{i=1}^n \mathbb{Z}/d_i \mathbb{Z} \) and let \( \rho: G \to (\mathbb{Z}/r \mathbb{Z})^n \) and \( \varphi: \mathbb{Z}^n \to G \) be the natural quotient maps. Let \( \bar{\rho} = \rho \circ \varphi(\rho) \) and \( K_{\rho} = \rho^{-1}(\bar{\rho}) \).

Note that
\[
|K_{\rho}| = \prod_{i=1}^n k_i, \quad \alpha \in \Delta_{\rho} \Leftrightarrow \alpha + a \in \Delta_{\rho} \quad \text{and} \quad \bar{\eta} \in K_{\rho} \Leftrightarrow \bar{\eta} + \varphi(\alpha) \in K_{\rho}.
\]

Let \( H \) be the cyclic subgroup of \( G \) generated by \( \varphi(\alpha) \) and let \( \{G_l\}_{l=1}^{(G:H)} \) be the orbits of \( G \) under addition by elements of \( H \): \( G = \bigsqcup_{l=1}^{(G:H)} G_l \).

We have \( K_{\rho} = \bigsqcup_{\alpha \in \Delta_{\rho}} G_l \) and \( \overline{\Delta}_{\rho} = \bigsqcup_{l=1}^{(G:H)} \overline{\Delta}_{\rho}(l) \), where \( \overline{\Delta}_{\rho}(l) = \{\alpha \in \overline{\Delta} | \varphi(\alpha) \in K_{\rho} \cap G_l\} \).

Let \( l \) be such that \( K_{\rho} \cap G_l \neq \emptyset \) and let \( \eta \in \overline{\Delta}_{\rho}(l) \) be such that \( J(\eta) \) is minimum. Let \( \varepsilon = |H| \); \( \varepsilon \) is the smallest integer such that
\( \varepsilon a_i \equiv 0 \pmod{d_i} \) for all \( i \). For any \( \alpha \in \Delta_p(l) \), there is a unique integer \( \mu \in \mathbb{N} \) such that \( 0 \leq \mu < \varepsilon \) and \( \alpha + \mu a_i \equiv \eta_i \pmod{d_i} \) for all \( i \), and we have \( J(\eta) \leq J(\alpha + \mu a) < J(\eta) + \varepsilon N \). Conversely, if \( \beta \in \Delta \) satisfies \( J(\eta) \leq J(\beta) < J(\eta) + \varepsilon N \) and \( \beta_i \equiv \eta_i \pmod{d_i} \) for all \( i \), there is a unique \( \nu \in \mathbb{N}, 0 \leq \nu < \varepsilon \) such that \( J(\eta) + \nu N \leq J(\beta) < J(\eta) + (\nu + 1)N \).

Let \( \gamma = \beta - \nu a \); then \( J(\eta) \leq J(\gamma) < J(\eta) + N \). If \( J(\gamma) \geq N \), then \( J(\gamma - a) \geq 0 \) and \( J(\gamma - a) < J(\eta) \), contradicting the minimality of \( J(\eta) \). Hence \( \gamma \in \Delta \).

Let \( D_p(l) = \{ \alpha \in \Delta | \alpha_i \equiv \eta_i \pmod{d_i} \forall i \text{ and } J(\eta) \leq J(\alpha) < J(\eta) + \varepsilon N \} \). Since \( w(\alpha + a) = w(\alpha) \) for all \( \alpha \in \mathbb{Z}^n \) we deduce that:

\[
\sum_{\alpha \in \Delta_p} w(\alpha) = \sum_{\alpha \in \Delta_p} w(\alpha) = \sum_{l=1}^{(G:H)} \sum_{\alpha \in D_p(l)} w(\alpha).
\]

It follows from Lemma 2.3 that \( D_p(l) = \{ \tau(\eta, J(\eta) + k) \mid 0 \leq k \leq \varepsilon N - 1 \} \). For each \( k \in \mathbb{N} \), let \( \alpha^{(k)} = \tau(\eta, J(\eta) + k) \), \( s_k = s(\alpha^{(k)}) \), \( J_k = J(\alpha^{(k)}) = J_0 + k \), \( \lambda_k = \lambda(\eta, J_k) \). By Lemma 2.3, \( \alpha^{(k)} = \alpha^{(k-1)} + d_{\lambda_k} U_{\lambda_k} \) and \( s_k = \alpha^{(k)}_{\lambda_k+1} / a_{\lambda_k+1} \). For each \( i \in \{1, \ldots, n\} \) let \( \mu_i \) be the integer satisfying \( \varepsilon a_i = \mu_i d_i \). Since \( \alpha^{(\varepsilon N)} = \eta + \varepsilon a \), it follows that \( \varepsilon a = \sum_{k=1}^{\varepsilon N} d_{\lambda_k} U_{\lambda_k} \) and \( \mu_i = \# \{ k \mid 1 \leq k \leq \varepsilon N \text{ and } \lambda_k = i \} \).

We have

\[
\sum_{k=0}^{\varepsilon N-1} s_i = \sum_{j=1}^{n} \sum_{\lambda_k = j} \alpha^{(k)}_{\lambda_k+1} / a_j = \sum_{j=1}^{n} \frac{1}{a_j} \left( \sum_{\nu=0}^{\mu_j-1} \eta_j + \nu d_j \right)
\]

\[
= \sum_{j=1}^{n} \left[ \frac{\mu_j}{a_j} \left( \eta_j + \frac{\mu_j - 1}{2} d_j \right) \right]
\]

\[
= \varepsilon \sum_{j=1}^{n} \left( \frac{\mu_j}{d_j} + \frac{\mu_j - 1}{2} \right) = \varepsilon \left( J_0 + \frac{\varepsilon N - n}{2} \right).
\]

On the other hand:

\[
\sum_{k=0}^{\varepsilon N-1} J_k = \varepsilon NJ_0 + \frac{N(\varepsilon N - 1)}{2}.
\]

Thus

\[
\sum_{\alpha \in D_p(l)} w(\alpha) = \sum_{k=0}^{\varepsilon N-1} (J_k - NS_k)
\]

\[
= \varepsilon N \left( \frac{n-1}{2} \right) = |K_p \cap G_l| N \left( \frac{n-1}{2} \right).
\]
Hence

\[ \sum_{\alpha \in \Delta_p} w(\alpha) = |K_p| N \frac{(n - 1)}{2}. \]

\[ \square \]


a. Definitions. Let \( K_r \) be the unramified extension of \( \mathbb{Q}_p \) in \( \Omega \) of degree \( r \), \( \zeta_p \in \Omega \) a primitive \( p \)-th root of unity, \( \Omega_0 = K_r(\zeta_p) \) and let \( \tau \in \text{Gal}(\Omega_0 \mid \mathbb{Q}_p(\zeta_p)) \) denote the Frobenius automorphism. Let \( \mathcal{O}_0 \) be the ring of integers of \( \Omega_0 \).

Let \( M = \text{l.c.m.}(a_1, \ldots, a_n) \) and, for \( m \in \mathbb{N}^* \):

(3.1) \( S_m = \{(\alpha; \gamma) \in \mathbb{N}^n \times \mathbb{Z} \mid \gamma \geq -mMs(\alpha) \}; \)

(3.2) \( E_m = \{(\alpha; \gamma) \in E \times \mathbb{Z} \mid \gamma \geq -mMs(\alpha) \}; \)

(3.3) \( A_m = \Omega_0\text{-algebra generated by } \{t^\alpha Y^\gamma \mid (\alpha; \gamma) \in S_m \}; \)

(3.4) \( P^{(m)} = t^a Y^{-mM} - 1 \);

(3.5) \( \overline{A}_m = A_m/(P^{(m)}); \)

(3.6) \( \mathcal{R}_m = \Omega_0\text{-span of } \{t^\alpha Y^\gamma \mid (\alpha; \gamma) \in E_m \}. \)

If \( \alpha \in \mathbb{Z}^n, \ \gamma \in \mathbb{Z} \), we set:

(3.7) \[ w_m(\alpha; \gamma) = J(\alpha) + \frac{N\gamma}{mM}. \]

Remarks.

(3.8) \[ w_m(\alpha; \gamma) \geq 0 \text{ for all } (\alpha; \gamma) \in S_m \]

(3.9) If \( W \in \mathbb{Q} \), the set \( \{(\alpha; \gamma) \in E_m \mid w_m(\alpha; \gamma) = W \} \) is finite.

If \( \alpha, \beta \in \mathbb{Z}^n \), there exist \( \delta = \delta(\alpha, \beta) \in E \), \( \lambda = \lambda(\alpha, \beta) \in \mathbb{Z} \) unique, such that \( \alpha + \beta = \delta + \lambda \alpha \) and we set:

(3.10) \[ t^\alpha *_m t^\beta = Y^{\lambda mM} t^\delta. \]

If \( (\alpha; \gamma) \) and \( (\beta; \varepsilon) \) are two elements of \( S_m \), \( \delta = \delta(\alpha, \beta), \ \lambda = \lambda(\alpha, \beta) \) as above, then \( (\delta, \gamma + \varepsilon + \lambda) \in E_m \). In particular, the operation \(*_m\) makes \( \mathcal{R}_m \) into an \( \Omega_0[Y] \) algebra and, if we set

(3.11) \[ \Phi_m(t^\alpha) = t_1^{\alpha_1} *_m t_2^{\alpha_2} *_m \cdots *_m t_n^{\alpha_n} \quad (\alpha \in \mathbb{Z}^n), \]

then \( \Phi_m \) extends to an \( \Omega_0[Y] \)-algebra homomorphism \( \Phi_m: A_m \to \mathcal{R}_m \).

Furthermore, \( \Phi_m \) induces an \( \Omega_0[Y] \)-algebra isomorphism.

(3.12) \[ \Phi_m: \overline{A}_m \xrightarrow{\sim} \mathcal{R}_m. \]
$A_m, \overline{A}_m, \mathcal{R}_m$ are graded algebras with
\begin{equation}
 w_m(Y^\gamma t^\alpha) = w_m(\alpha; \gamma).
\end{equation}

Both $\Phi_m$ and $\phi_m$ are homogeneous of degree 0.

**Note.** When no confusion can arise, we shall omit the subscript “$m$” and write $\ast$ instead of $\ast_m$.

For $b, c \in \mathbb{R}, b \geq 0$, let
\begin{equation}
 L(b, c) = \{ \eta = \sum A(\alpha)t^\alpha \mid \alpha \in \mathbb{N}^n, \ A(\alpha) \in \Omega_0, \ \text{ord} \ A(\alpha) \geq bJ(\alpha) + c \};
\end{equation}

\begin{equation}
 L(b) = \bigcup_{c \in \mathbb{R}} L(b, c).
\end{equation}

$L(b)$ and $L(b, c)$ are $p$-adic Banach spaces with the norm
\begin{equation}
 ||\eta|| = \sup_{\alpha} p^{-c_\alpha}, \quad c_\alpha = \text{ord} A(\alpha) - bJ(\alpha).
\end{equation}

Let
\begin{equation}
 L_m(b, c) = \{ \xi = \sum B(\alpha; \gamma)t^\alpha Y^\gamma \mid (\alpha; \gamma) \in E_m, \ B(\alpha; \gamma) \in \Omega_0, \ \text{ord} B(\alpha; \gamma) \geq bw_m(\alpha; \gamma) + c \};
\end{equation}

\begin{equation}
 L_m(b) = \bigcup_{c \in \mathbb{R}} L_m(b, c).
\end{equation}

$L_m(b)$ and $L_m(b, c)$ are $p$-adic Banach spaces with the norm
\begin{equation}
 ||\xi||_m = \sup_{(\alpha; \gamma)} p^{-c_{\alpha, \gamma}}, \quad c_{\alpha, \gamma} = \text{ord} B(\alpha; \gamma) - bw_m(\alpha; \gamma).
\end{equation}

Let
\begin{equation}
 R_m(b, c) = \Omega_0[[Y]] \cap L_m(b, c),
\end{equation}
\begin{equation}
 R_m(b) = \Omega_0[[Y]] \cap L_m(b) = \bigcup_{c \in \mathbb{R}} R_m(b, c).
\end{equation}

The operation $\ast_m$ described in (3.10) makes $L_m(b)$ into an $R_m(b)$-algebra. (3.9) ensures that this is well defined. Furthermore, if $\eta \in L_m(b)$, the mapping $\xi \mapsto \eta \ast_m \xi$ is a continuous endomorphism of $L_m(b)$. Note that $L_m(b)$ is the completion of $\mathcal{R}_m$ for the norm $|| \ ||_m$. 
For each $c \in \mathbb{R}$, there is a continuous $\Omega_0$-linear map from $L(b, c)$ into $L_m(b, c)$ whose action on monomials is given by (3.11). This map will again be denoted $\Phi_m$.

Let $\bar{c}_1, \ldots, \bar{c}_n$ be non-zero elements of $F_q$ and, for each $i$ let $c_i$ be the Teichmüller representative of $\bar{c}_i$ in $\Omega_0$ (so $c_i^q = c_i$).

Let:

(3.22) $f(t) = \sum_{i=1}^{n} c_i t_i^{k_i}.$

Let $\{\gamma_j\}_{j=0}^{\infty}$ be a sequence of elements of $\mathbb{Q}_p(\zeta_p)$ such that

(3.23)

\[
\begin{cases}
\text{ord } \gamma_0 = \frac{1}{p-1}, \\
\text{ord } \gamma_j \geq \frac{p^{j+1}}{p-1} - (j + 1), \quad j \geq 1.
\end{cases}
\]

If $t^a Y^\gamma$ is a monomial, we set

(3.24) $E_i(t^a Y^\gamma) = \left(\frac{\alpha_i}{a_i} - \frac{\alpha_n}{a_n}\right) t^a Y^\gamma,$ $i = 1, \ldots, n - 1.$

Note that $E_i(t^a \ast t^b) = E_i(t^a) \ast t^b + t^a \ast E_i(t^b)$ so that $E_i$ acts as a derivation on all the rings and Banach spaces which have been defined so far.

Let

(3.25) $H(t) = \gamma \circ f(t').$

(3.26) $H(t) = \sum_{l=0}^{\infty} \gamma_l f^{(r)}(t^r p') = \sum_{l=0}^{\infty} \gamma_l \left( \sum_{i=1}^{n} c_i^{p^l} t_i^{p^l d_i} \right);$

(3.27) $H_i = E_i(\bar{H}(t)) = \gamma_0 \left( \frac{d_i}{a_i} t_i^{d_i} - c_n \frac{d_n}{a_n} t_n^{d_n} \right),$ $i = 1, \ldots, n - 1;$

(3.28) $H_i = E_i H(t),$ $i = 1, \ldots, n - 1;$

(3.29) $D_i = E_i + H_i,$ $i = 1, \ldots, n - 1.$

From now on we assume:

(3.30) $\text{g. c. d.}(p, M) = \text{g. d. c.}(p, D) = 1,$

and we let

(3.31) $\epsilon_i = c_i \frac{d_i}{a_i}, \quad i = 1, \ldots, n.$

Each $\epsilon_i$ is therefore a unit in $\mathcal{O}_0$.

Let $e = b - 1/(p-1)$: we have $\bar{H}_i \in L(b, -e)$ and $\bar{H}_i \in L_m(b, -e) \forall m$.

Also, if $b \leq p/(p - 1)$, $H_i \in L(b, -e)$ and $H_i \in L_m(b, -e) \forall m$.

b. Reduction.
Lemma 3.1. Let $\alpha \in \mathbb{N}^n$, $K = J(\alpha)$, $\beta = \tau(\alpha, K)$; then $t^\alpha = u(\alpha) t^\beta + \gamma_0^{-1} \sum_{i=1}^{n-1} H_i p_{i,\alpha}$, where $u(\alpha) \in \mathcal{O}_0$ is a unit and, for each $i$, $p_{i,\alpha} \in \mathcal{O}_0[t_1, \ldots, t_n]$. Furthermore, $p_{i,\alpha}$ has unit coefficients and, if $t^\delta$ is any monomial of $p_{i,\alpha}$ having non-zero coefficient, then

(i) $J(\delta) = J(\alpha) - 1$

(ii) $s(\delta) \geq s(\alpha)$.

Proof. If $\delta \in \mathbb{Z}^n$, we can write

$$t^\delta = \epsilon_j e_i^{-1} t^{\alpha-d_i U_i + d_i U_j} + \gamma_0^{-1} e_i^{-1} (H_i - H_j) t^{\alpha-d_i U_i}, \quad i, j = 1, \ldots, n-1;$$

$$t^\delta = \epsilon_n e_i^{-1} t^{\alpha-d_i U_i + d_n U_n} + \gamma_0^{-1} e_i^{-1} H_i t^{\alpha-d_i U_i}, \quad i = 1, \ldots, n-1.$$

By assumption, there are integers $\lambda_1, \ldots, \lambda_n$ such that $\alpha = \beta + \sum_{i=1}^n \lambda_i d_i U_i$, with $\sum_{i=1}^n \lambda_i = 0$. The result follows immediately, except maybe for (ii): if $\alpha \neq \beta$, there is an index $i$ such that $\lambda_i > 0$; hence $\alpha_i \geq \beta_i + d_i$. Thus $(\alpha_i - d_i)/a_i \geq \beta_i/a_i \geq s(\beta)$ and $s(\beta) \geq s(\alpha)$ since $\beta \in \Delta$. \qed

Lemma 3.2. Let $Y^\gamma t^\alpha$ be a monomial in $\mathcal{R}_m$ and let $\tilde{\alpha} \in \tilde{\Delta}$, $\tau \in \mathbb{N}$, satisfying $\alpha \sim \tilde{\alpha} + \tau \alpha$ and $J(\alpha) = J(\tilde{\alpha}) + \tau N$. Then

$$Y^\gamma t^\alpha = u(\alpha) Y^\gamma t^\beta + \gamma_0^{-1} \sum_{i=1}^{n-1} H_i q_{i,\alpha,\gamma},$$

where $u(\alpha) \in \mathcal{O}_0$ is a unit and, for each $i$, $q_{i,\alpha,\gamma} \in \mathcal{R}_m$. Furthermore, each $q_{i,\alpha,\gamma}$ has unit coefficients and, if $Y^\delta t^\varepsilon$ is a monomial of $q_{i,\alpha,\gamma}$ with non-zero coefficient, then $w_m(\varepsilon; \delta) = w_m(\alpha; \gamma) - 1$.

Proof. Using Lemma 3.1 we can write:

$$Y^\gamma t^\alpha = u(\alpha) Y^\gamma t^\beta + \gamma_0^{-1} \sum_{i=1}^{n-1} H_i p_{i,\alpha,\gamma},$$

where $\beta$ is the unique element of $\Delta$ such that $\beta \mathcal{R} \alpha$, and $p_{i,\alpha,\gamma} = Y^\gamma p_{i,\alpha}$. Let $t^\delta$ be a monomial of $p_{i,\alpha}$ with non-zero coefficient:

Lemma 3.2 (ii) $\Rightarrow \gamma \geq -mM s(\delta)$ so that $p_{i,\alpha,\gamma} \in A_m$ and equation (3.32) is valid in $A_m$.

Applying the map $\Phi_m: A_m \rightarrow \mathcal{R}_m$ to equation (3.32) we obtain the desired result with $q_{i,\alpha,\gamma} = \Phi_m(p_{i,\alpha,\gamma})$. \qed

Let $V_m(b)$ be the $\mathcal{R}_m(b)$-vector space generated by

$$\{Y^{-M s(\alpha)} t^\alpha | \alpha \in \tilde{\Delta}\},$$

and let $V_m(b, c) = V_m(b) \cap L_m(b, c)$.
PROPOSITION 3.1.

\[ L_m(b, c) = V_m(b, c) + \sum_{i=1}^{n-1} H_i * L_m(b, c + e). \]

Proof. Let \( \xi = \sum_{(\alpha; \gamma) \in E_m} A(\alpha; \gamma) t^\alpha Y^\gamma \in L_m(b, c) \). We apply Lemma 3.2 to all the monomials in \( \xi \).

If \( \tilde{\alpha} \in \tilde{\Delta} \) and \( \nu \geq -mMs(\tilde{\alpha}) \) we let

\[ B^\sim(\nu) = A(\alpha; \gamma) u(\alpha), \]

where \( u(\alpha) \) has been defined in Lemma 3.2 and the sum is taken over the set

\[ E(\tilde{\alpha}, \nu) = \{ (\alpha; \gamma) \in E_m \mid \nu = \mu M + \gamma, \alpha \sim \tilde{\alpha} + \mu a, J(\alpha) = J(\tilde{\alpha}) + \mu N \}. \]

If \( (\alpha, \gamma) \in E(\tilde{\alpha}, \nu) \), then \( w_m(\alpha; \gamma) = w_m(\tilde{\alpha}; \nu) \); hence by (3.9) the sum (3.33) is finite and \( \text{ord} B^\sim(\nu) \geq bw_m(\tilde{\alpha}; \nu) + c. \)

Thus, for each \( \tilde{\alpha} \in \tilde{\Delta} \), \( B^\sim(\nu) t^{\tilde{\alpha}} = \sum_{\nu \geq -mMs(\tilde{\alpha})} B^\sim(\nu) Y^\nu t^{\tilde{\alpha}} \) is an element of \( V_m(b, c) \). On the other hand, let \( \zeta_i = \gamma_0^{-1} \sum_{(\alpha; \gamma) \in E_m} A(\alpha; \gamma) q_i, \alpha, \gamma \) and write

\[ \zeta_i = \sum_{(\beta, \nu) \in E_m} C_i(\beta; \nu) t^\beta Y^\nu, \quad i = 1, \ldots, n - 1. \]

If \( (\alpha; \gamma) \in E_m \) we can write \( q_i, \alpha, \gamma = \sum D_{i, \alpha, \gamma}(\varepsilon; \delta) t^\varepsilon Y^\delta \), the sum being taken over all \( (\varepsilon; \delta) \in E_m \) such that \( w_m(\varepsilon; \delta) = w_m(\alpha; \gamma) - 1 \). Thus

\[ C_i(\beta; \nu) = \gamma_0^{-1} \sum D_{i, \alpha, \gamma}(\beta, \nu) A(\alpha; \gamma), \]

the sum being over the set \( \{ (\alpha; \gamma) \in E_m \mid w_m(\alpha; \gamma) = w_m(\beta; \nu) + 1 \} \). This set is finite and

\[ \text{ord} C_i(\beta; \nu) \geq b[w_m(\beta; \nu) + 1] + c - \frac{1}{p - 1} = bw_m(\beta; \nu) + c + e. \]

Hence the sum (3.34) is meaningful, \( \zeta_i \in L_m(b, c + e) \), and we can write

\[ \xi = \sum_{\alpha \in \tilde{\Delta}} B^\sim(\nu) t^{\tilde{\alpha}} + \sum_{i=1}^{n-1} H_i * \zeta_i. \]

c.

PROPOSITION 3.2. \( V_m(b) \cap \sum_{i=1}^{n-1} H_i * L_m(b) = (0) \).
Proof. Let \( v \in V_m(b) \). For \( W \in \Omega \) we let \( v^{(W)} \) be the component of \( v \) which is of homogeneous weight \( W \): we can write \( v^{(W)} = \sum_{\alpha \in \Lambda} P_\alpha(Y)t^\alpha \), where each \( P_\alpha(Y) \) is a Laurent polynomial in \( Y \).

Let \( i: \overline{\Lambda} \to \Lambda \) be the map described in the proof of Lemma 2.4. Let \( Z = Y^{m,M} \) and, for \( \alpha \in \overline{\Lambda} \) let \( \beta = i(\alpha) = \alpha - \tau \alpha \ (\tau \in \mathbb{N}) \):

\[
t^\alpha = Z^\tau t^\beta + (t^a - Z)(t^{\alpha-a} + Zt^{\alpha-2a} + \cdots + Z^{\tau-1}t^{\alpha-\tau a}).
\]

Hence we can write:

\[
v^{(W)} = \sum_{\beta \in \overline{\Lambda}} Q_\beta(Y)t^\beta + (t^a - Z) \sum_{\beta \in \overline{\Lambda}} R_\beta(t, Y),
\]

where for each \( \beta \), \( Q_\beta(Y) \) is a Laurent polynomial in \( Y \) and \( R_\beta(t, Y) \) is a Laurent polynomial in \( Y, t_1, \ldots, t_n \). Furthermore:

(i) if \( y \in \Omega^x \) and \( \alpha \in \overline{\Lambda} \), then \( P_\alpha(y) = 0 \Leftrightarrow Q_i(\alpha)(y) = 0 \);

(ii) if \( Y^\tau t^\delta \) is any monomial in \( R_\beta(t, Y) \) with non-zero coefficient, then \( J(\delta) \geq 0 \).

Suppose \( v \in \sum_{i=1}^{n-1} H_i \ast L_m(b) \): we can write

\[
v^{(W)} = \sum_{i=1}^{n-1} H_i \ast \zeta_i,
\]

where, for each \( i \), \( \zeta_i \in \Omega_0[Y, 1/Y, t_1, \ldots, t_n] \) and is of homogeneous weight \( W - 1 \).

Let \( \alpha, \beta \in E \) and suppose \( \alpha + \beta = \delta + \tau \alpha \), with \( \delta \in E \) and \( \tau \in \mathbb{N} \):

\[
t^{\alpha} \ast_m t^\beta = t^{\alpha+\beta} - (t^{\alpha+\beta-a} + Zt^{\alpha+\beta-2a} + \cdots + Z^{\tau-1}t^{\alpha+\beta-\tau a})(t^a - Z).
\]

Hence we can write

\[
H_i \ast \zeta_i = H_i \zeta_i + \eta_i(t^a - Z), \quad \text{with } \eta_i \in \Omega_0\left[Y, 1/Y, t_1, \ldots, t_n\right].
\]

For each \( i = 1, \ldots, n \), fix \( \xi_i \in \Omega \) with \( \xi_i^{d_i} = e_n e_i^{-1} \) and let \( \mu_{d_i} \) be the group of \( d_i \)-th roots of unity in \( \Omega \).

Let \( s_i = \prod_{j \neq i} d_j \), \( s = \prod_{j=1}^n d_j \). Let \( \hat{v}(Y, t) = \sum_{\beta \in \overline{\Lambda}} Q_\beta(Y)t^\beta \) and suppose \( v^{(W)} \neq 0 \): there exists \( \alpha \in \overline{\Lambda} \) such that \( P_\alpha(Y) \neq 0 \); hence there exists \( \beta = i(\alpha) \in \overline{\Lambda} \) such that \( Q_\beta(Y) \neq 0 \). For such a fixed \( \beta \) let \( \overline{\Lambda}(\beta) = \{ \gamma \in \overline{\Lambda} \mid J(\gamma) = J(\beta) \} \) and let \( y \in \Omega^x \) such that \( Q_\beta(y) \neq 0 \).

We claim that there exists \( (\zeta_1, \ldots, \zeta_n) \in \prod_{i=1}^n \mu_{d_i} \) such that

\[
(3.37) \quad \hat{v}(y, u_1, \ldots, u_n) \neq 0,
\]

where \( u_i = \xi_i \zeta_i t_i^{\delta_i} \), \( i = 1, \ldots, n \).
Indeed, the coefficient of $t^sJ(\beta)$ in (3.37) is

$$\sum_{\gamma \in \Delta(\beta)} Q_\gamma(y) \zeta_1^{\gamma_1} \cdots \zeta_n^{\gamma_n}.$$

For each $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Delta(\beta)$, $\chi_\gamma: (\zeta_1, \ldots, \zeta_n) \mapsto \zeta_1^{\gamma_1} \cdots \zeta_n^{\gamma_n}$ is a character of $\prod_{i=1}^n \mu_d$.

The elements of $\Delta(\beta)$ all belong to distinct congruence classes, so these characters are all distinct, and therefore linearly independent. Our claim follows since $Q_\beta(y) \neq 0$.

Let now

$$S(Y; t) = \sum_{i=1}^n \eta_i - \sum_{\delta \in \Delta} R_\delta(Y; t),$$

$$u = \prod_{i=1}^n (\xi_i \zeta_i)^{a_i} \quad \text{and} \quad A = \sum_{i=1}^n a_i r_i = N \prod_{i=1}^n d_i.$$

We have:

(3.38) \hspace{1cm} \hat{v}(y; u_1, \ldots, u_n) = (ut^A_n - y^{mM})S(y; u_1, \ldots, u_n).

The left-hand side of (3.38) is a non-zero polynomial in $t_n$, of degree less than $A$, while the right-hand side vanishes for any choice of $t_n$ satisfying $t^A_n = u^{-1}y^{mM}$, a contradiction. Hence $v(W) = 0$. \hfill \Box

**Lemma 3.3.** Let $K$ be a field of arbitrary characteristic, $u_1, \ldots, u_n$ elements of $K^\times$, $\nu_1, \ldots, \nu_n$, $\lambda$ positive integers; let

$$B = K[t_1, \ldots, t_n, Y, Y^{-1}t^\lambda], \quad f = (Y^{-1}t^\lambda)^\lambda - 1,$$

$$\overline{B} = B/(f), \quad h_i = u_it_i^{\nu_i} - u_n^{\nu_n} (i = 1, \ldots, n-1); \text{ then the family } \{h_i\}_{i=1}^{n-1} \text{ in any order forms a regular sequence on } \overline{B}.$$

**Proof.** Let $I \subset \{1, \ldots, n-1\}$ and let $\mathfrak{A}_I$ be the ideal of $\overline{B}$ generated by $\{h_i\}_{i \in I}$. We must show that $(\mathfrak{A}_I; h_k) = \mathfrak{A}_I$ for any $k \notin I$. By relabelling we may assume that $I = \{1, \ldots, j\}$, with $j < n-1$, and that $k = j + 1$. Accordingly, we write $\mathfrak{A}_j$ instead of $\mathfrak{A}_I$. Let $B_1 = K[t_1, \ldots, t_n, Y, Z]$ and $\overline{B}_1 = B_1/(Z^\lambda - 1, YZ - t^a)$.

The mapping $Z \mapsto Y^{-1}t^a$ induces a ring isomorphism from $\overline{B}_1$ into $\overline{B}$. Thus, if $\mathfrak{B}_j$ is the ideal of $B_1$ generated by $\{h_1, \ldots, h_j, Z - 1, YZ - t^a\}$, we must show that $(\mathfrak{B}_j; h_{j+1}) = \mathfrak{B}_j$, or equivalently that $h_{j+1}$ does not belong to any associated prime of $\mathfrak{B}_j$. Since $\mathfrak{B}_j$ has $j + 2$ generators, its dimension is at least $n - j$. On the other hand,
the ring $B_1/\mathcal{B}_j$ is integral over $K[t_{j+1}, \ldots, t_n]$ (note that $Y^\lambda - t^{\lambda a} = 0$ in $B_1/\mathcal{B}_j$). Hence $\dim \mathcal{B}_j = n - j$. By Macaulay’s theorem [16, Ch. VII, §8], $\mathcal{B}_j$ is unmixed. Likewise, $\mathcal{B}_{j+1} = (\mathcal{B}_j, h_{j+1})$ is unmixed, of dimension $n - j - 1$. Let $p$ be an associated prime of $\mathcal{B}_j$ and suppose that $h_{j+1} \in p:p \supset (\mathcal{B}_j, h_{j+1}) = \mathcal{B}_{j+1}$; hence $\dim p \leq n - j - 1$, a contradiction since $\dim p = n - j$.

Let

\begin{align*}
R &= \Omega_0[t_1, \ldots, t_n, Y, Y^{-1}t^a] \\
f(m) &= (Y^{-1}t^a)^m M - 1 \\
\overline{R}^{(m)} &= R/f(m) \\
h_i^{(m)} &= \varepsilon_i t_i^{m M d_i} - \varepsilon_n t_n^{m M d_n}, \quad i = 1, \ldots, n - 1.
\end{align*}

For any monomial $t^\alpha Y^\gamma$ we set:

\begin{equation}
\tilde{w}_m(\alpha; \gamma) = \tilde{w}_m(t^\alpha Y^\gamma) = \frac{1}{m M}(J(\alpha) + N \gamma).
\end{equation}

\tilde{w}_m makes $\overline{R}^{(m)}$ into a graded ring, and each $h_i^{(m)}$ is homogeneous of weight 1.

**Lemma 3.4.** Let $I$ be a non-empty subset of $\{1, \ldots, n - 1\}$ and let $\{P_i\}_{i \in I}$ be a family of elements of $\overline{R}^{(m)}$ such that $\sum_{i \in I} P_i h_i^{(m)} = 0$. Then there exists a skew-symmetric set $\{\eta_{i,j}\}_{i, j \in I}$ such that $P_i = \sum_{j \in I} \eta_{i,j} h_j^{(m)}$ for each $i \in I$. Furthermore, if each $P_i$ is of homogeneous weight $\tilde{w}_m(P_i) = W$ independent of $i$:

(a) if $W \geq 1$, each $\eta_{i,j}$ may be chosen of homogeneous weight $\tilde{w}_m(\eta_{i,j}) = W - 1$ with $\min_{j \in I}\{\ord \eta_{i,j}\} \geq \ord P_i$ for all $i \in I$;

(b) if $W < 1$ then $P_i = 0$ for all $i \in I$ (i.e. each $\eta_{i,j}$ may be chosen to be zero).

**Proof.** To simplify notation, we write $h_i$ instead of $h_i^{(m)}$. We proceed by induction on the number of elements in $I$. By relabelling, we may assume that $I = \{1, \ldots, r + 1\}$, $r \geq 0$. If $r = 0$, then $P_i = 0$ and hence we can assume $r \geq 1$. Let $\mathfrak{A}_r$ be the ideal of $\overline{R}^{(m)}$ generated by $\{h_i\}_{i=1}^r$; by Lemma 3.3, $(\mathfrak{A}_r, h_{r+1}) = \mathfrak{A}_r$; hence $P_{r+1} \in \mathfrak{A}_r$. Thus there exist $y_1, \ldots, y_r \in \overline{R}^{(m)}$ such that

\begin{equation}
P_{r+1} = \sum_{i=1}^r y_i h_i.
\end{equation}
Now
\[
\sum_{i=1}^{r}(P_i + y_i h_{r+1})h_i = \sum_{i=1}^{r} P_i h_i + \left( \sum_{i=1}^{r} y_i h_i \right) h_{r+1}
\]
\[= \sum_{i=1}^{r+1} P_i h_i = 0.
\]

By induction hypothesis, there exists a skew-symmetric set \(\{\eta_{i,j}\}_{i,j=1}^{r}\) such that \(P_i + y_i h_{r+1} = \sum_{i=1}^{r} \eta_{i,j} h_j\) for \(i = 1, \ldots, r\).

We can now set \(\eta_{r+1,i} = y_i\) and \(\eta_{i,r+1} = -y_i, i = 1, \ldots, r\) and the first assertion follows.

If each \(P_i\) is of homogeneous weight \(W \geq 1\), in (3.44) we can choose each \(y_i\) to be of homogeneous weight \(W - 1\). If \(W < 1\), since \(\tilde{w}_m(h_i) = 1\) both sides of equation (3.44) must be zero and the induction hypothesis shows that each \(P_i = 0, i = 1, \ldots, r + 1\).

For the estimate on \(\text{ord} \eta_{i,j}\) we refer the reader to [7, Lemma 3.1] where a similar result is proved.

The argument of Lemmas 3.5 and 3.6 is due to S. Sperber and can be used to close a gap in the proof of directness of sum in [15, Theorem 3.9].

**Lemma 3.5.** Let \(T_m = \{(\alpha; \gamma) \in (mMZ)^n \times \mathbb{Z} \mid t^\alpha Y^\gamma \in R\};\) then the mapping \((\alpha; \gamma) \mapsto (mM\alpha; \gamma)\) establishes a bijection between \(S_m\) and \(T_m\). In particular, \(t_i \mapsto t_i^{mM}(i = 1, \ldots, n)\) maps \(A_m\) into a subring of \(R\) and \(\overline{A}_m\) into a subring of \(\overline{R}^{(m)}\).

**Proof.** Let \((\alpha; \gamma) \in S_m\) and let \(\beta = mM\alpha:\)
\[
t^\beta Y^\gamma = \left(Y^{-1} t^\alpha\right)^{s(\beta)} Y^\gamma + s(\beta) t^\beta - s(\beta) a.
\]
\(s(\beta) = mM s(\alpha)\) is an integer and, by assumption, \(\gamma \geq -mM s(\alpha)\) and \(\alpha_i \geq s(\alpha)a_i\) for all \(i\). Hence \(\gamma + s(\beta) \geq 0, \beta_i - s(\beta)a_i \geq 0 \forall i\) and \(t^\beta Y^\gamma \in R\).

Conversely, if \(t^\delta Y^\gamma\) is a monomial in \(R\), then \(\gamma \geq -s(\delta)\): this is clearly true of the generators of \(R\) and, for any \(\delta, \varepsilon \in \mathbb{Z}^n, s(\delta + \varepsilon) \geq s(\delta) + s(\varepsilon)\). Thus, if \((\beta; \gamma) \in T_m, with \beta = mM\alpha, then (\alpha; \gamma) \in S_m.\)

**Lemma 3.6.** Let \(I\) be a non-empty subset of \(\{1, \ldots, n-1\};\) then the family \(\{\tilde{H}_i\}_{i \in I}\) in any order forms a regular sequence in \(\mathcal{R}_m\). More precisely, if \(\{P_i(t,Y)\}_{i \in I}\) is a set of non-zero elements of \(\mathcal{R}_m, of\) homogeneous weight \(w_m(P_i) = W\) independent of \(i\), and such that
\[ \sum_{i \in I} H_i \ast P_i = 0, \]
then there exists a skew-symmetric set \( \{ \xi_{i,j} \}_{i,j \in I} \) of elements of \( \mathcal{R}_m \) such that

(i) \( P_i(t, Y) = \sum_{j \in I} H_j \ast \xi_{i,j}; \)
(ii) each \( \xi_{i,j} \) has homogeneous weight \( w_m(\xi_{i,j}) = W - 1 \) for all \( (i, j) \in I \times I; \)
(iii) \( \min_{j \in I} \{ \text{ord} \xi_{i,j} \} \geq \text{ord} P_i - 1/(p - 1) \) for all \( i \in I. \)

**Proof.** Assume that

\[ (3.45) \sum_{i \in I} H_i \ast P_i(t, Y) = 0. \]

Applying \( \Phi_m^{-1} \) to equation (3.45) we obtain the following equation in \( \overline{A}_m: \)

\[ (3.46) \sum_{i \in I} H_i P_i(t, Y) = 0. \]

Replacing \( t_i \) by \( t_i^{mM} (i = 1, \ldots, n) \), and multiplying by \( \gamma_0^{-1} \), we get

\[ (3.47) \sum_{i \in I} h_i^{(m)} P_i(t^{mM}, Y) = 0. \]

Let \( Q_i(t, Y) = P_i(t^{mM}, Y); \) by Lemma 3.5, \( Q_i(t, Y) \in \overline{R}_m \) and, if \( t^\alpha Y^\gamma \) is any monomial in \( Q_i(t, Y) \) with non-zero coefficient, then \( \tilde{w}_m(\alpha; \gamma) = W. \) Lemma 3.4 implies the existence of a skew-symmetric set \( \{ \eta_{i,j} \}_{i,j \in I} \) of elements of \( \overline{R}_m \) such that \( Q_i(t, Y) = \sum_{j \in I} \eta_{i,j} h_j^{(m)} \) for each \( i \in I, \) with \( \tilde{w}_m(\eta_{i,j}) = W - 1 \) and \( \text{ord} \eta_{i,j} \geq \text{ord} P_i \) for all \( i, j. \)

If \( t^\alpha Y^\gamma \) is any monomial in \( Q_i(t, Y) \) with non-zero coefficient then \( (\alpha; \gamma) \in T_m. \) The same is true of each \( h_i^{(m)}. \) Hence we may choose the elements \( \eta_{i,j} \) so that \( \eta_{i,j} = \xi_{i,j}^{t^{mM}, Y}: \)

\[ (3.48) P_i(t^{mM}, Y) = \sum_{j \in I} \xi_{i,j}^{t^{mM}, Y} h_j^{(m)}. \]

Therefore, letting \( \xi_{i,j}(t, Y) = \gamma_0^{-1} \xi_{i,j}(t, Y): \)

\[ (3.49) P_i(t, Y) = \sum_{j \in I} \xi_{i,j}(t, Y) H_j. \]

Equation (3.49) is now valid in \( \overline{A}_m \) and, for any monomial \( t^\alpha Y^\gamma \) in \( \xi_{i,j}(t, Y) \) with non-zero coefficient, \( w_m(\alpha; \gamma) = \tilde{w}_m(m M \alpha; \gamma) = W - 1. \) Applying \( \Phi_m \) to equation (3.49) yields the result.

Using the results already attained in this section, Lemmas 3.7 and 3.8 and Theorems 3.1, 3.2, and 3.3 can be obtained with a slight reworking of the arguments in [7, §3]. We shall therefore omit the proofs.
**Lemma 3.7** (see [7, Lemma 3.4]). If $b \leq p/(p - 1)$, then

$$L_m(b, c) = V_m(b, c) + \sum_{i=1}^{n-1} H_i * L_m(b, c + e).$$

**Lemma 3.8** (see [7, Lemma 3.5]). If $b < p/(p - 1)$, then

$$V_m(b) \cap \sum_{i=1}^{n-1} H_i * L_m(b) = (0).$$

**Theorem 3.1** (see [7, Lemma 3.6]). If $1/(p - 1) \leq b \leq p/(p - 1)$, then

$$L_m(b, c) = V_m(b, c) + \sum_{i=1}^{n-1} D_i * L_m(b, c + e).$$

**Theorem 3.2** (see [7, Lemma 3.10]). Let $I$ be a non-empty subset of $\{1, \ldots, n-1\}$ and assume that $1/(p - 1) < b \leq p/(p - 1)$; if $\{\xi_i\}_{i \in I}$ is a set of elements of $L_m(b, c)$ such that $\sum_{i \in I} D_i * \xi_i = 0$, then there exists a skew-symmetric set $\{\eta_{i,j}\}_{i,j \in I}$ in $L_m(b, c + e)$ such that $\xi_i = \sum_{j \in I} D_j * \eta_{i,j}$ for all $i \in I$. In particular, the family $\{D_i\}_{i=1}^{n-1}$ in any order forms a regular sequence on the $R_m(b)$-module $L_m(b, c)$.

**Theorem 3.3** (see [7, Lemma 3.11]). If $1/(p - 1) < b \leq p/(p - 1)$, then

$$V_m(b) \cap \sum_{i=1}^{n-1} D_i * L_m(b) = (0).$$

d. *A Comparison Theorem.*

We now undertake to compare reduction modulo

$$\sum_{i=1}^{n-1} H_i * L_m(b, c + e) \quad \left(\text{respectively} \sum_{i=1}^{n-1} D_i * L_m(b, c + e)\right)$$

with reduction modulo $\sum_{i=1}^{n-1} \overline{H}_i * L_m(b, c + e)$ studied in §2.
Fix $\xi \in L_m(b, c)$. Using Theorem 3.1, Lemma 3.8, and Proposition 3.1 we write:

\begin{equation}
(3.50) \quad \xi = v + \sum_{i=1}^{n-1} D_i \ast \zeta_i, \quad v \in V_m(b, c), \; \zeta_i \in L_m(b, c + e);
\end{equation}

\begin{equation}
(3.51) \quad \xi = \tilde{v} + \sum_{i=1}^{n-1} H_i \ast \tilde{\zeta}_i, \quad \tilde{v} \in V_m(b, c), \; \tilde{\zeta}_i \in L_m(b, c + e);
\end{equation}

\begin{equation}
(3.52) \quad \xi = \bar{v} + \sum_{i=1}^{n-1} \bar{H}_i \ast \bar{\zeta}_i, \quad \bar{v} \in V_m(b, c), \; \bar{\zeta}_i \in L_m(b, c + e).
\end{equation}

**Lemma 3.9.** Let $\xi, v, \zeta_1, \ldots, \zeta_{n-1}$ be as in (3.50); then in (3.51) $\tilde{v}$ satisfies $v - \tilde{v} \in V_m(b, c + e)$ and each $\tilde{\zeta}_i$ can be chosen so that $\zeta_i - \tilde{\zeta}_i \in L_m(b, c + 2e)$.

**Proof.**

\[
\sum_{i=1}^{n-1} D_i \ast \zeta_i - \sum_{i=1}^{n-1} H_i \ast \zeta_i = \sum_{i=1}^{n-1} E_i \zeta_i \in L_m(b, c + e).
\]

By Lemma 3.8, there exist $v' \in V_m(b, c + e)$ and $\zeta'_i \in L_m(b, c + 2e)$, $i = 1, \ldots, n - 1$, such that

\[
\sum_{i=1}^{n-1} E_i \zeta_i = v' + \sum_{i=1}^{n-1} H_i \ast \zeta'_i.
\]

Hence

\[
\xi = v + v' + \sum_{i=1}^{n-1} H_i \ast (\zeta_i + \zeta'_i)
\]

and we may set $\tilde{v} = v + v'$, $\tilde{\zeta}_i = \zeta_i + \zeta'_i$, $i = 1, \ldots, n - 1$. \(\square\)

In the rest of this section we fix $b = 1/(p - 1)$ (so $e = 1$).

**Lemma 3.10.** For each $i \in \{1, \ldots, n - 1\}$ there exist

\[
\Gamma_i \in L_m(p/(p - 1), 0) \quad \text{and} \quad G_i \in L_m(p/(p - 1), 0)
\]

such that $H_i = \bar{H}_i \ast G_i + \Gamma_i$. Furthermore, $G_i$ is invertible and $G_i^{-1} \in L_m(p/(p - 1), 0)$.

**Proof.** By definition,

\[
H_i = \sum_{l=0}^{\infty} p^l \gamma_l \left( c_i^{p^l} d_i a_i t^{p^l d_i} - c_i^{p^l} d_n a_n t^{p^l d_n} \right)
\]

(recall that $c_i^q = c_i$, and therefore $c_i^p = c_i^p$).
Let
\[ \Gamma_i = \sum_{l=0}^{\infty} p'^l \gamma_l \left[ \frac{d_i}{a_i} - \left( \frac{d_i}{a_i} \right) p' \right] c_i t_i' \Gamma_i' - \sum_{l=0}^{\infty} p'^l \gamma_l \left[ \frac{d_n}{a_n} - \left( \frac{d_n}{a_n} \right) p' \right] c_n t_n'. \]
Then
\[ H_i = \sum_{l=0}^{\infty} p'^l \gamma_l \left[ (e_i t_i'^d_i)^p' - (e_n t_n'^d_n)^p' \right] + \Gamma_i. \]
If we set
\[ G_i = 1 + \sum_{j=0}^{p'-1} \gamma_j p'^l \sum_{j=0}^{p'-1} (e_i t_i'^d_i) (e_n t_n'^d_n)^{p'-j-1}, \]
then formally: \( H_i = H_i G_i + \Gamma_i. \)
Since \( d_k/a_k \in \mathbb{Q} \) and \((p, M) = 1\) we have
\[ \text{ord} \left[ \frac{d_k}{a_k} - \left( \frac{d_k}{a_k} \right) p' \right] \geq 1 \quad \text{for all} \quad k = 1, \ldots, n. \]
Hence both \( \Gamma_i \) and \( G_i \) are elements of \( L(p/(p - 1), 0) \). \( G_i \) is of the form \( G_i = 1 - \sum_{\alpha \geq 0} C_{\alpha} t^\alpha ; \) such a series is invertible in \( L(p/(p - 1), 0) \), with inverse \( G_i^{-1} = 1 + \sum_{j=0}^{\infty} \left( \sum_{\alpha \geq 0} C_{\alpha} t^\alpha \right)^j. \)
Now apply \( \Phi_m : L(p/(p - 1)) \to L_m(p/(p - 1)) \). \( \square \)

**Lemma 3.11.** Let \( \xi, \tilde{v}, \tilde{\xi}_1, \ldots, \tilde{\xi}_{n-1} \) be as in (3.51); then in (3.52) \( \tilde{v} \)
satisfies \( \tilde{v} - v \in V_m(p/(p - 1), c + 1) \) and each \( \tilde{\xi}_i \) can be chosen so that
\[ \tilde{\xi}_i - G_i \ast \tilde{\xi}_i \in L_m \left( \frac{p}{p - 1}, c + 2 \right). \]

**Proof.** We construct a sequence \((\xi^{(\nu)}, v^{(\nu)}, \xi_1^{(\nu)}, \ldots, \xi_{n-1}^{(\nu)} ; \nu \in \mathbb{N})\) with
\[ \xi^{(\nu)} \in L_m \left( \frac{p}{p - 1}, c + \nu \right), \quad v^{(\nu)} \in V_m \left( \frac{p}{p - 1}, c + \nu \right), \]
\[ \xi_i^{(\nu)} \in L_m \left( \frac{p}{p - 1}, c + \nu + 1 \right) \]
by letting \( \xi^{(0)} = \xi, \; v^{(0)} = \tilde{v}, \; \xi_i^{(0)} = \tilde{\xi}_i \) and the following recursion. Given \( \xi^{(\nu)} \in L_m(p/(p - 1), c + \nu) \) we can write, using Lemma 3.8:
\[ \xi^{(\nu)} = v^{(\nu)} + \sum_{i=1}^{n-1} H_i \ast \xi_i^{(\nu)}, \quad v^{(\nu)} \in L_m \left( \frac{p}{p - 1}, c + \nu \right), \]
\[ \xi_i^{(\nu)} \in L_m \left( \frac{p}{p - 1}, c + \nu + 1 \right). \]
By Lemma 3.10,

\begin{equation}
(3.53) \quad \xi^{(\nu)} = v^{(\nu)} + \sum_{i=1}^{n-1} H_i * G_i * \xi_i^{(\nu)} + \xi^{(\nu+1)}, \quad \text{with}
\end{equation}

\[ \xi^{(\nu+1)} = \Gamma_i * \xi_i^{(\nu)} \in L_m \left( \frac{p}{p-1}, c + \nu + 1 \right). \]

Let \( s \in \mathbb{N} \). Writing equation (3.53) for \( 0 \leq \nu \leq s \) and adding yields, after cancellations:

\[ \xi = \sum_{\nu=0}^{s} v^{(\nu)} + \sum_{i=1}^{n-1} H_i * \sum_{\nu=0}^{s} G_i * \xi_i^{(\nu)} + \xi^{(s+1)}. \]

Letting \( s \to \infty \), \( \sum_{\nu=0}^{s} v^{(\nu)} \) converges to \( \overline{v} \in V_m(p/(p-1), c) \), \( \sum_{\nu=0}^{s} \xi_i^{(\nu)} \) converges to \( \overline{\xi}_i \in L_m(p/(p-1), c+1) \) and \( \xi^{(s+1)} \) converges to zero. \( \square \)

**Theorem 3.4.** Let \( \xi \in L_m(p/(p-1), c) \); if we express \( \xi \) in the form \( \xi = \overline{v} + \sum_{i=1}^{n-1} H_i * \xi_i \) on the one hand, with \( \overline{v} \in V_m(p/(p-1), c) \), \( \xi_i \in L_m(p/(p-1), c+1) \) and if we express \( \xi \) in the form \( \xi = v + \sum_{i=1}^{n-1} D_i * \xi_i \) on the other hand, with \( v \in V_m(p/(p-1), c) \), \( \xi_i \in L_m(p/(p-1), c+1) \), then \( v - \overline{v} \in V_m(p/(p-1), c+1) \) and \( \xi_i \) and \( \overline{\xi}_i \) may be chosen so that \( \xi_i - G_i * \xi_i \in L_m(p/(p-1), c+2) \) for all \( i \).

**Proof.** This is a consequence of Lemmas 3.9 and 3.11. \( \square \)

**4. Specialization.** In order to obtain estimates for the exponential sum (0.4), we need to specialize the spaces \( L_m(b, c) \) by setting \( Y = y \) for some \( y \in \Omega^\times \). We first observe that elements of \( L_m(b, c) \) are convergent for \( \text{ord} t_i > -b/d_i \) and \( Y > -Nb/mM \). Furthermore, if we fix \( Y = y \) with \( \text{ord} y > -Nb/mM \), the resulting series in \( t_1, \ldots, t_n \) are convergent for \( t_i \) satisfying \( \text{ord} t_i \geq (mM/d_i N) \text{ord} y \).

Throughout this section, we assume that \( (p, M) = 1 = (p, D) \) and \( 1/(p-1) < b \leq p/(p-1) \).

For \( \alpha \in \mathbb{Z}^n \) we let

\begin{equation}
(4.1) \quad w(\alpha) = J(\alpha) - Ns(\alpha).
\end{equation}

For \( x \in \Omega_0^\times \), let

\begin{equation}
(4.2) \quad L(x; b, c) = \left\{ \xi = \sum_{\alpha \in E} A(\alpha) t^\alpha \mid A(\alpha) \in \Omega_0, \right. \\
\left. \text{ord} A(\alpha) \geq bw(\alpha) - s(\alpha) \cdot \text{ord} x + c \right\};
\end{equation}
(4.3) \[ L(x; b) = \bigcup_{c \in \mathbb{R}} L(x, b, c); \]

(4.4) \[ V = \Omega_0 \text{-span of } \{ t^\alpha | \alpha \in \tilde{\Delta} \}; \]

(4.5) \[ V(x; b, c) = V \cap L(x, b, c). \]

\(L(x; b)\) is a Banach space with the norm

\[ \|\xi\|_x = \sup_{\alpha \in E} p^{-c_\alpha}, \quad c_\alpha = \operatorname{ord} A(\alpha) - bw(\alpha) + s(\alpha) \operatorname{ord} x. \]

We equip \(L(x; b, c)\) with an \(\Omega_0\)-algebra structure in the following way: if \(\alpha, \beta \in E\), there exist \(\delta \in E, \lambda \in \mathbb{N}\) unique such that \(\alpha + \beta = \delta + \lambda a\) and we set:

\[ t^\alpha \ast t^\beta = x^\lambda t^\delta. \]

If \(\eta = \sum_{\alpha \in E} B(\alpha)t^\alpha\) is an element of \(L(x; b, c')\), then \(\xi = \eta \ast \xi\) is a continuous mapping from \(L(x; b, c)\) into \(L(x; b, c + c')\). Note that \(\overline{H}_i\) and \(H_i\) (as defined in (3.27) and (3.28) respectively) can be viewed as elements of \(L(x; b, 0)\) and that \(\overline{H}_i, H_i, \) and \(D_i\) act continuously on \(L(x; b, c)\) for any \(c \in \mathbb{R}\). Given \(x \in \Omega_0^x\), \(\operatorname{ord} x^m > -N b\), we fix \(y \in \Omega_0^x\) with \(y^M = x\). Let \(L_m(b, c)', L_m(b)', V_m(b, c)', L(x; b, c)', L(x; b)', V'\) be defined as their unprimed counterparts, with the difference that the coefficients are allowed to lie in \(\Omega_0' = \Omega_0(y)\). We can define an \(\Omega_0'\)-linear specialization map

\[ S_y : L_m(b) \to L(m; b)' \]

by sending \(Y\) into \(y\). \(S_y\) is continuous of norm 1 and is surjective, sending \(V_m(b)\) onto \(V'\) and \(D_i \ast L_m(b)'\) onto \(D_i \ast L(x^m, b)'\) for all \(i\). Indeed, there is an \(\Omega_0'\)-linear section

\[ T_y : \sum_{\alpha \in E} A(\alpha)t^\alpha \to \sum_{\alpha \in E} x^{m(s(\alpha))}y^{-mM(s(\alpha))}t^\alpha. \]

**Proposition 4.1.** \(\operatorname{Ker}(S_y | L_m(b, c)') = (Y - y)L_m(b, c - \operatorname{ord} y)\).

In particular, \(L_m(b)'/(Y - y)L_m(b)' \cong L(x^m; b)'\).

**Proof.** Let \(\xi = \sum_{(\alpha; \gamma) \in E_m} A(\alpha; \gamma)t^\alpha Y^\gamma \in L_m(b, c)\) and assume that \(S_y(\xi) = 0\).
For each $\alpha \in E$ we must have $\sum_{\gamma \geq -mMs(\alpha)} A(\alpha; \gamma) y^{\gamma} = 0$. Multiplying by $y^{mMs(\alpha)}$ we obtain $\sum_{\gamma \geq 0} A(\alpha; \gamma - mMs(\alpha)) t^{\gamma} = 0$. Thus

$$\xi = \sum_{\alpha \in E} \left[ \sum_{\gamma \geq 0} A(\alpha; \gamma - mMs(\alpha))(Y^{\gamma} - y^{\gamma}) \right] y^{mMs(\alpha)} t^{\alpha} = (Y - y)\xi',$$

with

$$\xi' = \sum_{\alpha \in E} \left[ \sum_{\gamma \geq 0} A(\alpha; \gamma - mMs(\alpha)) \sum_{\lambda = 0}^{\gamma-1} Y^{\lambda} y^{\gamma-\lambda-1} \right] y^{mMs(\alpha)} t^{\alpha}.$$

$\xi' \in L_m(b, c - \text{ord } y)'$ since $\text{ord } y > -Nb/mM$. 

It follows from Theorem 3.2 that the operators $D_i$, $i = 1, \ldots, n - 1$, acting on the $R_m(b)$-module $L_m(b)$ (respectively the $R_m(b)'$-module $L_m(b)'$) form a completely secant family ([3, §9, n° 5, Proposition 5]). In other words, the associated Koszul complexes are acyclic: if

$$H_\mu(\{D_i\}_{i=1}^{n-1}, L_m(b))$$

(respectively $H_\mu(\{D_i\}_{i=1}^{n}, L_m(b)'$)

is the $\mu$-th homology group of the corresponding complex, then:

\begin{align}
(4.9) & \quad H_\mu(\{D_i\}_{i=1}^{n-1}, L_m(b)) = 0, \quad \mu \geq 1; \\
(4.10) & \quad H_\mu(\{D_i\}_{i=1}^{n}, L_m(b)') = 0, \quad \mu \geq 1.
\end{align}

**Lemma 4.1.** $(Y - y)$ is not a zero divisor in $L_m(b)'/\sum_{i=1}^{n-1} D_i*L_m(b)'$.

**Proof.** Let $\xi \in L_m(b)'$ and assume that

$$(4.11) \quad (Y - y)\xi = \sum_{i=1}^{n-1} D_i * \xi_i, \quad \xi_i \in L_m(b)' .$$

By Theorem 3.1, we can write

$$(4.12) \quad \xi = v + \sum_{i=1}^{n-1} D_i * \eta_i, \quad v \in V_m(b)', \ \eta_i \in L_m(b)' .$$

Thus (4.11), (4.12), and Theorem 3.3 imply $(Y - y)v = 0$; hence $v = 0$. 

**Theorem 4.1.**

(i) $H_\mu(\{D_i\}_{i=1}^{n-1}, L(x^m,b)') = 0$ for all $\mu \geq 1$;

(ii) $H_0(\{D_i\}_{i=1}^{n-1}, L(x^m,b)') \xrightarrow{\sim} V'$. 
Proof. (i) Let $D_m = Y - y$. As a consequence of Lemma 4.1, the family $\{D_i\}_{i=1}^n$ forms a regular sequence on the $R_m(b)'$-module $L_m(b)'$.

In particular,

\[(4.13) \quad H_\mu(\{D_i\}_{i=1}^n, L_m(b)') = 0 \quad \text{for all } \mu \geq 1.\]

Using [11, Ch. 8, Theorem 4] and Proposition 4.1, for all $\mu > 0$ there is an $\Omega'$-linear isomorphism.

\[(4.14) \quad H_\mu(\{D_i\}_{i=1}^n, L_m(b)') \cong H_\mu(\{D_i\}_{i=1}^{n-1}, L(x^m; b)').\]

(ii) $S_y$ maps $V_m(b, c)'$ onto $V(x^m; b, c)'$ and $D_i * L_m(b, c + e)'$ onto $D_i * L(x^m; b, c + e)'$ for all $i = 1, \ldots, n - 1$.

Hence using Theorems 3.1 and 3.3:

\[(4.15) \quad L(x^m; b, c)' = V(x^m; b, c)' + \sum_{i=1}^{n-1} D_i * L(x^m; b, c + e)'.\]

Now

\[H_0(\{D_i\}_{i=1}^{n-1}, L(x^m; b)') = L(x^m; b)' / \sum_{i=1}^{n-1} D_i * L(x^m; b)' .\]

**Proposition 4.2.** $L(x; b, c) = V(x; b, c) + \sum_{i=1}^{n-1} D_i * L(x; b, c + e)$.

**Proof.** Let $\eta = \sum_{\alpha \in E} A(\alpha) t^\alpha$ be an element of $L(x; b, c)$. Assume that, for any $\alpha \in E$ such that $A(\alpha) \neq 0$, $s(\alpha)$ is equal to some value $s$ independent of $\alpha$, and let $\xi = y^{-Ms} T_y(\eta)$.

Let $c_s = s \cdot \text{ord } x$; $\xi = \sum_{\alpha \in E} A(\alpha) t^\alpha Y^{-Ms}$ is an element of $L_1(b, c + c_s)$ and, by Theorem 3.1, there exist $v = \sum_{\beta \in \Delta} P_\beta(Y) t^\beta \in V_1(b, c + c_s)$ and $\zeta_i \in L_1(b, c + c_s + e)$ such that $\xi = v + \sum_{i=1}^{n-1} D_i * \zeta_i$. For each $\beta \in \Delta$, write $P_\beta(Y) = \sum_\gamma P_{\beta,\gamma} Y^\gamma$ and, for each $i = 1, \ldots, n - 1$, $\zeta_i = \sum_{(\alpha; \gamma)} \zeta_{i,\alpha; \gamma} t^\alpha Y^\gamma$.

For $l \in \mathbb{N}$, $0 \leq l < M$ we let:

\[P_{\beta, l}(Y) = \sum_{\gamma + Ms \equiv l \pmod M} P_{\beta,\gamma} Y^\gamma,\]

\[\zeta_{i, l} = \sum_{\gamma + Ms \equiv l \pmod M} \zeta_{i,\alpha; \gamma} t^\alpha Y^\gamma, \quad i = 1, \ldots, n - 1.\]

Note that if $t^\alpha Y^\gamma$ is any monomial in $D_i * \zeta_{i, l}$ with non-zero coefficient, then again $\gamma + Ms \equiv l \pmod M$. Thus, if $l \neq 0$:

\[\sum_{\beta \in \Delta} P_{\beta, l}(Y) + \sum_{i=1}^{n-1} D_i * \zeta_{i, l} = 0.\]
Applying Theorem 3.3, \( P_{\beta, \iota}(Y) = 0 \) for all \( \beta \in \widetilde{\Delta} \) and we may choose each \( \zeta_{i, \iota} \) to be zero. Therefore:

\[
\xi = \sum_{\beta \in \widetilde{\Delta}} P_{\beta, 0}(Y) t^\beta + \sum_{i=1}^{n-1} D_i \ast \zeta_{i, 0}.
\]

Certainly \( y^{M_\beta}P_{\beta, 0}(Y) \in \Omega_0 \) for all \( \beta \in \widetilde{\Delta} \) and \( y^{M_\beta}S_y(\zeta_{i, 0}) \) has its coefficients in \( \Omega_0 \) for all \( i = 1, \ldots, n - 1 \). Hence

\[
\eta \in V(x; b, c) + \sum_{i=1}^{n-1} D_i \ast L(x; b, c + e).
\]

Now observe that if \( \alpha \in E \), \( s(\alpha) \) can assume only a finite set of values. Finally, directness of sum follows from (4.15). \( \square \)

**Corollary 4.1.**

(i) \( \mathcal{H}_\mu(\{D_i\}_{i=1}^{n-1}, L(x^m; b)) = 0 \) for all \( \mu \geq 1 \).

(ii) \( \mathcal{H}_0(\{D_i\}_{i=1}^{n-1}, L(x^m; b)) \sim V. \)

**Proof.** (i) follows from Theorem 4.1 and the fact that

\[
\mathcal{H}_\mu(\{D_i\}_{i=1}^{n-1}, L(x^m; b)) = \mathcal{H}_\mu(\{D_i\}_{i=1}^{n-1}, L(x^m; b)) \otimes_{\Omega_0} \Omega'_0
\]

(ii) follows from Proposition 4.2 and the fact that

\[
\mathcal{H}_0(\{D_i\}_{i=1}^{n-1}, L(x^m; b)) = L(x^m; b)/\sum_{i=1}^{n-1} D_i \ast L(x^m; b). \quad \square
\]

**5. The Frobenius map.** We first review some of the definitions and results in [7, §4] concerning the lifting of characters. Let

\[
E(z) = \exp \left( \sum_{j=0}^{\infty} \frac{z^p^j}{p^j} \right)
\]

be the Artin-Hasse exponential series. For \( s \in \mathbb{N}^* \cup \{\infty\} \), fix \( \gamma_{s, 0} \in \mathbb{Q}_p(\zeta_p) \) satisfying

\[
\text{ord} \gamma_{s, 0} = \frac{1}{p - 1} \quad \text{and} \quad \sum_{j=0}^{s} \frac{\gamma_{s, 0}^j}{p^j} = 0,
\]

and let \( \theta_s \) be the splitting function

\[
(5.1) \quad \theta_s(z) = E(\gamma_{s, 0} z).
\]
Let
\[
(5.2) \quad a_s = \begin{cases} 
\frac{1}{p-1} - \frac{1}{p^s} \left( s + \frac{1}{p-1} \right) & \text{if } s \in \mathbb{N}^* , \\
\frac{1}{p-1} & \text{if } s = \infty .
\end{cases}
\]

As a power series in \( z \):
\[
(5.3) \quad \theta_s(z) = \sum_{l=0}^{\infty} B_l(s) z^l ,
\]
with
\[
\begin{cases} 
\text{ord } B_l(s) \geq la_{s+1} & \text{for all } l \geq 0 . \\
B_l(s) = \frac{\gamma_{s,0}^l}{l!} & \text{for } 0 \leq l \leq p-1 .
\end{cases}
\]

In particular:
\[
(5.5) \quad \text{ord } B_l(s) = \frac{l}{p-1} \quad \text{for } 0 \leq l \leq p-1 .
\]

For a fixed choice of \( s \), we can choose \( \gamma_{s,0} \) so that
\[
(5.6) \quad \theta_s(t) = \theta(i) \quad \text{whenever } t^p = t,
\]
where \( \theta \) is the additive character of \( \mathbb{F}_p \) chosen in (0.5). Let
\[
(5.7) \quad F(t) = \prod_{i=1}^{n} \theta_s(c_i t_i^{k_i});
\]
\[
G(t) = \prod_{j=0}^{\ell-1} \mathcal{F}^{t_j}(t^{p^s}).
\]

As a consequence of \([7, \S 4]\), for all \( m \geq 0 \):
\[
(5.8) \quad S_m(f, \mathbb{F}, \Theta, \rho) = \sum_{t \in \mathbb{F}_m} \left( \prod_{i=1}^{n} t_i^{-(q^m-1)r_i/r} \right) G(t) G(t^{q}) \cdots G(t^{q^m-1}).
\]

Clearly, \( F(t) \in L(ra_{s+1}, 0) \) and \( G(t) \in L(p^{-1} ra_{s+1}, 0) \).

Let \( \rho \in \mathbb{N}^n , \ 0 \leq \rho_i < r . \) We define elements \( \rho^{(0)} = \rho, \rho^{(1)}, \ldots, \rho^{(q')} = \rho \) satisfying:
\[
(5.9) \quad \begin{cases} 
p \rho_i^{(j+1)} - \rho_i^{(j)} \equiv 0 \pmod{r}, & i = 1, \ldots, n; \ j = 0, \ldots, q' . \\
0 \leq \rho_i^{(j)} < r,
\end{cases}
\]

For each of the Banach spaces which have been defined, we indicate by the subscript "\( \rho \)" the subspace where all monomials \( t^\alpha \) have zero coefficient unless \( \alpha \in Z^{(\rho)} \). Thus, for example,
\[
L_{m,\rho}(b,c)
= \left\{ \xi = \sum B(\alpha; \gamma) t^\alpha Y^\gamma \in L_m(b,c) \mid B(\alpha; \gamma) = 0 \text{ if } \alpha \notin E^{(\rho)} \right\}.
\]
Let \( X = Y^M \). If \( \alpha \in Z^{(\rho)} \) we set

\[
\psi(t^\alpha) = \begin{cases} 
t^{\alpha/p}, & \text{if } p \mid \alpha_i, \ 1 \leq i \leq n; \\
0, & \text{otherwise}. 
\end{cases}
\] (5.10)

\[
\psi_X(t^\alpha) = \begin{cases} 
X^{s(\alpha) - ps(\beta)}t^\beta, & \text{if } \exists \beta \in E(\rho^r) \text{ such that } \omega(\alpha) = p\omega(\beta); \\
0, & \text{otherwise}. 
\end{cases}
\] (5.11)

\[
\psi_X(t^\alpha) = S_y \circ \psi_X(t^\alpha).
\] (5.12)

\( \psi \) defines a continuous \( \Omega_0 \)-linear map \( \psi : L_\rho(b/p, c) \to L_\rho(b, c) \); \( \psi_X \) defines a continuous \( R_1(b) \)-linear map \( \psi_X : L_1,\rho(b/p, c) \to L_p,\rho'(b, c) \); \( \psi_X \) defines a continuous \( \Omega_0 \)-linear map \( \psi_X : L_p(x; b/p, c) \to L_p(x^p; b, c) \).

For all \( m \geq 0 \) the following diagram is commutative:

\[
\begin{array}{ccc}
L_\rho(b/p) & \xrightarrow{\phi_m} & L_{m,\rho}(b/p) \\
\downarrow \psi & & \downarrow \psi_X^m \\
L_\rho'(b) & \xrightarrow{\phi_m} & L_{p,\rho'}(b) \\
\end{array}
\]

\( \downarrow \psi_X^m \circ \text{id} \)

Let:

\[
\left\{ \begin{array}{l}
\psi'_X = \psi_{X_{a/p^2}} \circ \ldots \circ \psi_{X_{a/p}}; \\
\psi'_X = \psi_{X_{a/p^2}} \circ \ldots \circ \psi_{X_{a/p}}.
\end{array} \right.
\] (5.14)

\[
\begin{array}{l}
F_j(t, X) = [\phi_{p^j}(F(t'))]^t \in L_{p^j}(a_{s+1}, 0), \ 0 \leq j \leq \ell - 1; \\
G_0(t, X) = \phi_1(G(t')).
\end{array}
\] (5.15)

If \( b \leq pa_{s+1} \) we define maps

\[
\begin{array}{l}
\mathcal{F} : L_\rho(b, c) \to L_\rho(b/q, c) \xrightarrow{\times G(t')} L_\rho(b/q, c) \xrightarrow{\psi'_X} L_\rho(b, c); \\
\mathcal{F}_X : L_{1,\rho}(b/q, c) \to L_{1,\rho}(b/q, c) \xrightarrow{\psi'_X} L_{q,\rho}(b, c); \\
\mathcal{F}_X : L_\rho(x; b/c) \to L_\rho(x; b/c) \xrightarrow{\psi'_X} L_{q}(x; b, c).
\end{array}
\] (5.16)

By [12, §9], \( \mathcal{F} \) (respectively \( \mathcal{F}_X \), respectively \( \mathcal{F}_X \)) is a completely continuous \( \Omega_0 \)-linear map (respectively \( R_1(b) \)-linear, respectively \( \Omega_0 \)-linear).

Let \( \delta \) be the operator defined on \( 1 + T\Omega[[T]] \) by

\[
g(T)^\delta = \frac{g(T)}{g(qT)}.
\] (5.17)
If \( x \in \Omega_0^X \) is the Teichmüller lifting of \( \bar{x} \in \mathbb{F}_q \), it follows from Corollary 1.1 that
\[
(5.18) \quad L(f, \mathbb{Z}_\bar{x}, \Theta, \rho, T)^{(1-n)^{\gamma}} = \det(I - T\mathfrak{F}_x)^{\delta^{n-1}}.
\]

We now fix the choice of constants in (3.23) by setting
\[
(5.19) \quad \gamma_j = \begin{cases} \sum_{l=0}^{j} \frac{\gamma_l^{p^l}}{p^l}, & \text{if } j \leq s - 1, \\ 0, & \text{if } j \geq s. \end{cases}
\]

Let \( \hat{F}(t') = \exp H(t) \) (\( H(t) \) has been defined in (3.26)). We recall ([7, (4.22)]) that
\[
(5.20) \quad \begin{cases} F(t) = \frac{\hat{F}(t)}{\hat{F}(t')}, \\ G(t) = \frac{\hat{F}(t)}{\hat{F}(t')} \end{cases}
\]

As operators on \( L(0) \):
\[
(5.21) \quad D_i = \frac{1}{\hat{F}(t')} \circ E_i \circ \hat{F}(t'), \quad i = 1, \ldots, n - 1.
\]

On the other hand, \( \mathcal{F} = \psi^f \circ G(t') \) maps \( L(0) \) into itself, and it follows from (5.20) that
\[
(5.22) \quad \mathcal{F} = \frac{1}{\hat{F}(t')} \circ \psi^f \circ \hat{F}(t').
\]

Since \( \psi^f \circ E_i = qE_i \circ \psi^f \) for all \( i \), we deduce:
\[
(5.23) \quad \mathcal{F} \circ D_i = qD_i \circ \mathcal{F}, \quad i = 1, \ldots, n - 1,
\]

and this last equation is now valid in \( L(b) \subset L(0) \). Using (5.13) and the definition of \( \phi_m \) we deduce:
\[
(5.24) \quad \begin{cases} \mathcal{F}_x \circ D_i = qD_i \circ \mathcal{F}_x, \\ \mathcal{F}_x \circ D_i = qD_i \circ \mathcal{F}_x. \end{cases}
\]

Let
\[
(5.25) \quad \begin{cases} W_{x^m,\rho} = L_{m,\rho}(b)/ \sum_{i=1}^{n-1} D_i \ast L_{m,\rho}(b); \\ W_{x,\rho} = L_{\rho}(x; b)/ \sum_{i=1}^{n-1} D_i \ast L_{\rho}(x; b). \end{cases}
\]

As a consequence of (5.24), \( \mathcal{F}_x \) acts on the Koszul complex
Specifically, there is a commutative diagram:

\[
\begin{array}{c}
0 \to L_\rho(x; b) \to \cdots \to L_\rho(x; b)^{n-1} \to \cdots \to L_\rho(x; b) \to W_{x, \rho} \to 0 \\
\text{down to } q^{n-1}, \overline{\mathcal{F}}_x \\
0 \to L_\rho(x^q; b) \to \cdots \to L_\rho(x^q; b)^{n-1} \to \cdots \to L_\rho(x^q; b) \to W_{x^q, \rho} \to 0
\end{array}
\]

(5.26)

Corollary 4.1 implies that both rows of diagram (5.26) are exact. Therefore, taking the alternating product of the Fredholm determinants, we obtain

\[
\det(I - T\overline{\mathcal{F}}_x)^{\delta^{n-1}} = \det(I - T\overline{\mathcal{F}}_x).
\]

For \( j \geq 0 \) let

\[
\begin{align*}
\mathcal{F}(j) &= \psi \circ F^{(t')} (t') \\
\mathcal{F}_X(j) &= \psi_{x^{p_j}} \circ [F_j(t, X)] \\
\mathcal{F}_X(j) &= \psi_{x^{p_j}} \circ [F_j(t, x)]
\end{align*}
\]

\( \mathcal{F}_X(j) \) maps \( L_{p_i', \rho^{(j)}}(b, c) \) into \( L_{p_j^{(j)}, \rho^{(j+1)}}(b, c) \), while \( \mathcal{F}_X(j) \) maps \( L_{\rho^{(j)}}(x^{p_j}; b, c) \) into \( L_{\rho^{(j+1)}}(x^{p_{j+1}}; b, c) \). If we set:

\[
D_i^{(j)} = E_i + H_i^{(j)}, \quad i = 1, \ldots, n-1; \quad j = 0, \ldots, \ell,
\]

then, as above,

\[
\mathcal{F}(j) \circ D_i^{(j)} = p D_i^{(j+1)} \circ \mathcal{F}(j).
\]

Hence:

\[
\begin{align*}
\mathcal{F}_X(j) \circ D_i^{(j)} &= p D_i^{(j+1)} \circ \mathcal{F}_X(j) \\
\mathcal{F}_X(j) \circ D_i^{(j)} &= p D_i^{(j+1)}
\end{align*}
\]

Let

\[
\begin{align*}
W_{x, \rho}(j) &= L_{p_i', \rho^{(j)}}(b) / \sum_{i=1}^{n-1} D_i^{(j)} * L_{p_i, \rho^{(j)}}(b), \\
W_{x, \rho}(j) &= L_{\rho^{(j)}}(x^{p_j}; b) / \sum_{i=1}^{n-1} D_i^{(j)} * L_{\rho^{(j)}}(x^{p_j}; b)
\end{align*}
\]

\( \mathcal{F}_X(j) \) and \( \mathcal{F}_X(j) \) define quotient maps:

\[
\begin{align*}
\mathcal{F}_X(j) : W_{x, \rho}(j) &\to W_{x, \rho}(j+1) \\
\mathcal{F}_X(j) : W_{x, \rho}(j) &\to W_{x, \rho}(j+1)
\end{align*}
\]

(5.32)

With these notations, \( W_{x, \rho}^{(\ell)} = W_{x^q, \rho}^{(\ell)} = W_{x, \rho}^{(\ell)} \) and the following factorizations hold:

\[
\begin{align*}
\mathcal{F}_X = \mathcal{F}_X^{(\ell-1)} \circ \cdots \circ \mathcal{F}_X^{(1)} \circ \mathcal{F}_X^{(0)} \\
\overline{\mathcal{F}}_X = \overline{\mathcal{F}}_X^{(\ell-1)} \circ \cdots \circ \overline{\mathcal{F}}_X^{(1)} \circ \overline{\mathcal{F}}_X^{(0)}
\end{align*}
\]

We now fix:

\[
s = \infty; \quad b = \frac{p}{p-1}.
\]

(5.35)
PROPOSITION 5.1. (i) Let $C^{(j)}(Y) = (C^{(j)}_{\beta,\alpha}(Y))$ be the matrix of $F(j)_X: W_{X,p} \rightarrow W_{X,p}^{(j+1)}$ with respect to the bases $\{Y^{-p^i s(\alpha)} t^\alpha | \alpha \in \Delta_{p(i)}^{(j)}\}$ of $W_{X,p}^{(j)}$ and $\{Y^{-p^i s(\alpha)} t^\alpha | \alpha \in \Delta_{p(i+1)}^{(j)}\}$ of $W_{X,p}^{(j+1)}$ respectively; then for any $\alpha \in \Delta_{p(i)}^{(j)}$ and $\beta \in \Delta_{p(i+1)}^{(j)}$, $C^{(j)}_{\beta,\alpha}(Y)$ is analytic in the disk $\{y | \text{ord } y > -N/Mp^j(p - 1)\}$.

(ii) Let $x \in \Omega^X$ with ord $x = 0$ and let $A^{(j)} = (A^{(j)}_{\beta,\alpha}(x))$ be the matrix of $F(j)_X: W_{X,p} \rightarrow W_{X,p}^{(j+1)}$ with respect to the bases $\{t^\alpha | \alpha \in \Delta_{p(i)}^{(j)}\}$ of $W_{X,p}^{(j)}$ and $\{t^\alpha | \alpha \in \Delta_{p(i+1)}^{(j)}\}$ of $W_{X,p}^{(j+1)}$ respectively; then for any $\alpha \in \Delta_{p(i)}^{(j)}$ and $\beta \in \Delta_{p(i+1)}^{(j)}$, $\text{ord } A^{(j)}_{\beta,\alpha}(x) = (pw(\beta) - w(\alpha))/(p - 1)$.

Proof. (i) If $\alpha \in \Delta_{p(i+1)}^{(j)}$, then

$$Y^{-p^i s(\alpha)} t^\alpha \in L_{p^i} \left(\frac{1}{p - 1}, \frac{-w(\alpha)}{p - 1}\right)$$

so that

$$F(j)_X(Y^{-p^i s(\alpha)} t^\alpha) \in L_{p^{i+1}} \left(\frac{p}{p - 1}, \frac{-w(\alpha)}{p - 1}\right).$$

Using Theorem 3.1, we may write

\begin{equation}
(5.36) \quad F(j)_X(Y^{-p^i s(\alpha)} t^\alpha) = \sum_{\beta \in \Delta_{p(i+1)}^{(j)}} C^{(j)}_{\beta,\alpha}(Y) Y^{-p^{i+1} s(\beta)} t^\beta + \sum_{i=1}^{n-1} D^{(j+1)}_i * \xi_i(t, Y).
\end{equation}

with

$$C^{(j)}_{\beta,\alpha}(Y) \in R_{p^{i+1}} \left(\frac{p}{p - 1}, \frac{pw(\beta) - w(\alpha)}{p - 1}\right)$$

and

$$\xi_i(t, Y) \in L_{p^{i+1}} \left(\frac{p}{p - 1}, \frac{-w(\alpha)}{p - 1} + 1\right).$$

(ii) Applying the map $S_y$ to equation (5.36) and multiplying by $x^{p^i s(\alpha)}$ we obtain:

\begin{equation}
(5.37) \quad F(j)_X(t^\alpha) = \sum_{\beta \in \Delta_{p(i+1)}^{(j)}} C^{(j)}_{\beta,\alpha}(Y) x^{p^i s(\alpha) - p^{i+1} s(\beta)} t^\beta + \sum_{i=1}^{n-1} D^{(j+1)}_i * \xi_i(t, Y).
\end{equation}
Since $\mathcal{F}_x^{(j)}$ is defined over $\Omega_0$, Proposition 4.2 shows that in fact $C_{\beta,\alpha}^{(j)}(y)x^{p^{\ell} s(\alpha) - p^{\ell+1}s(\beta)} \in \Omega_0$ and we may write:

$$(5.38) \quad A_{\beta,\alpha}^{(j)}(x) = C_{\beta,\alpha}^{(j)}(y)x^{p^{\ell} s(\alpha) - p^{\ell+1}s(\beta)}.$$ 

The estimates now follow from the fact that

$$C_{\beta,\alpha}(y) \in L \left( x^{p^{\ell+1}}; \frac{p}{p-1}, \frac{pw(\beta) - w(\alpha)}{p-1} \right) \cap \Omega_0'. $$

**Theorem 5.1.** Let $\rho = (\rho_1, \ldots, \rho_n) \in \mathbb{Z}^n$, $0 \leq \rho_i < r$ and suppose that $\rho = 0$ or $\rho \equiv 1 \pmod{r}$; let $\mathcal{H}_p(T) = \prod_{\alpha \in \Delta_p} (1 - q^{w(\alpha)}T)$. Then the Newton polygon of $L(f, \theta, \rho, T)$ lies over the Newton polygon of $\mathcal{H}_p(T)$.

**Proof.** Let $\mathcal{F}$ be the completion of the maximal unramified extension of $\mathcal{Q}_p$ in $\Omega$. For $x \in \mathcal{F}(\zeta_p)$ satisfying $\text{ord} x \geq 0$ and $\tau(x) = x^p$ we can define

$$(5.39) \quad \tau^{-1}: W_{x,\rho}^{(1)} \rightarrow W_{x,\rho}^{(0)} = W_{x,\rho},$$

by sending $\xi = \sum_{\alpha \in E(\rho)} A(\alpha)t^{\alpha} \in L_{\rho}(x^p; b, c)$ into

$$\tau^{-1}(\xi) = \sum_{\alpha \in E(\rho)} \tau^{-1}(A(\alpha))t^{\alpha} \in L_{\rho}(x; b, c).$$

Certainly,

$$\tau^{-1}(D_{i}^{(1)} \ast_L L(x^p; b)) \subset D_{i} \ast_L L(x; b) \quad \text{for all } i,$$

so that $\tau^{-1}$ is defined on the quotient. Let $x \in \Omega_0^\times$ with $x^q = x$ and let

$$(5.40) \quad \mathcal{F}_x' = \tau^{-1} \circ \mathcal{F}_x^{(0)}.$$

If $p \equiv 1 \pmod{r}$, then $\rho^{(j)} = \rho$ for all $j \in \mathbb{N}$ and $\mathcal{F}_x'$ is a $\tau^{-1}$-semi-linear map and a completely continuous endomorphism of $L_{\rho}(x; b)$ over $\Omega_1 = \mathcal{Q}_p(\zeta_p)$. If we let

$$(5.41) \quad \mathcal{F}_x' = \tau^{-1} \circ \mathcal{F}_x^{(0)},$$

then:

$$(5.42) \quad \mathcal{F}_x = (\mathcal{F}_x')'.$$

It follows from [8, Lemma 7.1] that the Newton polygon of $\text{det}_{\Omega_0}(I - T \mathcal{F}_x)$ can be obtained from that of $\text{det}_{\Omega_1}(I - T \mathcal{F}_x')$ by
reducing both ordinates and abscissae by the factor $1/f$ and interpreting the ordinates as normalized so that ord $q = 1$. If $x \in \Omega_0^\chi$ is the Teichmüller representative of $\tilde{x} \in \mathbb{F}_q$, we let $\mathcal{A}(x) = (\mathcal{A}_{\beta, \alpha}(x))$ be the matrix of $\mathcal{F}_x': W_{x, \rho} \to W_{x, \rho}$ over $\Omega_0$ with respect to the basis $\{t^\alpha \mid \alpha \in \Delta_\rho\}$. By Proposition 5.1:

$$\text{(5.43)} \quad \text{ord}\mathcal{A}_{\beta, \alpha}(x) \geq \frac{pw(\beta) - w(\alpha)}{p - 1} \quad \text{for all } \alpha, \beta \in \Delta_\rho.$$

We fix an integral basis $\{\eta_i\}_{i=1}^f$ of $\Omega_0$ over $\Omega_1$ with the property that $\{\tilde{\eta}_i\}_{i=1}^f$ is a basis of $\mathbb{F}_q$ over $\mathbb{F}_p$. In particular, if $\omega \in \Omega_0$, $\omega = \sum_{i=1}^f \omega_i \eta_i$, $\omega_i \in \Omega_1$, then ord $\omega = \inf_{1 \leq i \leq f}\{\text{ord } \omega_i\}$. Write:

$$\text{(5.44)} \quad \mathcal{F}_x'(\eta_i t^\alpha) = \sum_{\beta \in \Delta_\rho} \sum_{1 \leq j \leq f} \mathcal{A}((\beta, j), (\alpha, i)) \eta_j t^\beta.$$

$\mathcal{F}_x'$ is an $\Omega_1$-linear endomorphism of $W_{x, \rho}$ with matrix

$$\mathcal{A}' = [\mathcal{A}((\beta, j), (\alpha, i))]$$

with respect to the basis $\{\eta_i t^\alpha \mid \alpha \in \Delta_\rho, 1 \leq i \leq f\}$. Furthermore:

$$\text{ord}\mathcal{A}((\beta, j), (\alpha, i)) \geq \frac{pw(\beta) - w(\alpha)}{p - 1} \quad \text{for all } i, j.$$

We now proceed as in [8, §7]:

$$\text{det}_{\Omega_1}(I - T\mathcal{F}_x') = 1 + \sum_{j=1}^Q m_j T^j,$$

where $Q = f N \prod_{i=1}^n k_i$ and $m_j$ is (up to sign) the sum of the $j \times j$ principal minors of the matrix $\mathcal{A}'$. Thus, ord $m_j$ is greater than or equal to the minimum of all $j$-fold sums $\sum_{i=1}^j w(\beta_{(i)})$, in which $\{\beta_{(i)}, i_{(i)}\}_{i=1}^j$ is a set of $j$ distinct elements in $\{(\beta, i) \mid \beta \in \Delta_\rho, 1 \leq i \leq f\}$.

PROPOSITION 5.2. For each $\alpha \in \Delta_{\rho(\delta)}$, let $\alpha' \in \Delta_{\rho(\delta+1)}$ and $\delta \in \mathbb{Z}^n$ be the unique elements such that $0 \leq \delta_i \leq p - 1$ and

$$p \left( \frac{\alpha_i'}{d_i} - s(\alpha') \frac{a_i}{d_i} \right) - \left( \frac{\alpha_i}{d_i} - s(\alpha) \frac{a_i}{d_i} \right) = \delta_i \quad \text{for all } i;$$

Let $C^{(j)} = (C^{(j)}_{\beta, \alpha}(Y))$ be the matrix of $\mathcal{F}_x^{(j)}: W_{x, \rho}^{(j)} \to W_{x, \rho}^{(j+1)}$. Then:

(i) $\text{ord } C^{(j)}_{\alpha', \alpha}(0) = \frac{pw(\alpha') - w(\alpha)}{p - 1} = \sum_{i=1}^n \delta_i.$
(ii) If $\beta \neq \alpha'$ then

$$\text{ord } C_{\beta,\alpha}^{(j)}(0) > \frac{pw(\beta) - w(\alpha)}{p - 1},$$

provided one of the following conditions holds:

(a) $\beta$ and $\alpha'$ lie in distinct congruence classes;
(b) $\beta \sim \alpha'$ and $s(\beta) \neq s(\alpha')$;
(c) $\beta \sim \alpha'$, $s(\beta) = s(\alpha')$, $w(\beta) < w(\alpha')$.

Proof. To simplify notation, we shall assume that $j = 0$. For each $l \in \mathbb{N}$ we write $B_l$ instead of $B_l^{(\infty)}$ in (5.3). For $\alpha \in \mathbb{N}^n$ let

$$B(\alpha) = \begin{cases} 
\prod_{i=1}^n c_i^{\alpha_i/d_i} B_{\alpha_i/d_i}, & \text{if } d_i | \alpha_i \text{ for all } i; \\
0, & \text{otherwise.}
\end{cases}$$

By (5.4), $\text{ord } B(\alpha) \geq J(\alpha)/(p - 1)$, and by (5.5), $\text{ord } B(\alpha) = J(\alpha)/(p - 1)$, if $\alpha_i/d_i \leq p - 1$ for all $i$.

With these notations:

$$F(t^\rho) = \sum_{\alpha \in \mathbb{N}^n} B(\alpha) t^\alpha,$$

$$F_0(t, X) = \sum_{\alpha \in E} \sum_{\lambda \in \mathbb{N}} B(\alpha + \lambda \alpha) t^\alpha Y^{\lambda M}.$$

Let $\alpha \in \tilde{\Delta}_p$:

$$\mathcal{S}_X^{(0)}(Y^{-Ms(\alpha)} t^\alpha)$$

$$= \sum_{\lambda \in \mathbb{N}} \sum_{\eta \in \mathbb{N}} B(\eta + \lambda \alpha) Y^{Ms(\alpha + \eta) - pMs(\sigma) - Ms(\alpha) + \lambda M} t^\sigma,$$

where the inner sum is indexed by the set

$$\{(\eta, \sigma) \in E^{(0)} \times E^{(\rho')} | \eta_i + \lambda a_i \equiv 0 \text{ mod } d_i, \omega(\alpha + \mu) = p \omega(\sigma)\}.$$

Let

$$\xi \in L_p \left( \frac{p}{p - 1}, c \right), \quad \xi = \sum_{(\alpha, \gamma) \in E_p} A(\alpha; \gamma) t^\alpha Y^\gamma.$$

If we write

$$\xi = \sum_{\beta \in \tilde{\Delta}} E_\beta(Y) t^\beta + \sum_{i=1}^{n-1} \overline{H}_i \ast \zeta_i,$$

we saw in the proof of Proposition 3.1 that the coefficient of $Y^{-pMs(\beta)}$ in $E_\beta(Y)$ is $\sum u(\widehat{\alpha}) A(\widehat{\alpha}; \gamma)$, where the sum is indexed by the set

$$\{(\widehat{\alpha}; \gamma) \in E \times \mathbb{N} | -pMs(\beta) = m \mu M + \gamma, \widehat{\alpha} \sim \beta + \mu a, \quad J(\widehat{\alpha}) = J(\beta) + \mu a, \mu \in \mathbb{N}\},$$
and where each \( u(\alpha) \) is a unit in \( \mathcal{O}_0 \). Thus, if we write

\[
\mathcal{F}_X^{(0)}(Y^{-M s(\alpha)} e^\alpha) = \sum_{\beta \in \Delta} \overline{C}_{\beta, \alpha}(Y) Y^{-p M s(\beta)} e^\beta + \sum_{i=1}^{n-1} H_i^* \zeta_i,
\]

then the constant coefficient of \( \overline{C}_{\beta, \alpha}(Y) \) is

\[
\overline{C}_{\beta, \alpha}(0) = \sum u(\sigma) B(\mu + \lambda a),
\]

where the sum is indexed by the set \( S(\beta, \alpha) \) of all \( (\eta, \sigma, \lambda) \in E(0) \times E(\rho') \times \mathbb{N} \) satisfying:

\[
\begin{cases}
ps(\beta) - s(\alpha) + s(\alpha + \eta) - ps(\sigma) + \lambda + p\mu = 0 \\
\sigma \sim \beta + \mu a, \quad \mu \in \mathbb{N} \\
J(\sigma) = J(\beta) + \mu a \\
\omega_{i,j}(\alpha + \eta) = p \omega_{i,j}(\sigma) \\
\eta_i + \lambda a_i \equiv 0 \mod d_i \quad i = 1, \ldots, n.
\end{cases}
\]

Let \( (\eta, \sigma, \lambda) \in S(\beta, \alpha) \). If \( \sigma \sim \beta + \mu a \) and \( J(\sigma) = J(\beta) + \mu a \) for some \( \mu \in \mathbb{N} \), then necessarily \( s(\sigma) \leq s(\beta) + \mu \). On the other hand, \( s(\alpha + \eta) \geq s(\alpha) + s(\eta) \). Hence:

\[
0 = ps(\beta) - s(\alpha) + s(\alpha + \eta) - ps(\sigma) + \lambda + p\mu \\
\geq s(\alpha + \eta) - s(\alpha) + \lambda \geq s(\eta) + \lambda \geq 0.
\]

We conclude that \( s(\alpha + \eta) = s(\alpha) \), \( s(\sigma) = s(\beta) + \mu \), \( \lambda = 0 \), \( s(\eta) = 0 \). Furthermore, since \( \sigma \) and \( \beta \) are elements of \( E \), \( s(\sigma) < 1 \) and \( s(\beta) < 1 \); hence \( \mu = 0 \). Thus

\[
\overline{C}_{\beta, \alpha}(0) = \sum u(\sigma) B(\eta),
\]

where the sum is indexed by the set \( T(\beta, \alpha) \) of all \( (\eta, \sigma) \in E(0) \times E(\rho') \) which satisfy

\[
\begin{cases}
s(\alpha + \eta) = s(\alpha) \\
s(\eta) = 0 \\
s(\sigma) = s(\beta) \\
\sigma \sim \beta, \\
J(\sigma) = J(\beta) \\
\omega_{i,j}(\alpha + \eta) = p \omega_{i,j}(\sigma) \quad \text{for all } i, j \\
\eta_i \equiv 0 \mod d_i \quad \text{for all } i.
\end{cases}
\]
Let \((\eta, \sigma) \in T(\beta, \alpha)\): there is an index \(l\) such that \(\eta_l = 0\) and \(s(\alpha) = s(\alpha + \eta) = \alpha_l/a_l\) and, by Remark 1.1, \(s(\sigma) = \sigma_l/a_l\). Hence:

\[ (5.53) \quad p \left( \frac{\sigma_i}{d_i} - s(\sigma) \frac{a_i}{d_i} \right) - \left( \frac{\alpha_i}{d_i} - s(\alpha) \frac{a_i}{d_i} \right) - \frac{\eta_i}{d_i} = \nu_i \in \mathbb{N} \quad \text{for all } i. \]

By assumption:

\[ (5.54) \quad p \left( \frac{\alpha_i'}{d_i} - s(\alpha') \frac{a_i}{d_i} \right) - \left( \frac{\alpha_i}{d_i} - s(\alpha) \frac{a_i}{d_i} \right) = \delta_i \in \mathbb{N} \quad \text{for all } i. \]

by Lemma 2.8, \(s(\alpha') = \alpha_l'/a_l\) and we deduce from (5.53) and (5.54) that

\[ pg_i \frac{(\sigma_i - \alpha_i)}{g_i} \in \mathbb{Z} \quad \text{for all } i = 1, \ldots, n. \]

Since \(\gcd(g_1, \ldots, g_n) = 1\) and \((p, M) = 1\), this implies \(\sigma_l \equiv \alpha_l' \mod g_l\); but \(\sigma\) and \(\alpha'\) are elements of \(E(\rho')\): \(\sigma_l/g_l < r, \alpha_l'/g_l < r\) and \(\sigma_l \equiv \alpha_l' \mod r\). Hence \(\sigma_l = \alpha_l'\) and \(s(\sigma) = s(\alpha')\). (5.53) and (5.54) now imply \(p(\sigma_i - \alpha_i') \equiv 0 \mod d_i\) for all \(i\); since \(p, D) = 1\) we deduce \(\alpha' \sim \sigma \sim \beta\). In particular, \(T(\beta, \alpha) = \emptyset\) if \(\beta\) and \(\alpha'\) lie in distinct congruence classes, or if \(s(\beta) \neq s(\alpha')\). Furthermore, since \(s(\sigma) = s(\beta)\), (5.53) yields

\[ (5.55) \quad p \left( \frac{\beta_i}{d_i} - s(\beta) \frac{a_i}{d_i} \right) - \left( \frac{\alpha_i}{d_i} - s(\alpha) \frac{a_i}{d_i} \right) = \varepsilon_i \in \mathbb{Z} \quad \text{for all } i. \]

Suppose \(\beta \neq \alpha'\): by Lemma 2.8 there exists an index \(j\) such that \(\varepsilon_j < 0\) or alternatively an index \(k\) such that \(\varepsilon_k > p - 1\).

If \(\varepsilon_j < 0\), (5.53) and (5.54) imply \(p(\sigma_j/d_j - \beta_j/d_j) = \nu_j - \varepsilon_j > 0\), hence \(\sigma_j > \beta_j\) and therefore \(\sigma_j \geq \beta_j + d_j\); but \(J(\sigma) = J(\beta)\), hence there exists an index \(m\) such that \(\beta_m \geq \sigma_m + d_m\). Subtracting (5.53) from (5.54) then yields \(\varepsilon_m - \nu_m \geq p\); hence \(\varepsilon_m > p - 1\). Now subtracting (5.54) from (5.55) we obtain

\[ p \left( \frac{\beta_m}{d_m} - \frac{\alpha_m'}{d_m} \right) = \varepsilon_m - \delta_m > 0, \]

hence \(\beta_m > \alpha_m'\). If \(\beta \sim \alpha'\), this last inequality implies that \(\beta_i \geq \alpha_i'\) for all \(i\) (Lemma 2.3) and therefore \(w(\beta) > w(\alpha')\) since \(s(\beta) = s(\alpha')\).

Thus, if \(\beta \sim \alpha', \beta \neq \alpha'\), \(s(\beta) = s(\alpha')\), and \(w(\beta) \leq w(\alpha')\) the set \(T(\beta, \alpha)\) is empty and \(\overline{C}_{\beta, \alpha}(0) = 0\).

Suppose finally that \(\beta = \alpha'\). Since \(J(\sigma) = J(\alpha')\), if \(\sigma \neq \alpha'\) there is an index \(i\) such that \(\alpha_i' \geq \sigma_i + d_i\); but this implies \(\delta_i - \nu_i \geq p\) in (5.53) and (5.54); hence \(\delta_i \geq p\), a contradiction. Hence \(\sigma = \alpha'\) and the set \(T(\alpha', \alpha)\) contains the single element \((\eta, \alpha')\) with \(\eta = (\delta_1 d_1, \ldots, \delta_n d_n)\). In particular, \(\ord_{\alpha', \alpha}(0) = \sum_{i=1}^{n} \delta_i\).
Summarizing:

(i) \( \text{ord} C_{\alpha',\alpha}(0) = (pw(\alpha') - w(\alpha))/(p - 1); \)

(ii) if \( \beta \neq \alpha' \) then \( C_{\beta,\alpha}(0) = 0 \) whenever one of the following holds:

(a) \( \beta \) and \( \alpha' \) lie in distinct congruence classes;

(b) \( \beta \sim \alpha' \) and \( s(\beta) \neq s(\alpha') \);

(c) \( \beta \sim \alpha' \), \( s(\beta) = s(\alpha') \), and \( w(\beta) \leq w(\alpha') \).

The proposition now follows from the fact that, by (5.36) and Theorem 3.4:

\[
(5.56) \quad C_{\beta,\alpha}(Y) - \overline{C}_{\beta,\alpha}(Y) \in Rp \left( \frac{p}{p - 1}, \frac{pw(\beta) - w(\alpha)}{p - 1} + 1 \right) \quad \forall \alpha, \beta \in \Delta. \]

Let \( \pi \) be a uniformizer of \( \mathbb{Q}_p(\zeta_p) \) and let \( \pi' \) be a root of \( Z^{MD} - \pi \) in \( \Omega \). If \( \mathcal{F} \) is the completion of the maximal unramified extension of \( \mathbb{Q}_p \) in \( \Omega \), we let \( \mathcal{F} = \mathcal{F}(\pi') \) and we extend \( \tau \) to \( \mathcal{F}' \) by setting \( \tau(\pi') = \pi' \).

Let \( \mathcal{G}^{(j)}(Y) \) be the matrix of \( \mathcal{F}^{(j)}_X: W^{(j)}_{x,\rho} \rightarrow W^{(j+1)}_{x,\rho} \) with respect to the bases \( \{\pi^{w(\alpha)}_X Y^{-p^{s(\alpha)}} t^\alpha | \alpha \in \tilde{\Delta}_{\rho^{(j)}}\} \) of \( W^{(j)}_{x,\rho} \) and \( \{\pi^{w(\beta)} Y^{-p^{s(\beta)}} t^\beta | \beta \in \tilde{\Delta}_{\rho^{(j+1)}}\} \) of \( W^{(j+1)}_{x,\rho} \).

For \( x \in \Omega_0^x \), with \( \text{ord} x = 0 \), let also \( \mathcal{A}^{(j)}(x) \) be the matrix of \( \mathcal{F}^{(j)}_X: W^{(j)}_{x,\rho} \rightarrow W^{(j+1)}_{x,\rho} \) with respect to the bases \( \{\pi^{w(\alpha)} t^\alpha | \alpha \in \tilde{\Delta}_{\rho^{(j)}}\} \) of \( W^{(j)}_{x,\rho} \) and \( \{\pi^{w(\beta)} t^\beta | \beta \in \tilde{\Delta}_{\rho^{(j+1)}}\} \) of \( W^{(j+1)}_{x,\rho} \).

By Proposition 5.2, the following estimates hold:

\[
(5.57) \quad \begin{cases} 
\text{ord} \mathcal{G}^{(j)}_{\beta,\alpha}(0) \geq w(\beta) & \text{for all } (\alpha, \beta) \in \tilde{\Delta}_{\rho^{(j)}} \times \tilde{\Delta}_{\rho^{(j+1)}}; \\
\text{ord} \mathcal{G}^{(j)}_{\alpha',\alpha}(0) = w(\alpha') & \text{for all } \alpha \in \tilde{\Delta}_{\rho^{(j)}}; \\
\mathcal{G}^{(j)}_{\beta,\alpha}(0) = 0 & \text{if } \beta \text{ and } \alpha \text{ satisfy condition (a),} \\
& \text{(b), or (c) of Proposition 5.2 (ii).}
\end{cases}
\]

\[
(5.58) \quad \begin{cases} 
\text{ord} \mathcal{G}^{(j)}_{\beta,\alpha}(x) \geq w(\beta) & \text{for all } (\alpha, \beta) \in \tilde{\Delta}_{\rho^{(j)}} \times \tilde{\Delta}_{\rho^{(j+1)}}; \\
\text{ord} \mathcal{G}^{(j)}_{\alpha',\alpha}(x) = w(\alpha') & \text{for all } \alpha \in \tilde{\Delta}_{\rho^{(j)}}; \\
\text{ord} \mathcal{G}^{(j)}_{\beta,\alpha}(x) > w(\beta) & \text{if } \beta \text{ and } \alpha \text{ satisfy condition (a),} \\
& \text{(b), or (c) of Proposition 5.2 (ii).}
\end{cases}
\]

If \( \alpha \in \tilde{\Delta} \), we let \( Z(\alpha) = w(\alpha) + w(\alpha') + \cdots + w(\alpha^{(J-1)}) \) and, for fixed \( \rho \), we let

\[
\mathcal{H}_\rho(T) = \prod_{\alpha \in \tilde{\Delta}_\rho} (1 - p^{Z(\alpha)} T) \in \Omega_1[T].
\]

Let \( Q = \mathcal{N} \prod_{i=1}^n k_i \).
THEOREM 5.2. The Newton polygon of \( L(\bar{f}, \Theta, \rho, T) \) lies below the Newton polygon of \( H_\rho(T) \) and their endpoints coincide at \((0,0)\) and \((Q, Q(n-1)/2)\).

Proof. Let \( R = N \prod_{i=1}^{n} k_i = \dim_{\Omega_0}(W_{X, \rho}) \). We can write

\[
det_{\Omega_0}(I - T\bar{f}_X | W_{X, \rho}) = 1 + \sum_{i=1}^{R} m_i(Y)T^i,
\]
and by Proposition 5.1 each \( m_i(Y) \) is analytic in the disk \( \{ y \mid \ord y > -Np/Mq(p-1) \} \). If \( y \) satisfies \( \ord y = 0 \), by the maximum modulus theorem, \( \ord(m_i(y)) \leq \ord(m_i(0)) \). Observe that if \( \alpha, \beta \in \tilde{\Delta} \) satisfy \( \alpha \sim \beta \), \( s(\alpha) = s(\beta) \) and \( w(\alpha) \leq w(\beta) \), then \( w(\alpha') \leq w(\beta') \). Thus, using (5.57), we can order the elements of \( \tilde{\Delta}_{\rho(j)} \) for each \( j, 0 \leq j \leq \mathcal{f} - 1 \), so that the matrices \( \mathcal{C}(j)(0) \) are simultaneously upper triangular, with diagonal entries \( \{ \mathcal{C}_{\alpha(j+1), \alpha(j)}(0) \mid \alpha \in \tilde{\Delta}_\rho \} \) and \( \ord \mathcal{C}_{\alpha(j+1), \alpha(j)}(0) = w(\alpha(j+1)) \). Hence for each \( i, 1 \leq i \leq R, \ord(m_i(0)) \) is the infimum of all the \( i \)-fold sums \( \sum \mathcal{Z}(\alpha) \), where \( \alpha \) runs over a subset of \( i \) distinct elements of \( \tilde{\Delta}_\rho \). This establishes the first assertion. By Lemma 2.9, \( \sum_{\alpha \in \tilde{\Delta}_\rho} w(\alpha) = R(n-1)/2 \) for any \( \rho \). Hence \( \ord m_Q(0) = \mathcal{f}R(n-1)/2 \).

On the other hand, estimates (5.58) imply that, for all \( j, 0 \leq j \leq \mathcal{f} - 1 \),

\[
\ord(\det \mathcal{C}(j)(x)) = \sum_{\alpha \in \tilde{\Delta}_\rho(j)} w(\alpha).
\]

The second assertion follows. \( \square \)

COROLLARY 5.1. If \( p \equiv 1 \mod{r} \), the endpoints of the Newton polygons of \( L(\bar{f}, \Theta, \rho, T) \) and of \( H_\rho(T) \) coincide.

THEOREM 5.3. If \( p \equiv 1 \mod{r} \), \( (\text{or } \rho = (0, \ldots, 0)) \), and \( pg_i \equiv g_i \mod{k_ig_j} \) for all \( i, j \in \{1, \ldots, n\} \), the Newton polygons of \( L(\bar{f}, \Theta, \rho, T) \) and of \( H_\rho(T) \) coincide.

Proof. Under our assumptions, the permutation \( \alpha \leftrightarrow \alpha' \) of Lemma 2.8 is the identity on \( \tilde{\Delta}_\rho \). Using the estimates (5.58), the remainder of the proof is identical to that of [15, Theorem 5.46]. \( \square \)

REMARK. Theorem 5.3 holds in particular when \( p \equiv 1 \mod{MD} \).
REFERENCES


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