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SHEAVES AND FUNCTIONAL CALCULUS

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Let A be a commutative Banach algebra with identity over the complex field, \mathbb{C} . Let a_1, \dots, a_n be elements of A , and $\text{sp}(a)$ their joint spectrum. In this paper we seek to characterize the functional calculus

$$T_a: \mathcal{O}(\text{sp}(a), A) \rightarrow A$$

as part of a cohomology sequence of certain sheaves, and the algebra A as the algebra of sections

$$H^0(\text{sp}(a), \mathcal{A}) = A$$

of a sheaf \mathcal{A} , which is related to the Putinar structural sheaf. This is obtained under certain conditions on a_1, \dots, a_n . The problem is related also to the unique extension property and to the local analytic spectrum $\sigma(a, x)$ of x with respect to a .

Section 2 is devoted to attacking this problem. In §1, some preliminary results are obtained. We also prove that if $\sigma(a, x)$ is empty, then x is nilpotent.

1. Let us start by briefly recalling (some details may be found in [2], [5]) the construction of a holomorphic functional calculus morphism

$$T_a: \mathcal{O}(\text{sp}(a), A) \rightarrow A.$$

Let U be an open neighborhood of $\text{sp}(a)$, and u_1, \dots, u_n, ψ infinitely differentiable A -valued functions defined on U and verifying:

- (i) $\sum_{i=1}^n u_i(z)(z_i - a_i) + \psi(z) = 1$, for all z in U .
- (ii) ψ has compact support contained in U .
- (iii) $\psi = 1$ in some neighborhood of $\text{sp}(a)$.

Then

$$T_a^U(f) = f(a) = n!(2\pi i)^{-n} \int_U f du_1 dz_1 \cdots du_n dz_n$$

defines a continuous A -linear morphism from $\mathcal{O}(U, A)$ to A . The compatibility of these morphisms as U varies over open neighborhoods of $\text{sp}(a)$ produces T_a . We have the following theorem, where U denotes a neighborhood of $\text{sp}(a)$.

THEOREM 1.1. *Let $f \in \mathcal{O}(U, A)$, and suppose there are g_1, \dots, g_n in $C^\infty(U, A)$ such that*

$$f(z) = \sum_{i=1}^n g_i(z)(z_i - a_i) \quad \text{for all } z \text{ in } U.$$

Then there exists an A -valued differential $2n$ -form α over U , verifying:

- (i) $(z_i - a_i)\alpha = 0$ for $i = 1, \dots, n$; and $f\alpha = 0$.
- (ii) For every h in $\mathcal{O}(U, A)$.

$$n!(2\pi i)^{-n} \int_U h\alpha = f(a)^n h(a).$$

Proof. Let u_1, \dots, u_n, ψ be as above, and let $f_k = u_k f$, $q_k = g_k \psi$, and $r_{jk} = g_k u_j - u_k g_j$. Then

$$\begin{aligned} g_k - f_k &= g_k \left(\sum_{j=1}^n u_j(z_j - a_j) + \psi \right) - u_k \sum_{j=1}^n g_j(z_j - a_j) \\ &= \sum_{j=1}^n r_{jk}(z_j - a_j) + q_k. \end{aligned}$$

Also

$$\sum_{j=1}^n f_j(z_j - a_j) = (1 - \psi)f = f - \psi f$$

and therefore, differentiating and multiplying by $dz_1 \cdots dz_n$,

$$\sum_{j=1}^n (z_j - a_j) df_j dz_1 \cdots dz_n = -d(\psi f) dz_1 \cdots dz_n.$$

Since $\text{supp}(\psi f)$ and $\text{supp}(q_k)$ are compact sets contained in U , we may proceed as in [5] (III, 4.9), and obtain an $n - 1$ differential form τ with $\text{supp}(\tau)$ contained in U and such that

$$\begin{aligned} (1) \quad d\tau dz_1 \cdots dz_n &= dg_1 dz_1 \cdots dg_n dz_n - df_1 dz_1 \cdots df_n dz_n \\ &= dg_1 dz_1 \cdots dg_n dz_n - f^n du_1 dz_1 \cdots du_n dz_n \end{aligned}$$

Now set

$$(2) \quad \alpha = dg_1 dz_1 \cdots dg_n dz_n \quad \text{and}$$

$$(3) \quad \omega = du_1 dz_1 \cdots du_n dz_n.$$

Differentiating f we obtain

$$df = \sum_{j=1}^n g_j dz_j + \sum_{j=1}^n (z_j - a_j) dg_j$$

and multiplying by $dg_1 dz_1 \cdots \widehat{dg_k} dz_k \cdots dg_n dz_n$,

$$0 = (z_k - a_k)\alpha.$$

Multiplying by g_k and adding gives $f\alpha = 0$ and so, (i) is proved.

Now let h be an element of $\mathcal{O}(U, A)$. By (1), (2), and (3) we have

$$h\alpha - hf^n\omega = h d\tau dz_1 \cdots dz_n = d(h\tau) dz_1 \cdots dz_n.$$

Hence, by construction of the functional calculus,

$$n!(2\pi i)^{-n} \int_U h - h(a)f(a)^n = n!(2\pi i)^{-n} \int_U d(h\tau) dz_1 \cdots dz_n$$

but this is zero by Stokes' theorem, for $\text{supp}(h\tau)$ is contained in U .

COROLLARY 1.2. *Under the hypothesis of the theorem, $f(a)^{n+1} = 0$.*

Proof. Simply put $h = f$.

COROLLARY 1.3. *Let U be an open neighborhood of $\text{sp}(a)$, and $f \in \mathcal{O}(U, A)$. Suppose that for every z^0 in U , there are f_1, \dots, f_n infinitely differentiable functions near z^0 such that*

$$f(z) = \sum_{i=1}^n f_i(z)(z_i - a_i) \quad \text{for } z \text{ near } z^0.$$

Then $f(a)^{n+1} = 0$.

Proof. A partition of unity will put us in a situation where the theorem is applicable.

Now suppose x is an element of A and consider $\sigma(a, x)$, the local analytic spectrum of x with respect to a ([1], [4]). Putting $f = x$, we obtain that if $\sigma(a, x)$ is empty, then $x^{n+1} = 0$. The conclusion $x = 0$ is known only under additional hypotheses [4].

2. Let \mathcal{O}^A be the sheaf of germs of holomorphic A -valued functions over \mathbb{C}^n . If $a = (a_1, \dots, a_n) \in A^n$, the morphism $\lambda_a: (\mathcal{O}^A)^n \rightarrow \mathcal{O}^A$ defined by

$$\lambda_a(f_1, \dots, f_n) = \sum_{i=1}^n (z_i - a_i) f_i$$

induces an exact sequence of sheaves

$$0 \rightarrow \mathcal{N}_a \rightarrow (\mathcal{O}^A)^n \rightarrow \mathcal{O}^A \rightarrow \mathcal{A} \rightarrow 0.$$

Here the stalk of \mathcal{N}_a over z^0 , $\mathcal{N}_{a_{z^0}}$, consists of germs of n -tuples (g_1, \dots, g_n) of functions analytic near z^0 and verifying $\sum (z_i - a_i)g_i = 0$ in some neighborhood of z^0 . $\mathcal{A}_{z^0} = \mathcal{O}_{z^0}^A / \mathcal{I}_{z^0}$, where $\mathcal{I} \subset \mathcal{O}^A$ is the sheaf of ideals generated by $(z_i - a_i)$ for $i = 1, \dots, n$. Note that if z^0 is not in $\text{sp}(a)$, $\mathcal{I}_{z^0} = \mathcal{O}_{z^0}^A$, and therefore $\mathcal{A}_{z^0} = 0$.

Clearly, if U is an open, holomorphically convex subset of C^n , then

$$\begin{aligned} H^0(U, \mathcal{N}_a) &= \mathcal{N}_a(U) \\ &= \left\{ (f_1, \dots, f_n) \in \mathcal{O}(U, A)^n : \sum_{i=1}^n (z_i - a_i)f_i = 0 \right\}. \end{aligned}$$

On the other hand, if $I(U)$ denotes the ideal generated by $(z_i - a_i)$, $i = 1, \dots, n$ in $\mathcal{O}(U, A)$, then $I(U) \subset H^0(U, \mathcal{I})$, but the equality does not, in general, hold.

Suppose U contains the joint spectrum of a_1, \dots, a_n . Since $T_a^U(z_i - a_i) = 0$ for $i = 1, \dots, n$; we have the inclusion $I(U) \subset \text{Ker } T_a^U$. The following proposition shows that the ideals are the same.

PROPOSITION 2.1. *Let U be a holomorphically convex open neighborhood of $\text{sp}(a)$. Then $\text{Ker } T_a^U = I(U)$.*

Proof. The ideal of $\mathcal{O}(U \times U, \mathbb{C})$ generated by the functions

$$(z, w) \mapsto z_i - w_i, \quad i = 1, \dots, n,$$

is the ideal of functions analytic on $U \times U$ and zero on the diagonal $\Delta \subset U \times U$, for both ideals are closed, and they coincide locally.

Since $\mathcal{O}(U \times U, A) = \mathcal{O}(U \times U, \mathbb{C}) \otimes_{\mathbb{C}} A$, it follows from [3] that all $g : U \times U \rightarrow A$ null over Δ belong to the ideal generated by $(z_i - w_i)$, for $i = 1, \dots, n$.

Therefore, if $f \in \mathcal{O}(U, A)$, there are analytic $g_k : U \times U \rightarrow A$, such that

$$f(z) - f(w) = \sum_{k=1}^n g_k(z, w)(z_k - w_k).$$

Applying the functional calculus morphism in the w -variable,

$$f(z) - f(a) = \sum_{k=1}^n g_k(z, a)(z_k - a_k).$$

Hence, if $f(a) = 0$, $f \in I(U)$.

Now we can relate this fact with the homological approach of Putinar ([7]; see also [6]); to do this we consider the presheaf \mathcal{P} over \mathbb{C}^n defined by

$$\mathcal{P}(U) = \mathcal{O}(U) \hat{\otimes}_{\mathcal{O}(\mathbb{C}^n)} A \quad (U \text{ open, } U \in \mathbb{C}^n),$$

and let \mathcal{F} be the sheaf defined by \mathcal{P} . The standard definitions ([6]) give the identification

$$(*) \quad \mathcal{P}(U) = \mathcal{O}^A(U) / I(U).$$

Hence looking at the germs we have

LEMMA 2.2. *The sheaf \mathcal{A} is the sheaf \mathcal{F} defined by the presheaf $\mathcal{P} = \mathcal{O} \hat{\otimes}_{\mathcal{O}(\mathbb{C}^n)} A$.*

We also have the following fact:

PROPOSITION 2.3. *Let U be a holomorphically convex open neighborhood of $\text{sp}(a)$. Then*

(i) *The functional calculus induces a topological isomorphism*

$$\mathcal{P}(U) \approx A$$

(ii) *The kernel of the canonical map*

$$\mathcal{P}(U) \rightarrow \mathcal{A}(U)$$

consists of nilpotent elements.

Proof. The first assertion follows easily from Proposition 2.1 and (*) above, since $I(U)$ is closed in $\mathcal{O}(U, A)$. For the second, let $f \in \mathcal{O}^A(U)$ and assume that the image of f is zero in $\mathcal{A}(U)$; this means that the class of the germ of f in \mathcal{A}_z is zero for every $z \in U$.

Then for every $z^0 \in U$ we have an n -tuple $(g_{g_1}^{z^0}, \dots, g_n^{z^0})$ of functions analytic near z^0 such that $f = \sum (z_i - a_i) g_i^{z^0}$ in some neighborhood of z^0 .

Using a partition of unity we are in the situation of Corollary 1.3; hence $f(a)^{n+1} = 0$. But this implies $f^{n+1} \in I(U)$ and this means that $f^{n+1} = 0$ in $\mathcal{P}(U)$.

We shall now study, for a neighborhood U of $\text{sp}(a)$, the cohomology sequence resulting from the exact sequence of sheaves

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{O}^A \rightarrow \mathcal{A} \rightarrow 0.$$

Note that when U is holomorphically convex, $H^p(U, \mathcal{O}^A) = 0$, for all $p > 0$, due to [3] and the well-known case $A = \mathbb{C}$. We have then the commutative diagram

$$\begin{array}{ccccccc}
 0 & \rightarrow & I(U) & \rightarrow & \mathcal{O}(U, A) & \xrightarrow{T_a^A} & A \rightarrow 0 \\
 & & \downarrow & & \parallel & & \downarrow L \\
 0 & \rightarrow & H^0(U, \mathcal{F}) & \rightarrow & \mathcal{O}(U, A) & \rightarrow & H^0(U, \mathcal{A}) \rightarrow H^1(U, \mathcal{F}) \rightarrow 0
 \end{array}$$

The ideal $\text{Ker } L$ is isomorphic, because of the snake lemma construction, to the A -module $H^0(U, \mathcal{F})/I(U)$. In fact, $H^0(U, \mathcal{F}) \simeq \text{Ker } L \oplus I(U)$. We obtain also the exact sequence,

$$0 \rightarrow I(U) \rightarrow H^0(U, \mathcal{F}) \xrightarrow{T_a^U} A \xrightarrow{L} H^0(U, \mathcal{A}) \rightarrow H^1(U, \mathcal{F}) \rightarrow 0$$

and therefore, $H^1(U, \mathcal{F}) \simeq H^0(U, \mathcal{A})/\text{Im } L$.

On the other hand, the exact sequence of sheaves

$$0 \rightarrow \mathcal{N}_a \rightarrow (\mathcal{O}^A)^n \xrightarrow{\lambda_a} \mathcal{F} \rightarrow 0$$

produces the exact cohomology sequence

$$\begin{aligned}
 0 \rightarrow \mathcal{N}_a(U) \rightarrow \mathcal{O}(U, A) \xrightarrow{n} H^0(U, \mathcal{F}) \rightarrow H^1(U, \mathcal{N}_a) \rightarrow 0 \rightarrow \dots \\
 \dots \rightarrow 0 \rightarrow H^{p-1}(U, \mathcal{F}) \rightarrow H^p(U, \mathcal{N}_a) \rightarrow 0 \rightarrow \dots
 \end{aligned}$$

We then have

$$H^1(U, \mathcal{N}_a) \simeq H^0(U, \mathcal{F})/I(U) \simeq \text{Ker } L$$

and

$$H^p(U, \mathcal{N}_a) \simeq H^{p-1}(U, \mathcal{F}), \quad \text{for } p > 1.$$

We have proved:

- PROPOSITION 2.4.** *The morphism $L: A \rightarrow H^0(U, \mathcal{A})$ is*
- (i) *a monomorphism iff $H^0(U, \mathcal{F}) = 0$ iff $H^1(U, \mathcal{N}_a) = 0$*
 - (ii) *an epimorphism iff $H^1(U, \mathcal{F}) = 0$ iff $H^2(U, \mathcal{N}_a) = 0$.*

Note that $\text{Ker } L$ consists of the elements x in A whose local analytic spectrum is empty. Therefore, $L(x) = 0$ implies $x^{n+1} = 0$. If A has no nilpotent elements, $\text{Ker } L = 0$ and $H^0(U, \mathcal{F}) = I(U)$.

DEFINITION. We shall say that A is a -representable if

- (i) $\text{sp}(a)$ is holomorphically convex, and
- (ii) $H^1(\text{sp}(a), \mathcal{N}_a) = 0, H^2(\text{sp}(a), \mathcal{N}_a) = 0$.

Note that the first condition ensures the existence of a basis for neighborhoods of $\text{sp}(a)$ made up of holomorphically convex open sets,

while the second says that L is an isomorphism for a basis of neighborhoods of $\text{sp}(a)$. Hence, $A = H^0(\text{sp}(a), \mathcal{A})$.

If $n = 1$, and $\text{sp}(a)$ has no interior, A is a -representable: in this case, $\mathcal{N}_a = 0$, for if $g \in \mathcal{N}_a(V)$, then $g|_{V \cap (\mathbb{C} - \text{sp}(a))} = 0$, and hence, $g = 0$.

Finally, we wish to compare a -representability and the unique extension property [4].

THEOREM 2.5. *Suppose that $\text{sp}(a)$ is holomorphically convex, and that the n -tuple $a = (a_1, \dots, a_n)$ (considered as a family of operators from A to A) has the unique extension property. Then A is a -representable.*

Proof. Consider the sheaf complex $K = K(\mathcal{O}, \alpha)$, where, for each open set V ,

$$K^r(V) = \mathcal{O}(V, \Lambda_A^r(A^n))$$

consists of analytic A -valued r -forms over V , and

$$\alpha_r: K^r \rightarrow K^{r+1}$$

is induced by the exterior product $\eta \rightarrow \sum_{j=1}^n (z_j - a_j) dz_j \wedge \eta$. For $n - 1$, α may be written as

$$\begin{aligned} \alpha_{n-1} \left(\sum_{i=1}^n f_i dz_1 \cdots d\hat{z}_i \cdots dz_n \right) \\ = \sum_{i=1}^n (-1)^{i+1} f_i (z_i - a_i) dz_1 \cdots dz_n \end{aligned}$$

so that $\text{Ker } \alpha_{n-1}(z)$ is the stalk \mathcal{N}_{a_z} (save a sign), and $\text{Ker } \alpha_{n-1} = \mathcal{N}_a$.

Now the unique extension property expresses that cohomology $H^r(K) = 0$, for $r = 0, \dots, n - 1$, that is, the sequence of sheaves

$$0 \rightarrow K^0 \xrightarrow{\alpha_0} K^1 \xrightarrow{\alpha_1} \dots \rightarrow K^{n-2} \xrightarrow{\alpha_{n-2}} \mathcal{N}_a \rightarrow 0$$

is exact. Since the sheaves K^r are acyclic, that is, $H^i(U, K^r) = 0$ for holomorphically convex U and $i > 0$, we obtain $H^p(U, \mathcal{N}_a) = 0$ for all $p > 0$.

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