COMPLEMENTATION OF CERTAIN SUBSPACES OF $L^\infty(G)$ OF A LOCALLY COMPACT GROUP

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Let $G$ be a locally compact group, $\text{WAP}(G)$ be the space of continuous weakly almost periodic functions on $G$ and $C_0(G)$ the space of continuous functions on $G$ vanishing at infinity. We prove in this paper, among other things, that if $G$ is infinite and $X$ is any subspace of $\text{WAP}(G)$ (or $\text{CB}(G)$, the space of bounded continuous functions in case $G$ is nondiscrete) containing $C_0(G)$, then $X$ is uncomplemented in $L_\infty(G)$. If $G$ is non-compact, then $\text{WAP}(G)$ is uncomplemented in $L_{\text{UC}}(G)$. Furthermore, $\text{AP}(G)$, the space of continuous almost periodic functions on $G$, is complemented in $L_{\text{UC}}(G)$ if and only if $G/N$ is compact, where $N$ is the intersection of the kernels of all finite-dimensional continuous unitary representations of $G$. We also prove that if $A$ is any left translation invariant $C^*$-subalgebra of $C_0(G)$, then $A$ is the range of a continuous projection commuting with left translations.

1. Introduction and some preliminaries. Let $G$ be a locally compact group and $\text{CB}(G)$ be the space of bounded continuous complex-valued functions on $G$ with supremum norm. Let $L_{\text{UC}}(G)$ denote the space of bounded left uniformly continuous complex-valued functions on $G$, i.e. all $f \in \text{CB}(G)$ such that the map $g \to l_g f$ from $G$ into $\text{CB}(G)$ is continuous when $\text{CB}(G)$ has the norm topology where $l_g f(x) = f(gx)$, $x \in G$. Let $\text{WAP}(G)$ (respectively $\text{AP}(G)$) denote the space of continuous weakly almost periodic (respectively almost periodic) functions on $G$ i.e. all $f \in \text{CB}(G)$ such that $\{l_a f; a \in G\}$ is relatively compact in the weak (resp. norm) topology of $\text{CB}(G)$. Let $L_{\infty}(G)$ denote the Banach space of essentially bounded complex-valued functions on $G$ with the essential supremum norm $\| \cdot \|_\infty$ as defined in [12, p. 141]. Then $\text{CB}(G)$, $L_{\text{UC}}(G)$, $\text{WAP}(G)$ and $\text{AP}(G)$ are translation invariant subalgebras of $L_{\infty}(G)$ with $\text{AP}(G) \subseteq \text{WAP}(G) \subseteq L_{\text{UC}}(G) \subseteq \text{CB}(G)$. Furthermore, $C_0(G) \cap \text{AP}(G) = \{0\}$ unless $G$ is compact, where $C_0(G)$ is the closed subalgebra of $\text{CB}(G)$ consisting of all $f \in \text{CB}(G)$ vanishing at infinity. Recall that an application of the Ryll-Nardzewski fixed point theorem ([21]) shows that $\text{WAP}(G)$ has a unique invariant mean $m_G$ i.e. $m_G$ is a positive linear functional on $\text{WAP}(G)$ of norm one and $m_G(l_a f) = m_G(r_a f) = m_G(f)$ for all $f \in \text{WAP}(G)$, where
Let \( W_0(G) = \{ f \in WAP(G); m_G(|f|) = 0 \} \). Then \( WAP(G) = AP(G) \oplus W_0(G) \) (see \([6]\) or \([2]\)) i.e. \( AP(G) \) is always complemented in \( WAP(G) \).

B. B. Wells proved in \([26]\) that \( AP(\mathbb{R}) \) and \( WAP(\mathbb{R}) \) are uncomplemented in \( LUC(\mathbb{R}) \), where \( \mathbb{R} \) denotes the additive group of the reals. It was also shown by I. Glicksberg \([9]\) that if \( G \) is a compact group, \( A \) is a closed translation invariant subalgebra of \( C(G) \) (continuous complex-valued functions on \( G \)) and \( A \) is not self-adjoint, then \( A \) is uncomplemented in \( C(G) \). More recently, Y. Takahashi \([23]\) proves that a weak*-closed non-self-adjoint translation invariant subalgebra of \( L_\infty(G) \) is uncomplemented in \( L_\infty(G) \) (see \([14]\) for proof of Lemma 4 in \([23]\)). Furthermore, \([24, \text{Theorem 1}]\) if \( G \) is an infinite maximally almost periodic group, then \( WAP(G) \) and \( AP(G) \) are uncomplemented in \( L_\infty(G) \). Also, as shown by Lau in \([13]\), if \( G \) is an amenable locally compact group, then any weak*-closed self-adjoint left translation invariant subalgebra of \( L_\infty(G) \) is the range of a continuous projection commuting with left translations.

In this paper, we prove among other things, (Corollary 3) that if \( G \) is an infinite locally compact group and \( X \) is any closed subspace of \( WAP(G) \) containing \( C_0(G) \), then \( X \) is uncomplemented in \( L_\infty(G) \). If \( G \) is non-discrete and \( X \) is any closed subspace of \( CB(G) \) containing \( C_0(G) \), then \( X \) is not complemented in \( L_\infty(G) \) (Theorem 4). Furthermore, (Theorem 6), if \( G \) is a locally compact non-compact group, then \( WAP(G) \) is not complemented in \( LUC(G) \). We prove that (Theorem 7) if \( H \) is a closed subgroup of a locally compact group \( G \), then \( CB(G/H) \) (when identified as a closed subspace of \( CB(G) \)) is always complemented in \( CB(G) \). This result is used to show that (Theorem 8) \( AP(G) \) is complemented in \( LUC(G) \) if and only if \( G/N \) is compact where \( N \) is the intersection of the kernels of all finite dimensional continuous unitary representations of \( G \). In particular, if \( G \) is maximally almost periodic, then \( AP(G) \) is complemented in \( LUC(G) \) if and only if \( G \) is compact. However (Theorem 11), if \( A \) is a left translation invariant \( C^* \)-subalgebra of \( C_0(G) \), then there exists a continuous projection \( P \) from \( C_0(G) \) onto \( A \) and \( P \) commutes with left translations.

2. Uncomplemented subspaces of \( L_\infty(G) \). In this section we show that if \( G \) is an infinite locally compact group, then any subspace \( X \) of \( WAP(G) \) containing \( C_0(G) \) is uncomplemented in \( L_\infty(G) \). We first establish the following lemma which follows directly from the corollary
in Losert and Rindler [16, p. 74] when $G$ contains a countable dense subset.

**Lemma 1.** Let $G$ be an infinite $\sigma$-compact locally compact group. Then there exists a sequence $\{\mu_n\}$ of probability measures on $G$ such that for each $f \in WAP(G)$

$$\lim_{n \to \infty} \int r_y f \, d\mu_n = m_G(f)$$

and the convergence is uniform with respect to $y$, $y \in G$.

**Proof.** We may assume that $G$ is nondiscrete (otherwise, $G$ is countable, and the lemma follows directly from Losert and Rindler [16, p. 74]).

Let $K$ be a compact normal subgroup such that $G/K$ is metrizable separable (see Remark 14(b)). For each $x \in G$, $f \in WAP(G)$, let $f^K$ be a function on $G$ defined by

$$f^K(x) = m_K(f_x), \quad x \in G,$$

where $f_x(k) = f(xk)$.

Then $f^K$ is constant on each coset of $K$, $f^K \in WAP(G/K)$ and $m_G(f) = m_{G/K}(f^K)$ (see Chou [4, Lemma 2.3]). By the corollary in [16, p. 74], there exists a sequence $\{\bar{x}_n\}$ in $G/K$ such that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} r_{\bar{y}}(f^K)(\bar{x}_n) = m_G(f)$$

holds uniformly in $\bar{y} \in G/K$.

For each $n$, let $\theta_n = (1/N) \sum_{n=1}^{N} \delta_{\bar{x}_n}$, $\bar{x} \in G/K$, where $\delta_{\bar{x}}(f) = f(\bar{x})$. Let $\mu_n$ denote the probability measure on $G$ defined by the functional $\hat{\theta}_n$ on $C_0(G)$, where $\hat{\theta}_n(f) = \theta_n(f^K)$, $f \in C_0(G)$. If $f \in WAP(G)$, $y \in G$, then

$$m_G(f) = m_{G/K}(f^K) = \lim_n \theta_n(r_y f^K) = \lim_n \theta_n((r_y f)^K) = \int r_y f \, d\mu_n$$

and the convergence is uniform in $y$. \hfill $\Box$

**Theorem 2.** Let $G$ be a locally compact group. The following are equivalent:

(a) $G$ is finite.

(b) There exists a continuous linear operator $S$ from $L_\infty(G)$ into $WAP(G)$ such that $S(f) = f$ for all $f \in C_0(G)$.

**Proof.** (a) implies (b) is clear.
(b) implies (a). Let $G_0$ be an infinite open and closed subgroup of $G$ which is $\sigma$-compact. For $f \in L_\infty(G)$, define $(\pi f)(x) = f(x)$ for $x \in G_0$ (restriction to $G_0$). Then $\pi$ is a norm decreasing linear map from $L_\infty(G)$ onto $L_\infty(G_0)$.

Given $h \in L_\infty(G_0)$, write $h' \in L_\infty(G)$, where $h'(x) = h(x)$ if $x \in G_0$ and $h'(x) = 0$ if $x \notin G_0$. Define $S'(g) = \pi S(h')$. Then $S'$ is a bounded linear map from $L_\infty(G_0)$ into $L_\infty(G_0)$. Also if $x \in G_0$, then $l_x S'(h) = \pi(l_x S(h'))$. In particular, the range of $S'$ is contained in $\text{WAP}(G_0)$. Furthermore, if $h \in C_0(G_0)$, then $h' \in C_0(G)$, and $S'(h) = \pi(Sh') = \pi(h') = h$.

Let $\{\mu_n\}$ be a sequence of probability measures on $G_0$ satisfying the conclusion of Lemma 1. Let $\tilde{\mu}_n(f) = \int S'(f) \, d\mu_n$, $f \in L_\infty(G_0)$. Then for each $f \in L_\infty(G_0)$,

$$\lim_n \tilde{\mu}_n(f) = \lim_n \int S'(f) \, d\mu_n = m_{G_0}(S'(f)).$$

Let $\tilde{m}_{G_0}(f) = m_{G_0}(S'(f))$, $f \in L_\infty(G)$. Since $f \in L_\infty(G_0)$ is an abelian $\mathcal{W}^*$-algebra, its spectrum $\Omega$ is Stonean (see [22, p. 46] or [25, p. 109]). Since $C(\Omega)$ and $L_\infty(G_0)$ are isometrically isomorphic via the Gelfand transform, it follows from Theorem 9 [121, p. 168] that weak* convergence of a sequence in $L_\infty(G_0)^*$ implies weak convergence. Consequently $\tilde{m}_{G_0}$ is the weak limit of the sequence $\tilde{\mu}_n$. Let $K$ be the convex hull of $\{\tilde{\mu}_n; n = 1, 2, \ldots\}$ in the Banach space $L_\infty(G_0)^*$; then there exists a sequence $\psi_n$ in $K$ such that $\|\psi_n - \tilde{m}_{G_0}\| \to 0$. For $\psi \in L_\infty(G_0)^*$, let $\psi'$ denote the restriction of $\psi$ to $C_0(G_0)$. Since $S'$ is the identity on $C_0(G_0)$, it follows that for $\psi \in L_\infty(G_0)^*$, $f \in C_0(G_0)$, we have $\tilde{\psi}(f) = \psi(S'(f)) = \psi(f)$ i.e. $\tilde{\psi}' = \psi'$. In particular if $G_0$ is non-compact, then $\tilde{m}_{G_0}' = 0$. Now for each $n$, there exists a continuous function $f$ on $G$ with compact support, $0 \leq f \leq 1$, $f(x) = 1$, if $x \in \text{supp} \mu_i$, $i = 1, \ldots, n$. Since $\tilde{\mu}'_i = \mu'_i$ (as shown above), it follows (by linearity) that if $\varphi = \sum_{i=1}^n \lambda_i \tilde{\mu}'_i$, $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$, then $\varphi(f) = 1$. Hence $\|\varphi\| = 1$. Consequently, each $\varphi$ in $K' = \{\psi'; \psi \in K\}$ has norm one. But this is impossible. Hence $G_0$ is again finite. This implies that $G$ is discrete (otherwise take $G_0 = \bigcup_{n=1}^\infty U^n$ where $U$ is a compact symmetric neighbourhood of the identity) and then that $G$ is finite.

If $G_0$ is compact and infinite (hence not discrete), we may assume that the measures $\mu_n$ are singular with respect to the Haar measure $m_{G_0}$. Then for each $n$, there exists $f \in C_0(G_0)$ with $0 \leq f \leq 1$, $\int f(x) \, d\mu_i(x) = 0$ for $i = 1, \ldots, n$ and $\int f(x) \, dm_{G_0}(x) > m_{G_0}(G_0)/2$. 298 ANTHONY TO-MING LAU AND VIKTOR LOSERT
It follows that \( \| \varphi - m'_{G_0} \| > m_{G_0}(G_0)/2 \) for each \( \varphi \in K \), which is impossible. So \( G_0 \) must again be finite. \( \square \)

The following is a generalization of Theorem 1 (i) \( \leftrightarrow \) (ii) in [24]:

**Corollary 3.** Let \( G \) be a locally compact group. The following are equivalent:

(a) \( G \) is finite.

(b) There exists a closed subspace \( X \) of \( WAP(G) \), \( X \supseteq C_0(G) \) and \( X \) is complemented in \( L_\infty(G) \).

When \( G \) is non-discrete, we have a much stronger result:

**Theorem 4.** Let \( G \) be a locally compact group. The following are equivalent:

(a) \( G \) is discrete.

(b) There exists a closed subspace \( X \) of \( CB(G) \), \( X \supseteq C_0(G) \), and \( X \) is complemented in \( L_\infty(G) \).

**Proof.** (a) implies (b) is clear.

(b) implies (a). If \( G \) is not discrete, let \( U \) be a compact symmetric neighbourhood of the identity of \( G \) and \( G_0 = \bigcup_{n=1}^{\infty} U^n \). Then \( G_0 \) is an infinite open and closed compactly generated subgroup of \( G \). Let \( K \) be a compact normal subgroup of \( G_0 \) such that \( G_0/K \) is metrizable and not discrete (see [12, p. 71]). Then \( G_0/K \) is open in \( G/K \). In particular, \( H = G/K \) is also metrizable. By Corollary 3, \( G \) is non-compact. Since \( H \) is locally compact and not discrete, there exists an infinite compact subset \( L \) of \( H \). By the Borsuk-Dugundji Theorem [7, Theorem 5.1], there exists a continuous linear extension operator \( S_0 : CB(L) \to CB(H) \). Let \( f \) be a continuous real-valued function on \( H \) with compact support satisfying \( f(x) = 1 \) for all \( x \in L \) and let \( \pi : G \to H \) be the canonical mapping. Then \( S(g) = [f \cdot S_0(g)] \circ \pi \) defines a continuous linear mapping from \( CB(L) \) into \( C_0(G) \). Let \( \lambda \) be the normalized Haar measure of \( K \). If \( g \in CB(G) \), let \( R(g) \) denote the restriction of \( g^K \) to \( L \), where \( g^K(x) = m_K(f_x), \ x \in G \). Observe that \( R \circ S \) is the identity on \( CB(L) \); hence \( S \circ R : X \to X \) is a continuous projection on \( Y = \Im S \), i.e., \( Y \) is a complemented subspace of \( X \). Now if \( X \) is complemented in \( L_\infty(G) \), then the same is true for \( Y \). Since \( L \) is infinite and metrizable, \( CB(L) \) is infinite dimensional and separable. Hence \( Y \) (being isomorphic to \( CB(L) \)) is also infinite dimensional and separable. However, as in the proof of Theorem 1, \( L_\infty(G) \), being an abelian von Neumann algebra, is isometrically isomorphic to \( C(\Omega) \).
of a Stonean space $\Omega$. This is impossible by Corollary 2 in [11, p. 169].

3. Uncomplemented subspaces in LUC($G$). B. B. Wells proved in [26] that if $G = \mathbb{R}$, then the space WAP($\mathbb{R}$) is not complemented in LUC($\mathbb{R}$) using Phillips' lemma [21] (or [25, p. 117]). We now show that this result also holds for all locally compact non-compact groups.

**Lemma 5.** Let $G$ be a non-compact group, $\{F_n; n = 1, 2, \ldots\}$ be a family of compact subsets of $G$ and $U$ be a compact neighbourhood of the identity $e$ of $G$. There exists a sequence $\{y_n\}$ in $G$ and a sequence $g_n$ of continuous functions on $G$ with compact support, $0 \leq g_n \leq 1$ such that

(a) $\{UF_n y_n\}$ is pairwise disjoint,
(b) $g_n(x) = 1$ for each $x \in F_n y_n$ and $g_n(x) = 0$ for each $x \notin UF_n y$.
(c) For any subset $E$ of $N = \{1, 2, \ldots\}$, the function $g_E(x) = \sum g_n(x); n \in E$ is left uniformly continuous.

**Proof.** By induction, we can construct a sequence $\{y_n\}$ in $G$ such that $\{UF_n y_n\}$ is pairwise disjoint. Let $V$ be a compact symmetric neighbourhood of $e$ such that $V^3 \subseteq U$. By Urysohn’s Lemma, there exists a continuous function $f: G \rightarrow [0, 1]$ such that $f(e) = 1$ and $f(G \setminus V) = \{0\}$. Define a pseudometric $d$ on $G$ by

$$d(x, y) = \|l_x f - l_y f\|, \quad x, y \in G.$$ 

Also for each $n = 1, 2, \ldots$, define

$$g_n(x) = 1 - d(x, F_n y_n).$$

Clearly, each $g_n$ is continuous, $0 \leq g_n \leq 1$ and $g_n(x) = 1$ for all $x \in F_n y_n$. Furthermore, if $g_n(x) > 0$, then $x \in V^2 F_n y_n$. (Indeed, in this case, $d(x, y) < 1$ for some $y \in F_n y_n$, and hence $Vx \cap Vy \neq \emptyset$. For otherwise $(l_x f)(x^{-1}) = 1$ and $(l_y f)(x^{-1}) = 0$ and $d(x, y) = 1$ i.e. (b) holds.)

Finally, since $\{UF_n y_n\}$ is pairwise disjoint, the function $g_E, E \subseteq N$ is well defined. To see that $g_E$ is left uniformly continuous, let $x \in V, t \in G$ be such that $|g_E(x t) - g_E(t)| > 0$. If $g_E(x t) \neq 0$, then $x t \in V^2 F_n y_n$ for some unique $n, n \in E$, and this gives $t \in V^3 F_n y_n$. Similarly, if $g_E(t) \neq 0$, then both $x t$ and $t$ are in $UF_n y_n$ for some unique $n, n \in E$. Thus

$$|g_E(x t) - g_E(t)| = |g_n(x t) - g_n(t)| = |d(x t, F_n y_n) - d(t, F_n y_n)| \leq d(x t, t) = \|l_x f - f\|. $$
Consequently \( \| l_x g_E - g_E \| \leq \| l_x f - f \| \). Hence \( g_E \in \text{LUC}(G) \) since \( f \in \text{LUC}(G) \).

**Theorem 6.** Let \( G \) be a non-compact group. Then \( \text{WAP}(G) \) is not complemented in \( \text{LUC}(G) \).

**Proof.** We first assume that \( G \) is \( \sigma \)-compact. Let \( \{ \mu_n \} \) be the sequence of probability measures on \( G \) constructed in Lemma 1. Let \( F_n = \text{supp} \mu_n \). Let \( \{ y_n \} \) be a sequence of elements in \( G \) and \( 0 \leq g_n \leq 1 \) be a sequence of continuous functions of \( G \) satisfying the conditions in Lemma 5. Define for each \( f \in \text{WAP}(G) \)

\[
\psi_n(f) = m_G(f) - \int r_{y_n} f \, d\mu_n.
\]

Then, by Lemma 1, \( \lim_{n \to \infty} \psi_n(f) = 0 \) for each \( f \in \text{WAP}(G) \). Assume that \( P \) is a continuous projection of \( \text{LUC}(G) \) onto \( \text{WAP}(G) \) and define for each subset \( E \subset \mathbb{N} \)

\[
\nu_n(E) = \psi_n(P(g_E)).
\]

Then \( \nu_n \) is a finitely additive function on the algebra of subsets of \( \mathbb{N} \) and

\[
\lim_{n} \nu_n(E) = 0 \quad \text{for all} \ E \subseteq \mathbb{N}.
\]

But if \( n \in \mathbb{N} \), \( g_n \in \text{WAP}(G) \) and hence

\[
\nu_n(\{ n \}) = \psi_n(Pg_n) = \psi_n(g_n) = \int r_{y_n} g_n \, d\mu_n = 1
\]

since \( 0 \leq r_{y_n} g_n \leq 1 \), and \( r_{y_n} g_n(x) = 1 \) for each \( x \in F_n = \text{supp} \mu_n \). This contradicts Phillips' Lemma [20].

If \( G \) is not \( \sigma \)-compact, let \( H \) be an open \( \sigma \)-compact but non-compact subgroup of \( G \). For each \( f \in \text{LUC}(H) \), let \( f' \) be the continuous function on \( G \) which agrees with \( f \) on \( H \) and is zero outside \( H \). Then \( f' \in \text{LUC}(G) \). Also, if \( f \in \text{WAP}(H) \), then \( f' \in \text{WAP}(G) \) (see Chou [3, Lemma 2.4] or Milnes [17, Theorem 2]).

Assume once more that \( P \) is a continuous projection of \( \text{LUC}(G) \) onto \( \text{WAP}(G) \). Define for each \( f \in \text{LUC}(H) \)

\[
Qf = P(f')|_H.
\]

Since \( h|_H \in \text{WAP}(H) \) for each \( h \in \text{WAP}(G) \), it follows that \( Q \) is a continuous projection of \( \text{LUC}(H) \) onto \( \text{WAP}(H) \). By the first part, this is impossible.

\[\square\]
B. B. Wells [26, Theorem 3.2] also proved that if $G = \mathbb{R}^n$, then $\text{AP}(G)$, the space of almost periodic functions on $G$, is uncomplemented in $\text{LUC}(G)$. Of course, if $\text{AP}(G)$ is finite dimensional (e.g. $G = \text{SL}(2,\mathbb{R})$), then $\text{AP}(G)$ is complemented in $\text{LUC}(G)$. It also follows from Takahashi [24, Theorem 2] that if $G$ is a discrete group, then $\text{AP}(G)$ is complemented in $l_\infty(G)$ if and only if $\text{AP}(G)$ is finite dimensional. We shall prove an extension of these results. First we establish the following theorem that we need:

**Theorem 7.** Let $G$ be a locally compact group, $H$ a closed subgroup of $G$. Then there exists a contractive linear projection $P$ from $\text{CB}(G/H)$ onto $\text{CB}(G/H)$. In particular, $\text{CB}(G/H)$ is complemented in $\text{CB}(G)$.

**Proof.** Let $\pi: G \to G/H$ be the canonical mapping. We consider $\text{CB}(G/H)$ as a subspace of $\text{CB}(G)$ by identifying $f \in \text{CB}(G/H)$ and $f \circ \pi \in \text{CB}(G)$. First we show that it is sufficient to prove the theorem for almost connected groups. Indeed, assume that $G$ is an open, almost connected subgroup of $G$. Then for $x \in G$, we have $\pi(Gx) = GxH/H$ and this is homeomorphic to $G_1/(G_1 \cap xHx^{-1})$. Now let $R$ be a set of representatives for the $G_1 - H$-double cosets in $G$ and assume that for each $x \in R$, we have a linear contractive projection $P_x: \text{CB}(G_1) \to \text{CB}(G_1/G_1 \cap xHx^{-1})$ (i.e. $P_x(f \circ \pi_x) = f$ for $f \in \text{CB}(G_1/(G_1 \cap xHx^{-1}))$, if again $\pi_x: G_1 \to G_1/(G_1 \cap xHx^{-1})$ denotes the canonical mapping). $P_x$ gives rise to a continuous projection $P'_x: \text{CB}(G_1x) \to \text{CB}(\pi(G_1x))$: for $f \in \text{CB}(G_1x)$, $y \in G_1x$, we put

$$P'_x(f)(\pi(y)) = P_x(r_x f)(yx^{-1}(G_1 \cap xHx^{-1})).$$

If $f \in \text{CB}(\pi(G_1x))$, then $r_x(f \circ \pi)$ is right-$G_1 \cap xHx^{-1}$ periodic (i.e. $r_k(r_x(f \circ \pi)) = r_x(f \circ \pi)$ for all $k \in G_1 \cap xHx^{-1}$). Hence $P'_x(f \circ \pi) = f$. Observe also that $G/H = \bigcup \{\pi(G_1x); x \in R\}$. For $y \in G_1x$, $f \in \text{CB}(G)$, put

$$P(f)(yH) = P_x(f|_{G_1x})(yH).$$

Then $P$ is a contractive linear projection onto $\text{CB}(G/H)$.

If $G$ is almost connected, let $K$ be a compact normal subgroup of $G$ such that $G/K$ is a Lie group. By convolution with the normalized Haar measure of $K \cap H$, we get a contractive linear projection from $\text{CB}(G)$ to $\text{CB}(G/(K \cap H))$ (compare with proof of Lemma 1). Hence, it is sufficient to construct a contractive linear projective from $\text{CB}(G/(K \cap H))$ to $\text{CB}(G/H)$. 

Let $\pi_K: G \to G/K$ be the canonical mapping, similarly $\pi_H$ and $\pi_{K \cap H}$ are defined. Let $v_1, \ldots, v_n$ be a basis for the Lie algebra of $G/K$ such that $v_{k+1}, \ldots, v_n$ span the Lie algebra of $\pi_K(H) = HK/K$ for some $k$. Let $x_i(t) (1 \leq i \leq n)$ be the corresponding one parameter subgroups of $G/K$. By [19], 4.15, Theorem 1, there are continuous one-parameter subgroups $x_i(t)$ in $G (1 \leq i \leq n)$ such that $\pi_K(x_i(t)) = x_i(t)$. For $k < i \leq n$, we can even accomplish that $x_i(t) \in H$. There exists $\varepsilon > 0$ such that $(t_1, \ldots, t_n) \mapsto \dot{x}_1(t_1) \cdots \dot{x}_n(t_n)$ is a homeomorphism of the cube $C$

$$\{(t_1, \ldots, t_n) \in \mathbb{R}^n : |t_i| \leq \varepsilon \text{ for } i = 1, \ldots, n\}$$

onto a neighbourhood $V$ of $e$ ($= K$) in $G/K$ and $V \cap (HK)/K$ corresponds to $\{(t_1, \ldots, t_n) \in C : t_1 = \cdots = t_k = 0\}$. Put

$$M_1 = \{x_1(t_1) \cdots x_k(t_k) : |t_i| \leq \varepsilon \text{ for } i = 1, \ldots, k\}$$

and

$$M_2 = \{x_{k+1}(t_{k+1}) \cdots x_n(t_n) : |t_i| \leq \varepsilon \text{ for } i = k + 1, \ldots, n\}.$$

(If $n = 0$, i.e. $K$ is open in $G$, we put $M_1 = M_2 = \{e\}$, $V = \{e\}$. Similarly if $k = 0$ or $k = n$.) Then $(x, y) \mapsto xy$ maps $M_1 \times M_2$ homeomorphically to $M_1 M_2$, the restriction of $\pi_K$ to $M_1 M_2$ is a homeomorphism onto $V$ and the restriction of $\pi_{HK}$ to $M_1$ is a homeomorphism onto $\pi_{HK}(V)$. Put $W = \pi_K^{-1}(V)$, $U = \pi_H(W)$. Then

$$W = \{abc : a \in M_1, b \in K, c \in M_2\}$$

and the elements $a, b, c$ are uniquely determined by $x = abc$. Assume that $x, x' \in W$ are decomposed as above: $x = abc$, $x' = a'b'c'$, and that $\pi_H(x) = \pi_H(x')$. Then $\pi_{HK}(x) = \pi_{HK}(x')$ and, since $\pi_{HK}(x) = \pi_{HK}(a)$, $\pi_{HK}(x') = \pi_{HK}(a')$, it follows that $a = a'$. Hence $\pi_H(bc) = \pi_H(b'c')$ and this gives $\pi_{H \cap K}(b) = \pi_{H \cap K}(b')$ (recall that $M_2 \subseteq H$). Given $\pi_H(x) \in U$ with $x = abc \in W$, we put $\psi(\pi_H(x)) = \pi_{K \cap H}(ab)$. It follows from the above argument that $\psi : U \to G/K \cap H$ is well defined. Also $\psi$ is continuous. This follows easily from the compactness of $M_1$, $M_2$ and $K$ and from the fact that $a, b, c$ depend continuously on $x = abc$. Furthermore, $\psi \circ \pi_H = \pi_{K \cap H}$ on $M_1 K$ and the canonical mapping $\pi_{H, K \cap H} : G/K \cap H \to G/H$ maps $\psi(\pi_H(ab)) = \pi_{K \cap H}(ab)$ to $\pi_H(ab)$. Since $\pi_H(M_1 K) = U$, we conclude that $\pi_{H, K \cap H} \circ \psi$ is the identity on $U$. The covering $\{x U ; x \in G\}$ of $G/H$ has a locally finite refinement. Let $\{\varphi_x : x \in G\}$ be a partition of unity, subordinate to this covering, i.e. $\varphi_x \in C_0(G/H)$, $0 \leq \varphi_x \leq 1,$
supp $\varphi_x \subseteq xU$ for each $x \in G$ and $\sum_{x \in G} \varphi_x(y) = 1$ for all $y \in G/H$, where the sum is finite on each compact subset of $G/H$.

For $f \in \text{CB}(G/(K \cap H))$ define

$$Pf = \sum_{x \in G} \varphi_x \cdot l_x^{-1}((l_x f) \circ \psi).$$

(The sum is actually finite on each compact subset of $G/H$.)

Then it is easy to see that $P$ is a contractive linear projection from $\text{CB}(G/(K \cap H))$ to $\text{CB}(G/H)$.

If $G$ is a locally compact group, the von Neumann-kernel is defined as the intersection of the kernels of all finite-dimensional (continuous, unitary) representations of $G$. It coincides with the kernel of the canonical mapping of $G$ into its Bohr compactification $bG$. The quotient group $G/N$ is maximally almost periodic (for short: $G/N \in \text{MAP}$).

**Theorem 8.** Let $G$ be a locally compact group. The following statements are equivalent:

(a) $\text{AP}(G)$ is complemented in $\text{LUC}(G)$.

(b) $G/N$ is compact, where $N$ denotes the von Neumann kernel of $G$.

(c) The canonical mapping of $G$ into its Bohr compactification $bG$ is surjective.

**Proof.** The equivalence of (b) and (c) is almost immediate.

If (b) holds, then (a) follows from Theorem 7, since $\text{AP}(G) = \text{AP}(G/N) = \text{CB}(G/N)$ (we get a contractive linear projection even from $\text{CB}(G)$ to $\text{AP}(G)$).

For the proof of (a) $\rightarrow$ (b) assume that $\text{AP}(G)$ is complemented in $\text{LUC}(G)$. We start with three observations:

If $G_1$ is a subgroup of $G$ with finite index, and $f \in \text{AP}(G_1)$ is extended to $G$ by putting $f(x) = 0$ for $x \notin G_1$, then $f \in \text{AP}(G)$. In this way, $\text{AP}(G_1)$ becomes a subspace of $\text{AP}(G)$ and it follows now as in the proof Theorem 2 that $\text{AP}(G_1)$ is complemented in $\text{LUC}(G_1) \subseteq \text{LUC}(G)$.

For the second observation assume that $G = H + K$ is the direct sum of closed subgroups $H$ and $K$. Let $\pi : G \to H$ be the corresponding projection. If $P : \text{LUC}(G) \rightarrow \text{AP}(G)$ is a projection, then $Qf = P[(f \circ \pi)]|_H$ (where $f \in \text{LUC}(H)$) defines a projection from $\text{LUC}(H)$ to $\text{AP}(H)$. 
For the third observation, assume that $G_1$ is an open subgroup of $G$ that is also closed for the Bohr topology, i.e. the topology induced by $bG$ (in particular $N \subseteq G_1$). We claim that (under the assumption that $\text{AP}(G)$ is complemented in $\text{LUC}(G)$) $G_1$ has finite index in $G$. Let $L$ be the closure of the image of $G_1$ in $bG$. Then the isomorphism between $\text{AP}(G)$ and $\text{CB}(bG)$ maps $\text{AP}(G) \cap \text{CB}(G_1 \setminus G)$ onto $\text{CB}(L \setminus bG)$ (where $G_1 \setminus G$ resp. $L \setminus bG$ denote the spaces of right cosets). As in the proof of Theorem 7, $\text{CB}(L \setminus bG)$ is complemented in $\text{CB}(bG) = \text{AP}(G)$. It follows that $\text{CB}(L \setminus bG)$ is complemented in $\text{LUC}(G)$. Since $\text{AP}(G) \cap \text{CB}(G_1 \setminus G) \subseteq \text{CB}(G_1 \setminus G) \subseteq \text{LUC}(G)$ and $G_1 \setminus G$ is discrete (hence $\text{CB}(G_1 \setminus G) = l^\infty(G_1 \setminus G)$), there exists a bounded linear projection from $l^\infty(G_1 \setminus G)$ to $\text{CB}(L \setminus bG)$ and also to $\text{CB}((KL) \setminus bG)$ if $K$ is any compact normal subgroup of $bG$. If $(KL) \setminus bG$ is metrizable, it follows from Corollary 2, p. 169 of [11] that $\text{CB}((KL) \setminus bG)$ can be complemented in $l^\infty(G_1 \setminus G)$ only if it is reflexive, hence, only if $(KL) \setminus bG$ is finite. Now if $L \setminus bG$ would happen to be infinite, there would exist $f \in \text{CB}(L \setminus bG) \subseteq \text{CB}(bG)$ such that $f(L \setminus bG)$ is infinite. Then, by the Kakutani-Kodaira theorem, there would exist a closed normal subgroup $K$ of $G$ such that $bG/K$ is metrizable and $f$ is $K$-periodic i.e. $f \in \text{CB}(bG/K)$. This would imply that $f \in \text{CB}((KL) \setminus bG)$. But by the argument above, this is impossible. This shows that $L \setminus bG$ is finite, and since $G_1$ is the preimage of $L$ in $G$, it follows that $G_1 \setminus G$ is finite too.

To prove (b), we can assume that $G \in \text{MAP}$ (otherwise replace $G$ by $G/N$ and observe that $\text{AP}(G) = \text{AP}(G/N) \subseteq \text{LUC}(G/N) \subseteq \text{LUC}(G)$). We want to show that $G$ is compact.

Let $H$ be an open, almost connected subgroup of $G$. Then $H \in \text{MAP}$; hence by Theorem 2.9 of [10], it has an open subgroup of finite index which is a direct sum $V + L$ of a compact group $L$ and a vector group $V$ (i.e. $V \simeq \mathbb{R}^n$ for some $n \geq 0$). Replacing $H$ by this open subgroup, we may assume that $H = V + L$.

Let $V_1$ be the closure of $V$ in $G$ with respect to the Bohr topology. Then (by continuity) $L$ centralizes $V_1$; hence $V_1L$ is an open subgroup of $G$ which is closed for the relative topology of $bG$. From the third observation above, it follows that $V_1L$ has finite index in $G$ and, by the first observation above, we can assume that $G = V_1L$ (The Bohr topology induces on a subgroup of finite index again the Bohr topology). This implies that $L$ is normal in $G$.

Let $\pi: G \to G/L$ be the canonical projection. Since $L$ is compact, $\pi(V)$ is closed in $G/L$ and, since $\pi(V_1) = G/L$, it follows that $G/L$
is abelian. Assume that $\pi(V) \neq G/L$. Take $\hat{x} \notin \pi(V)$. Then there exists a continuous character $\chi \in (G/L)^\wedge$ such that $\chi(\hat{x}) \neq 1$ and $\chi(\pi(V)) = \{1\}$. Then $\chi \circ \pi \in \text{AP}(G)$ and if $x \in V_1$ satisfies $\pi(x) = \hat{x}$, then $\chi(\pi(x)) \neq 1$. But this would imply that $x$ does not belong to the closure of $V$ with respect to the Bohr topology, which is a contradiction. Thus $\pi(V) = G/L$ and hence $G = V \oplus L$. If it would happen that $n > 0$, then we could write $G$ as a direct sum of two groups, one of them being isomorphic to $\mathbb{R}$. By the second observation above, this would imply that $\text{AP}(\mathbb{R})$ is complemented in $\text{LUC}(\mathbb{R})$, contradicting Theorem 3.2 of Wells [26]. Hence $n = 0$, i.e. $G = L$ is compact. 

\begin{corollary}
If $G$ is a locally compact, maximally almost periodic group, then $\text{AP}(G)$ is complemented in $\text{LUC}(G)$ if and only if $G$ is compact.
\end{corollary}

\begin{remark}
In general, the conditions of Theorem 8 do not imply that $N$ is minimally almost periodic group (i.e. that $\text{AP}(N)$ contains only the constant functions). Take e.g. $G = C \ltimes_T \mathbb{R}$ (semidirect product), where $T = \mathbb{R}/\mathbb{Z}$ and the multiplication is defined by $(z,s)(w, t) = (z + e^{2\pi i s}w, s + t)$. Then $N = C$ and $G/N \approx T$ is compact (see also Theorem 2.3 in [18]).
\end{remark}

\section{Subspaces of WAP(G)}
Let $G$ be a locally compact group. For each $m, n \in \text{WAP}(G)^*$, define a multiplication

\[\langle m \circ n, f \rangle = \langle m, n_l(f) \rangle, \quad f \in \text{WAP}(G),\]

where $n_l(f)(g) = \langle n, l_g f \rangle$, $g \in G$. Then $n_l(f) \in \text{WAP}(G)$ (see [2, p. 36]) and, as readily checked, $\text{WAP}(G)^*$ with $\circ$ is a Banach algebra. Furthermore, for each $g \in G$, let $\delta_g$ denote the point evaluation at $g$. Then the map $g \rightarrow \delta_g$ is a natural embedding of $G$ into $\text{WAP}(G)^*$.

Let $X$ be a Banach space and $\mathcal{B}(X)$ be the space of bounded linear operators from $X$ into $X$. Let $\{U_g; g \in G\}$ be continuous representation of $G$ on $X$ i.e. for each $g \in G$, $U_g \in \mathcal{B}(X)$, $U_{g_1}U_{g_2} = U_{g_1g_2}$, $g_1, g_2 \in G$, and for each $x \in X$, the map $g \rightarrow U_g(x)$ from $G$ into $X$ is continuous. We say that $\{U_g; g \in G\}$ is weakly almost periodic if for each $x \in X$, $\{U_gx, g \in G\}$ is a relatively weakly compact subset of $X$.

\begin{lemma}
Let $G$ be a locally compact group and $\{U_g; g \in G\}$ be a weakly almost periodic continuous representation of $G$. Then there
exists a representation \( \{U(m); m \in WAP(G)^*\} \subseteq \mathcal{B}(X) \) of the Banach algebra \( WAP(G)^* \) on \( X \) such that:

(i) \( \|U(m)\| \leq K\|m\| \) for each \( m \in WAP(G)^* \) and some fixed \( K > 0 \).

(ii) \( U(\delta_g) = U_g \) for each \( g \in G \).

(iii) \( P = U(m_G) \) is a projection of \( X \) onto the closed subspace \( F_X = \{x \in X; U_g x = x \text{ for all } g \in G\} \).

(iv) \( P \) commutes with any continuous linear operator \( T \) from \( X \) into \( X \) which commutes with \( \{U_g, g \in G\} \).

**Proof.** Since \( \{U_g; g \in G\} \) is weakly almost periodic, it follows from the principle of uniform boundedness that there exists \( K > 0 \) such that \( \|U_g\| \leq K \) for all \( g \in G \). For each \( x \in X \), \( \varphi \in X^* \), define \( h_{x,\varphi}(g) = \langle U_g x, \varphi \rangle \), \( g \in G \). Then, it is well known [2, p. 36] that \( h_{x,\varphi} \in WAP(G) \). Given \( m \in WAP(G)^* \), let \( \langle U(m)x, \varphi \rangle = \langle m, h_{x,\varphi} \rangle \). Then, it is readily checked that \( U(m) \) is a continuous linear operator on \( X \), and \( \|U(m)\| \leq K\|m\| \). Furthermore \( U(m \circ n) = U(m) \circ U(n) \), \( m, n \in WAP(G)^* \), and \( U(\delta_g) = U_g \) for each \( g \in G \).

Now if \( x \in X \), \( g \in G \), then

\[
U_g P(x) = U(\delta_g) \circ U(m_G)(x) = U(\delta_g \circ m_G)(x) = U(m_G)(x) = P(x)
\]

i.e. \( P(x) \in F_X \). Also if \( x \in F_X \), \( \varphi \in X^* \)

\[
\langle P(x), \varphi \rangle = \langle m_G, h_{x,\varphi} \rangle = \langle x, \varphi \rangle.
\]

Hence \( P \) is a projection from \( X \) onto \( F_X \).

Finally if \( T \in \mathcal{B}(X) \) and \( TU_g = U_g T \), let \( m_\alpha = \sum_{i=1}^{n_\alpha} \lambda_i^\alpha \delta^\alpha \) denote a convex combination of point evaluations such that \( m_\alpha \) converges to \( m_G \) in the weak*-topology of \( WAP(G)^* \), then for each \( x \in X \), and \( \varphi \in X^* \), \( \langle U(m_\alpha)x, \varphi \rangle \rightarrow \langle U(m_G)x, \varphi \rangle \), i.e. \( U(m_\alpha) \) converges to \( U(m_G) \) in the weak operator topology of \( \mathcal{B}(X) \). Replacing by a different net if necessary, we may assume that \( U(m_\alpha) \) even converges to \( U(m_G) \) in the strong operator topology of \( (X) \). Hence for each \( x \in X \),

\[
T \circ P(x) = \lim_\alpha TU(m_\alpha)(x) = \lim_\alpha U(m_\alpha)T(x) = PT(x).
\]

**Theorem 11.** Let \( G \) be a locally compact group and \( X \) be a closed translation invariant subspace of \( WAP(G) \). Let \( N \) be a closed subgroup of \( G \) and

\[
A = \{f \in X; r_g f = f \text{ for all } g \in N\}.
\]
There exists a projection $P$ from $X$ onto $A$ and $P$ commutes with any continuous linear operator from $X$ into $X$ which commutes with right translations. In particular, $P$ commutes with any left translations.

Proof. This follows directly from Lemma 10 with the observation that left translation always commutes with right translation. \qed

Parts of the following Lemma were proved in [5, Theorem 5.1] for $G$ abelian.

**Lemma 12.** Let $G$ be a locally compact group. Then $A$ is a non-zero left translation invariant $C^*$-subalgebra of $C_0(G)$ if and only if there exists a unique compact subgroup $N_A$ of $G$ such that

$$A = \{ f \in C_0(G); r_g f = f \text{ for all } g \in N_A \}.$$  

Furthermore, $A$ is translation invariant if and only if $N_A$ is normal.

Proof. Let $N$ be a compact subgroup of $G$, it is easy to see that

$$A = \{ f \in C_0(G); r_g f = f \text{ for each } g \in N \}$$

is a left translation invariant $C^*$-subalgebra of $C_0(G)$. Also, since $C_0(G/N) \simeq A$ (using the identification $f \leftrightarrow f \circ \pi$, where $\pi$ is the canonical mapping of $G$ onto $G/N$), $A \neq \{0\}$.

Conversely, if $A$ is a left translation invariant $C^*$-algebra of $C_0(A)$ let

$$N = N_A = \{ g \in G; r_g f = f \text{ for all } f \in A \}.$$  

Then $N$ is a closed subgroup of $G$. Also, if $f \in A$, and $f \neq 0$, let $g_0 \in G$ such that $f(g_0) = \lambda \neq 0$. Then for each $g \in N$, $f(g_0 g) = f(g_0) = \lambda$. Consequently $N$ is compact.

Let $B = \{ f \in C_0(G); r_g f = f \text{ for each } g \in N \}$. Clearly $B \supseteq A$. To prove equality, we observe that each $f \in B$ may be regarded as a function $\tilde{f}$ in $C_0(G/N)$. Let $\mathcal{A} = \{ \tilde{f}; f \in A \}$ and $\mathcal{B} = \{ \tilde{f}; f \in B \}$. Clearly $\mathcal{B} \supseteq \mathcal{A}$. However as in the proof of Theorem 5.1 in [5], an application of the Stone-Weierstrass theorem shows that $\mathcal{A} = \mathcal{B}$.

Suppose $N_0$ is another compact subgroup of $G$ such that $A = \{ f \in C_0(G); r_g f = f \text{ for each } g \in N_0 \}$ then $N_0 \subseteq N$. If $a \in N$, $a \notin N_0$, there exists $h \in C_0(G/N_0)$ such that $h(a N_0) \neq h(N_0)$. Let $f \in C_0(G)$ such that

$$\tilde{f}(x) = \int_{N_0} f(x \xi) d\xi = h(x).$$
Then \( f \in A \) and \( r_a f \neq f \), which is impossible. Hence \( N_0 = N \).

Finally if \( A \) is translation invariant, \( g \in G, a \in N \), then

\[
 r_{g^{-1}ag}(f) = r_{g^{-1}a}(rgf) = r_{g^{-1}rgf} = f
\]

since \( rgf \in A \). Hence \( N \) is normal. Conversely, if \( N \) is normal, \( f \in A \) and \( g \in G \), then for each \( a \in N \), \( r_a(rgf) = r_agf = r_gb = r_gf \) where \( b = g^{-1}ag \in N \). In particular, \( r_gf \in A \).

\[\Box\]

The following is an analogue of Theorem 3.3 in [13]:

**Theorem 13.** Let \( G \) be any locally compact group and \( A \) be a left translation invariant \( C^* \)-subalgebra of \( C_0(G) \). Then there exists a continuous projection \( P \) from \( C_0(G) \) onto \( A \) and \( P \) commutes with any continuous linear operator from \( C_0(G) \) into \( C_0(G) \) which commutes with right translations. In particular, \( P \) commutes with any left translations.

**Remark 14.** (a) Let \( N = N_A \), then the projection \( P \) in Theorem 13 corresponds to the mapping \( T_N(f)(x) = \int_N f(x\xi) d\xi, x \in G \), which maps \( C_0(G) \) onto \( C_0(G/N) \) [8, p. 261] and \( C_0(G/N) \simeq A \).

(b) Lemma 12 can be applied to obtain a well-known result of Kakutani-Kodaira: If \( G \) is a \( \sigma \)-compact group, there exists a compact normal subgroup \( N \) of \( G \) such that \( G/N \) is metrizable. Let \( f \in C_0(G), f \neq 0 \). Since \( G \) is \( \sigma \)-compact, the translation invariant \( C^* \)-subalgebra \( A \) of \( C_0(G) \) generated by \( f \) is separable. Let \( N = N_A \). Then \( C_0(G/N) \simeq A \) is also separable. In particular, \( G/N \) is metrizable.

**References**


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