KAPLANSKY’S THEOREM AND BANACH PI-ALGEBRAS

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By the theorem of Kaplansky a bounded operator in a Banach space is algebraic if and only if it is locally algebraic. We prove a generalization of this theorem. As a corollary we obtain the analogous result for finite (or countable) families of operators. Further we prove that a Banach algebra is PI (i.e. it satisfies a polynomial identity) if and only if it is locally PI.

Let $T$ be a bounded operator on a Banach space $X$. The classical theorem of Kaplansky [5] states that $T$ is algebraic (i.e. $p(T) = 0$ for some polynomial $p \neq 0$) if and only if it is locally algebraic (i.e. for every $x \in X$ there exists a non-zero polynomial $p_x$ such that $p_x(T)x = 0$). In this paper we prove (Theorem 1) a generalized version of this theorem. As its corollaries it is possible to obtain the original theorem of Kaplansky, the theorem of Sinclair [9] and also new analogical results for finite or countable families of operators.

In the second part of the paper we deal with Banach PI-algebras (i.e. Banach algebras satisfying a polynomial identity). PI-rings and PI-algebras were studied intensely from the algebraic point of view, see e.g. [4], [8]. On the other hand Banach PI-algebras are much less known even though they form a very interesting class of Banach algebras. They are a natural generalization of commutative Banach algebras and it is possible to develop the complete analogy of the Gelfand theory, see [6].

In this paper we prove a theorem of Kaplansky’s type for Banach PI-algebras. This result is closely related to earlier results of Grabiner [2] and Dixon [1].

The author wishes to thank Professor B. Silbermann for calling his attention to the interesting field of Banach PI-algebras and fruitful discussions about it.

Let $n$ be a positive integer. We denote by $\mathcal{P}^{(n)}$ the set of all complex polynomials in $n$ non-commutative indeterminates i.e. the free algebra over $\mathbb{C}$ with $n$ generators and with the unit element. Similarly we denote by $\mathcal{P}^{(\infty)} = \bigcup_{n=1}^{\infty} \mathcal{P}^{(n)}$ the set of all complex polynomials with countably many indeterminates.
Let $X$ and $Y$ be Banach spaces. Then $B(X,Y)$ denotes the set of all bounded operators from $X$ to $Y$; we write shortly $B(X)$ instead of $B(X,X)$.

Let $X$ be a Banach space, $1 \leq n < \infty$ and let $T_1, \ldots, T_n \in B(X)$. We say that the $n$-tuple $(T_1, \ldots, T_n)$ is algebraic if $p(T_1, \ldots, T_n) = 0$ for some $p \in \mathcal{P}(n)$, $p \neq 0$. We say that $(T_1, \ldots, T_n)$ is locally algebraic if, for every $x \in X$, there exists a non-zero polynomial $p_x \in \mathcal{P}(n)$ such that $p_x(T_1, \ldots, T_n)x = 0$.

These definitions can be used also for an infinite sequence $\{T_i\}_{i=1}^{\infty}$ of bounded operators on $X$ (for $p \in \mathcal{P}(n) \subset \mathcal{P}(\infty)$ we have $p(T_1, T_2, \ldots) = p(T_1, \ldots, T_n)$). Equivalently, the sequence $\{T_i\}_{i=1}^{\infty}$ is locally algebraic if, for every $x \in X$ there exist $n$ and $0 \neq p \in \mathcal{P}(n)$ such that $p(T_1, \ldots, T_n)x = 0$.

We start with the following generalization of Kaplansky’s theorem.

**Theorem 1.** Let $M$ be a linear space of countable (infinite) dimension, let $Y, Z$ be Banach spaces and let $R: M \rightarrow B(Y, Z)$ be a linear mapping with the property that for every $y \in Y$ there exists $m \in M$, $m \neq 0$ such that $R(m)y = 0$. Then there exists $m \in M$, $m \neq 0$ such that $R(m)$ is a finite-dimensional operator.

**Proof.** Let $e_1, e_2, \ldots$ be a basis in $M$. Put $M_0 = \{0\}$ and denote by $M_k$ ($k = 1, 2, \ldots$) the linear subspace of $M$ spanned by the vectors $e_1, \ldots, e_k$.

Let $F$ be a finite-dimensional subspace of $Z$. For $j = 1, 2, \ldots$ denote by $Y_{F,j}$ the set of all $y \in Y$ for which there exists $m \in M_j$, $m \neq 0$, such that $R(m)y \in F$ and $R(m')y \notin F$ for every $m' \in M_{j-1}$, $m' \neq 0$. By the assumption $\bigcup_{j=1}^{\infty} Y_{F,j} = Y$ so there exists $k = k(F)$ such that $Y_{F,k}$ is of the second category and $Y_{F,l}$ is of the first category for every $l < k$. Fix a finite-dimensional subspace $F \subset Z$ with the property that

$$k = k(F) = \min_{G \subset Z} \min_{\dim G < \infty} k(G).$$

We have $Y_{F,k} = \bigcup_{s=1}^{\infty} Y^{(s)}_{F,k}$ where

$$Y^{(s)}_{F,k} = \left\{ y \in Y_{F,k}, \text{ there exists } m = e_k + \sum_{i=1}^{k-1} \alpha_i e_i \in M_k \text{ such that } \sum_{i=1}^{k-1} |\alpha_i| \leq s \text{ and } R(m)y \in F \right\}.$$
We prove that $Y_{F,k}^{(s)}$ is a closed set for every $s$. Let $y_j \in Y_{F,k}^{(s)}$ ($j = 1, 2, \ldots$), $y_j \to y$. Then there exist elements $m_j \in M_k$, $m_j = e_k + \sum_{i=1}^{k-1} \alpha_j e_i$ such that $\sum_{i=1}^{k-1} |\alpha_j| \leq s$ and $R(m_j)y_j \in F$. Using the compactness argument it is possible to find a subsequence \{$y_{j_r}$\}$_{r=1}^{\infty}$ and a vector $m \in M_k$ such that $m_{j_r} \to m$ coordinate-wise and $R(m_{j_r}) \to R(m)$ in the norm topology. It is easy to show that

$$R(m)y = \lim_{r \to \infty} R(m_{j_r})y_{j_r} \in F;$$

hence $y \in Y_{F,k}^{(s)}$ and $Y_{F,k}^{(s)}$ is closed. Therefore there exists $w \in Y$, $r > 0$ and a positive integer $s$ such that

$$\{y \in Y, \|y - w\| < r\} \subset Y_{F,k}^{(s)} \subset Y_{F,k}.$$

Let $a = e_k + \sum_{i=1}^{k-1} \alpha_i e_i$ be the element of $M_k$ satisfying

$$(1) \quad R(a)w \in F.$$

Denote by $F' = F \vee \bigvee_{i=1}^{k} \{R(e_i)w\}$. Clearly $\dim F' \leq \dim F + k < \infty$. Put $V = Y_{F,k} - \bigcup_{l<k} Y_{F,l}$. It follows from the choice of the subspace $F$ that $V$ is of the second category. Let $v \in V$. Then $v \in Y_{F,k}$ and

$$(2) \quad R(b)v \in F$$

for some $b \in M_k$, $b = e_k + \sum_{i=1}^{k-1} \beta_i e_i$.

Further $w + \lambda v \in Y_{F,k}$ for some complex number $\lambda \neq 0$, i.e. there exists $c = e_k + \sum_{i=1}^{k-1} \gamma_i e_i \in M_k$ such that

$$(3) \quad R(c)(w + \lambda v) = R(c)w + \lambda R(c)v \in F.$$

This implies $R(c)v \in F'$ and together with (2) $R(c - b)v \in F'$ where $c - b = \sum_{i=1}^{k-1} (\gamma_i - \beta_i)e_i \in M_{k-1}$. Since $v \notin \bigcup_{l<k} Y_{F',l}$, we conclude $c - b = 0$, $c = b$.

By (2), (3) and (1) we have $R(c)v = R(b)v \in F$, $R(c)w \in F$ and $R(c - a)w \in F$, where $c - a \in M_{k-1}$. Since $w \notin \bigcup_{l<k} Y_{F,l}$ we conclude again that $c = a$, i.e. $R(a)v \in F$ for every $v \in V$. Thus $R(a)^{-1}F \supset V$ and $R(a)^{-1}F$ is a linear subspace of the second category in $Y$, therefore $R(a)^{-1}F = Y$, $R(a)Y \subset F$ and $R(a)$ is a finite dimensional operator.

**Remark.** One is tempted to expect in Theorem 1 that there exists $m \in M$, $m \neq 0$, such that $R(m) = 0$. However, the following example shows that this is not true in general. Let $Y = Z$ be a separable
Hilbert space with an orthonormal basis \( \{ h_i \}_{i=1}^{\infty} \). Define operators \( R(m) \), \( m \in M \), by

\[
R(e_1)h_1 = h_1, \quad R(e_1)h_j = 0 \quad (j \geq 2),
R(e_2)h_1 = 0, \quad R(e_2)h_2 = h_1, \quad R(e_2)h_j = 0 \quad (j \geq 3),
R(e_i)h_j = \delta_{ij}h_j \quad (i \geq 3; \delta_{ij} \text{ means the Kronecker's symbol}).
\]

It is easy to show that the conditions of Theorem 1 are satisfied and \( R(m) \neq 0 \) (\( m \neq 0 \)).

**Theorem 2.** Let \( X \) be a Banach space, \( 1 \leq n \leq \infty \). Let \( T = \{ T_i \}_{i=1}^{n} \) be a (finite or infinite) sequence of bounded operators on \( X \). Then \( T \) is algebraic if and only if it is locally algebraic.

**Proof.** Suppose \( T \) is locally algebraic. We prove that it is algebraic (the converse implication is trivial). Put \( M = \mathcal{P}^{(n)}, Y = Z = X \). For \( p \in \mathcal{P}^{(n)} \) put \( R(p) = p(T) \). By Theorem 1 there exist a polynomial \( p \in \mathcal{P}^{(n)}, p \neq 0 \), such that \( \dim p(T)X < \infty \). Hence \( (q \circ p)(T) = 0 \) where \( q \in \mathcal{P}^{(1)} \) is the characteristic polynomial of the finite-dimensional operator \( p(T)|_{p(T)X} \).

In [9], the following generalization of the Kaplansky's theorem was proved: Let \( T \in B(X) \) be a non-algebraic operator. Then there exists a sequence \( x_1, x_2, \ldots \) of elements of \( X \) such that \( \sum_{i=1}^{k} p_i(T)x_i \neq 0 \) for every \( k \geq 0 \) and for every polynomial \( p_1, \ldots, p_k \in \mathcal{P}^{(1)} \) not all of which are equal to 0.

This result can be extended to the case of more than one operator.

**Theorem 3.** Let \( X \) be a Banach space, \( 1 \leq n \leq \infty \). Let \( T = \{ T_i \}_{i=1}^{\infty} \) be a (finite or infinite) sequence of bounded operators on \( X \) which is not algebraic. Then there exist vectors \( x_1, x_2, \ldots \in X \) such that \( \sum_{i=1}^{k} p_i(T)x_i \neq 0 \) for every \( k \) and for every polynomial \( p_1, \ldots, p_k \in \mathcal{P}^{(n)} \) not all of which are equal to 0.

**Proof.** Suppose on the contrary that for every sequence \( x_1, x_2, \ldots \) of elements of \( X \) there exist \( k \) and polynomials \( p_1, \ldots, p_k \in \mathcal{P}^{(n)}, (p_1, \ldots, p_k) \neq (0, \ldots, 0) \) such that \( \sum_{i=1}^{k} p_i(T)x_i = 0 \).

Let \( M \) be the linear space of all sequences \( \{ p_i \}_{i=1}^{\infty} \) of polynomials \( p_i \in \mathcal{P}^{(n)} \) only a finite number of which are non-zero. Put \( Z = X \) and

\[
Y = \{ \{ x_i \}_{i=1}^{\infty}, x_i \in X \ (i = 1, 2, \ldots), \sup\{ \| x_i \|, i = 1, 2, \ldots \} < \infty \}.
\]
Then \( Y \) with the norm \( \| \{ x_i \}_{i=1}^{\infty} \| = \sup\{ \| x_i \|, i = 1, 2, \ldots \} \) is a Banach space. For \( p = \{ p_i \}_{i=1}^{\infty} \in M \) and \( y = \{ x_i \}_{i=1}^{\infty} \in Y \) put \( R(p)y = \sum_{i=1}^{\infty} p_i(T)x_i \) (in fact the sum is finite). By Theorem 1 there exist a finite-dimensional subspace \( F \subset X \), a positive integer \( k \) and polynomials \( p_1, \ldots, p_k \in \mathcal{P}^{(n)} \), \( (p_1, \ldots, p_k) \neq (0, \ldots, 0) \), such that

\[
\sum_{i=1}^{k} p_i(T)x_i \in F \quad \text{for every } x_1, \ldots, x_k \in X.
\]

Choose \( j \in \{1, \ldots, k\} \) such that \( p_j \neq 0 \). Let \( x \in X \) be arbitrary. If we put \( x_j = x \), \( x_i = 0 \) (\( i \neq j \)) then we get \( p_j(T)x \in F \) for every \( x \in X \), i.e. \( p_j(T) \) is a finite-dimensional operator. The rest is the same as in the proof of Theorem 2.

**Remark.** Theorem 1 unifies some of the results of Kaplansky's type (cf. problem of Halmos [3]). On the other hand there are some results of this type which do not fit into this frame (see e.g. [10] where bounded analytic functions are used instead of polynomials or "approximative" results of Kaplansky's type [7], [11]). Another example will be the result for Banach PI-algebras which we prove in the following section.

Let \( A \) be a Banach algebra with the unit (we shall always assume that a Banach algebra has a unit element although this assumption is not essential). We say that \( A \) is PI if there exist a positive integer \( n \) and a non-zero polynomial \( p \in \mathcal{P}^{(n)} \) such that \( p(a_1, \ldots, a_n) = 0 \) for every \( a_1, \ldots, a_n \in A \). We say that \( A \) is locally PI if for every sequence \( \{a_i\}_{i=1}^{\infty} \) of elements of \( A \) there exist \( n \) and a non-zero polynomial \( p \in \mathcal{P}^{(n)} \) such that \( p(a_1, \ldots, a_n) = 0 \) (both \( n \) and \( p \) depend on the sequence \( \{a_i\}_{i=1}^{\infty} \)).

**Theorem 4.** Let \( A \) be a Banach algebra with the unit. Then \( A \) is PI if and only if \( A \) is locally PI.

**Proof.** The implication PI \( \Rightarrow \) locally PI is trivial. Suppose that \( A \) is locally PI. Denote by \( \tilde{A} \)

\[
\tilde{A} = \{ \{a_i\}_{i=1}^{\infty}, \ a_i \in A, \ i = 1, 2, \ldots, \sup\{ ||a_i||, i = 1, 2, \ldots \} < \infty \}.
\]

Then \( A \) with the norm \( \| \{a_i\}_{i=1}^{\infty} \| = \sup\{ ||a_i||, i = 1, 2, \ldots \} \) is a Banach space. Further \( \tilde{A} = \bigcup_{n=1}^{\infty} \tilde{A}_n \) where

\[
\tilde{A}_n = \{ \{a_i\}_{i=1}^{\infty} \in \tilde{A}, \there exists \ p \in \mathcal{P}^{(n)}, \ \deg p \leq n, \ n^{-1} \leq |p| \leq n, \ p(a_1, \ldots, a_n) = 0 \}
\]
(we denote by $\deg p$ the degree of a polynomial $p$ and $|p|$ denotes the sum of moduli of coefficients of $p$).

Since $\tilde{A}_n$ is a closed subset for every $n$, Baire's theorem implies that there exist a positive integer $n$, $\tilde{y} \in \tilde{A}$ and $r > 0$ such that

$$\{\tilde{a} \in \tilde{A}, \|\tilde{a} - \tilde{y}\| < r\} \subset \tilde{A}_n.$$ 

Let $\tilde{x} = \{z_i\}_{i=1}^\infty \in \tilde{A}_n$. Then $p(z_1, \ldots, z_n) = 0$ for some $p \in \mathcal{P}(n)$, $p \neq 0$, $\deg p \leq n$, i.e. the set

$$C = \{z_i, \ldots, z_k, 0 \leq k \leq n, i_1, \ldots, i_k \in \{1, \ldots, n\}\}$$

is linearly dependent and $\sum_{c \in C} \alpha_c c = 0$ where $\alpha_c$ denotes the coefficient of $p$ standing at the term $c$. Therefore $\sum_{c \in C} \alpha_c (cz_{n+1} - z_{n+1} c) = 0$. Let $C = \{c_1, \ldots, c_s\}$. Denote by

$$e_s(x_1, \ldots, x_s) = \sum_{\sigma \in S_s} (-1)^{\text{sign } \sigma} x_{\sigma(1)} \cdots x_{\sigma(s)}$$

the standard polynomial (the sum is taken over all permutations of the set $\{1, \ldots, s\}$). Clearly,

$$e_s(c_1 z_{n+1} - z_{n+1} c_1, \ldots, c_s z_{n+1} - z_n c_s) = 0,$$

i.e. there exists a non-zero polynomial $p_n \in \mathcal{P}(n+1)$ such that $p_n(z_1, \ldots, z_{n+1}) = 0$ for every sequence $\{z_i\}_{i=1}^\infty \in \tilde{A}_n$. Let $\tilde{a} = \{a_i\}_{i=1}^\infty \in \tilde{A}$ be arbitrary. Then $\tilde{y} + \lambda \tilde{a} \in \tilde{A}_n$ for all complex $\lambda$, $|\lambda|\|\tilde{a}\| < r$, i.e.

$$p_n(y_1 + \lambda a_1, \ldots, y_{n+1} + \lambda a_{n+1}) = 0.$$ 

We can write

$$p_n(y_1 + \lambda a_1, \ldots, y_{n+1} + \lambda a_{n+1})$$

$$= p_n(y_1, \ldots, y_{n+1}) + \lambda q^{(1)}(y_1, \ldots, y_{n+1}, a_1, \ldots, a_{n+1})$$

$$+ \cdots + \lambda^{\deg p_n - 1} q^{(\deg p_n - 1)}(y_1, \ldots, y_{n+1}, a_1, \ldots, a_{n+1})$$

$$+ \lambda^{\deg p_n} p_n(a_1, \ldots, a_{n+1}).$$

Since this expression is equal to 0 for all $\lambda$ such that $|\lambda|\|\tilde{a}\| < r$, we conclude that $p_n(a_1, \ldots, a_{n+1}) = 0$ for every $(n+1)$-tuple $a_1, \ldots, a_{n+1}$ of elements of $A$. Thus $A$ is a PI-algebra.

**Remark.** In [2], S. Grabiner proved that a nil Banach algebra (i.e. consisting of nilpotent elements) is nilpotent (i.e. $A^n = 0$ for some $n$). The previous theorem is closely related to this result.

An algebra $A$ is called algebraic if every element $a \in A$ is algebraic, i.e. $p(a) = 0$ for some non-zero polynomial $p \in \mathcal{P}(1)$. An algebra
is called locally finite if every finite subset of $A$ generates a finite-dimensional subalgebra.

Clearly, a locally finite algebra is algebraic.

As an easy corollary of the previous theorem we can obtain the following result of Dixon [1] that the converse implication is true for Banach algebras.

**Corollary 5.** Let $A$ be a Banach algebra with the unit. Then $A$ is algebraic if and only if $A$ is locally finite.

**Proof.** If $A$ is algebraic then $A$ is locally PI and thus PI by Theorem 4. An algebraic PI-algebra is locally finite (see [4], X/12, Theorem 1).

**References**


Received March 23, 1988.

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The Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924 (ISSN 0030-8730) publishes 5 volumes per year. Application to mail at Second-class postage rates is pending at Carmel Valley, California, and additional mailing offices. Postmaster: send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

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