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**THE WAVE FRONT SET AND THE ASYMPTOTIC SUPPORT
FOR p -ADIC GROUPS**

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We prove that for p -adic groups the notion of the wave front set of a representation coincides with the notion of the asymptotic support.

1. The wave front sets of finite sums of homogeneous distributions.

Let Ω be a p -adic field of characteristic zero, with valuation $|\cdot|$. Let \mathfrak{g} be a finite dimensional vector space over Ω . Fix a non-trivial character χ of the additive group Ω , and a non-degenerate symmetric bilinear form β on \mathfrak{g} with values in Ω .

For $f \in C_c^\infty(\mathfrak{g})$ (compactly supported, locally constant functions on \mathfrak{g}) define a Fourier Transform by

$$(1.1) \quad \hat{f}(Y) = \int_{\mathfrak{g}} \chi(\beta(Y, X)) f(X) dX \quad (Y \in \mathfrak{g}).$$

Here dX is a Haar measure on the additive group of \mathfrak{g} (normalized so that the formula $(\hat{f})^\wedge(x) = f(-x)$ holds). Then $f \rightarrow \hat{f}$ is a bijective mapping of $C_c^\infty(\mathfrak{g})$ onto itself (see [Ha1] or [W, p. 107]). If T is a distribution \mathfrak{g} then its Fourier transform \hat{T} is given by

$$(1.2) \quad \hat{T}(f) = T(\hat{f}) \quad (f \in C_c^\infty(\mathfrak{g})).$$

Let $n = \dim_{\Omega}(\mathfrak{g})$. For $f \in C_c^\infty(\mathfrak{g})$ define

$$(1.3) \quad f_\lambda(X) = |\lambda|^{-n} f(\lambda^{-1} X) \quad (X \in \mathfrak{g}, \lambda \in \Omega^\times).$$

Fix an open subgroup Λ of Ω^\times with $[\Omega^\times : \Lambda] < \infty$.

DEFINITION 1.4. A distribution T on \mathfrak{g} is Λ -homogeneous of degree $d \in \mathbf{C}$ if

$$T(f_\lambda) = |\lambda|^d T(f) \quad (f \in C_c^\infty(\mathfrak{g}), \lambda \in \Lambda).$$

Notice that

$$(1.5) \quad (f_\lambda)^\wedge = |\lambda|^{-n} (\hat{f})_{\lambda^{-1}} \quad (f \in C_c^\infty(\mathfrak{g}), \lambda \in \Omega^\times),$$

so that if T is Λ -homogeneous of degree d then \hat{T} is a Λ -homogeneous of degree $-n - d$. Clearly if T is a function:

$$T(f) = \int_{\mathfrak{g}} T(X) f(X) dX,$$

then T is Λ -homogeneous of degree d iff for any $\lambda \in \Lambda$,

$$T(\lambda X) dX = |\lambda|^d T(X) dX.$$

The reader may safely focus on the case $\Lambda = \Omega^x$. In order to justify the generality of Definition 1.4 we mention that a distribution homogeneous with respect to a quasicharacter of Ω^x is Λ -homogeneous for a suitable Λ (see for example [G-G-PS, Ch. II]).

By fixing a base of \mathfrak{g} we can identify it with Ω^n and use the norm

$$(1.6) \quad |(\lambda, \lambda_2, \dots, \lambda_n)| = \max\{|\lambda_1|, |\lambda_2|, \dots, |\lambda_n|\}.$$

The following simple fact will be used later.

LEMMA 1.7. *Let F and V be open-compact subsets of \mathfrak{g} . Then there is $\delta > 0$ such that for any $\lambda \in \Omega$ with $|\lambda| < \delta$ the following inclusion holds:*

$$\lambda F + V \subseteq V.$$

It is known that any compactly supported distribution on \mathfrak{g} has a locally constant function as a Fourier Transform.

We are going to use (1.2) to analyze the singularities of T near zero.

DEFINITION 1.8 ([He] §2). A distribution T on \mathfrak{g} is Λ -smooth at $Y_0 \in \mathfrak{g} \setminus \{0\}$ if there is an open neighborhood W of 0 and an open neighborhood V of Y_0 such that for any $f \in C_c^\infty(W)$ there is $N > 0$ for which $\lambda \in \Lambda$ and $|\lambda| > N$ imply

$$(fT)^\wedge(\lambda Y) = 0 \quad \text{for any } Y \in V.$$

The complement of the set of Λ -smooth points of T in $\mathfrak{g} \setminus \{0\}$ is called the Λ -wave front set of T at zero and is denoted $WF_\Lambda^0(T)$.

The function $(fT)^\wedge$, (1.9), is sometimes called a localized Fourier Transform of T (because $\text{supp}(fT) \subseteq \text{supp}(f)$). Of course this function can be expressed in terms of the convolution

$$(1.10) \quad \begin{aligned} (fT)^\wedge &= \hat{f} * \hat{T}, \quad \text{where for } X, Y \in \mathfrak{g}, \\ \hat{f} * \hat{T}(X) &= \hat{T}(L_X \hat{f}), \quad L_X \hat{f}(Y) = \hat{f}(X - Y). \end{aligned}$$

Using (1.10) and the notion of a lattice in \mathfrak{g} [W, p. 28] we rephrase the Definition 1.8. For a subset $U \subseteq \mathfrak{g}$, let f_U denote the characteristic function of U .

LEMMA 1.11. *Let T be a distribution on \mathfrak{g} and let V be an open-compact subset of $\mathfrak{g} \setminus \{0\}$. Then the following conditions on V are equivalent:*

- (a) $V \cap WF_{\Lambda}^0(T)$ is empty.
- (b) There is a lattice U in \mathfrak{g} and a constant $c > 0$, such that

$$f_U * \widehat{T}(\lambda Y) = 0 \quad \text{for } \lambda \in \Lambda, |\lambda| > c, Y \in V.$$

(c) *There is a lattice W in \mathfrak{g} and for any constant $1 > \varepsilon > 0$ a constant $c_{\varepsilon} > 0$ such that for any $f \in C_c^{\infty}(W)$,*

$$(*) \quad (f_{\gamma} T) \wedge (\lambda Y) = 0 \quad \text{for } \lambda, \gamma \in \Lambda, |\lambda| > c_{\varepsilon}, \varepsilon < |\gamma| < 1, Y \in V.$$

Proof. Clearly $(*)$ implies (a). The equivalence of (a) and (b) was shown by Heifetz [He, Lemma 2.2]. We shall recall his proof to see that (b) implies $(*)$. Let W be the lattice dual to U , $f \in C_c^{\infty}(W)$, and let $F = -\text{supp } \hat{f}$. Lemma 1.7 applied to the sets F and V provides a constant $\delta > 0$. Put $c_{\varepsilon} = \max\{\delta^{-1}\varepsilon^{-1}, c\}$. Since by (1.5) $\text{supp}(f_{\gamma}) \wedge = \gamma^{-1} \text{supp } \hat{f}$ we see that (under the assumptions of $(*)$)

$$\begin{aligned} (f_{\gamma} T) \wedge (\lambda Y) &= (f_{\gamma} f_W T) \wedge (\lambda Y) \\ &= \int_{\mathfrak{g}} (f_{\gamma}) \wedge (Z) f_W T \wedge (\lambda(-\lambda^{-1}Z + Y)) dZ = 0. \quad \square \end{aligned}$$

The reader may compare this proof with [Hö, 8.1.1] to see that the analogous argument in the classical situation is more complex.

Lemma 1.11 has the following immediate

COROLLARY 1.12. *The wave front set $WF_{\Lambda}^0(T)$ contains the set A of those $Y \in \mathfrak{g} \setminus \{0\}$ satisfying the condition that for any lattice $U \subseteq \mathfrak{g}$ and any constant $c > 0$ there is $\lambda \in \Lambda$ with $|\lambda| > c$ such that $f_U * \widehat{T}(\lambda Y) \neq 0$.*

Clearly Lemma 1.11 implies that

$$(1.13) \quad WF_{\Lambda}^0(T) \subseteq \Lambda \cdot \text{supp } \widehat{T}.$$

Also, since for any lattice $U \subseteq \mathfrak{g}$ the support of $f_U \widehat{T}$ is compact, the wave front set of T is the same as that associated to the truncation T_U of T at infinity, defined by $\widehat{T}_U = \widehat{T} - f_U \widehat{T}$. Therefore we have another

COROLLARY 1.14. *The wave front set $WF_{\Lambda}^0(T)$ is contained in the set B , the intersection of all $\Lambda \cdot \text{supp } \widehat{T}_U$, where U varies over all lattices in \mathfrak{g} .*

Next we define a p -adic analog of the classical notion of an asymptotic cone (see [Hö, 8.1.7]). For any subset E of $\mathfrak{g} \setminus \{0\}$ define its Λ -asymptotic cone to be the set

$$(1.15) \quad AC_\Lambda(E) = \left\{ \lim_{j \rightarrow \infty} \lambda_j Z_j \mid \lambda_j \in \Lambda, \lim_{j \rightarrow \infty} \lambda_j = 0, Z_j \in E \right\}.$$

By a Λ -conical subset of \mathfrak{g} we will mean a subset closed under multiplication by elements of Λ . Then $AC_\Lambda(E)$ is a closed Λ -conical subset of \mathfrak{g} .

THEOREM 1.16. *For any distribution T on \mathfrak{g} define the sets A and B as in Corollaries 1.12 and 1.14 respectively. Then*

$$(1.17) \quad A \subseteq WF_\Lambda^0(T) \subseteq B \subseteq AC_\Lambda(\text{supp } \widehat{T}).$$

Moreover all these sets (1.17) coincide if T is Λ -homogeneous.

Proof. Only the last inclusion in (1.17) remains to be verified. It is obvious, however, if we realize that for any lattice U in \mathfrak{g} the support of \widehat{T}_U is contained in the intersection of the support of \widehat{T} with the complement of U in \mathfrak{g} .

LEMMA 1.18. *For any finite sequence of real numbers $d_1 < d_2 < \dots < d_r$ and a sequence a_1, a_2, \dots, a_r of complex numbers define the function*

$$F(x) = a_1 x^{d_1} + a_2 x^{d_2} + \dots + a_r x^{d_r} \quad (x > 0).$$

Then either F is identically equal to zero or F has at most $r - 1$ zeros.

We omit the elementary proof.

THEOREM 1.19. *Let T_1, T_2, \dots, T_r be Λ -homogeneous distributions on \mathfrak{g} of degrees $d_1 < d_2 < \dots < d_r$ respectively. Put $T = T_1 + T_2 + \dots + T_r$. then*

$$WF_\Lambda^0(T) = \bigcup_{j=1}^r WF_\Lambda^0(T_j).$$

Proof. Since the wave front set of a finite sum of distributions is clearly contained in the union of the wave front sets of the summands, it will suffice to verify the inclusion

$$(1.20) \quad WF_\Lambda^0(T) \supseteq \bigcup_{j=1}^r WF_\Lambda^0(T_j).$$

Take V disjoint with $\text{WF}_\Lambda^0(T)$ as in Lemma 1.11 (a). Then by (c)

$$(1.21) \quad 0 = (f_\gamma T) \wedge (\gamma^{-1} \lambda Y) = \sum_{j=1}^r |\gamma|^{d_j} (f T_j) \wedge (\lambda Y)$$

for $f \in C_c^\infty(W)$, $\lambda \in \Lambda$, $\gamma \in \Lambda$, $|\gamma| > c_\varepsilon$, $\varepsilon < |\gamma| < 1$, $Y \in V$.

Choose $\varepsilon > 0$ so that there are at least r elements in the set $(\varepsilon, 1] \cap \{|\gamma| \mid \gamma \in \Lambda\}$. Then Lemma 1.18 implies that each summand in (1.21) is zero. \square

2. P -adic wave front sets of group representation. Let \mathbf{G} be a connected, reductive Ω -group and G the subgroup of all Ω -rational points in \mathbf{G} . Then G with its usual topology is a locally compact, totally disconnected, unimodular group. Let \mathfrak{g} be the Lie algebra of G . Then \mathfrak{g} is a vector space over Ω of finite dimension and G operates on \mathfrak{g} by means of the adjoint representation. Assume that the form β in (1.1) is G -invariant.

Let π be an irreducible admissible representation of G and

$$\Theta_\pi(f) = \text{tr } \pi(f) \quad (f \in C_c^\infty(G))$$

be its character.

Let N be the set of all elements of \mathfrak{g} which are nilpotent. Then N is the union of a finite number of G -orbits which are called the nilpotent orbits. For all this see [Ha1], [Ha2]. Harish-Chandra [He 1, p. 180] has shown that one can choose an open neighborhood U of zero in \mathfrak{g} and, for each nilpotent orbit \mathbf{O} , a complex constant $c_{\mathbf{O}}$ such that

$$(2.1) \quad \Theta_\pi(\exp(X)) = \sum_{\mathbf{O}} c_{\mathbf{O}} \hat{\mu}_{\mathbf{O}}(X) \quad (X \in U).$$

Here $\mu_{\mathbf{O}}$ is a Radon measure on \mathfrak{g} given by

$$\mu_{\mathbf{O}}(f) = \int_{G/G_0} f(\text{Ad } g \cdot X_0) dg^* \quad (f \in C_c^\infty(\mathfrak{g}))$$

where $X_0 \in \mathbf{O}$ and G_0 is the stabilizer of X_0 in G (see [R]).

It follows from Theorem 1 in [R], that $\mu_{\mathbf{O}}$ is a Ω^\times -homogeneous distribution on \mathfrak{g} of degree $d = -n + \dim_\Omega(\mathbf{O})/2$. Therefore, via statement (1.5), $\hat{\mu}_{\mathbf{O}}$ is a homogeneous distribution of degree $-\dim_\Omega(\mathbf{O})/2$.

Let π be an admissible representation of G of finite length. Put

$$T = \Theta_\pi \cdot \exp.$$

Then (2.1) implies that

$$T = \sum_{j=1}^r T_j$$

where the T_j 's are homogeneous distributions on \mathfrak{g} of degrees d_j ($j = 1, 2, \dots, r$). Explicitly

$$T_j = \sum_{\dim \mathbf{O}/2 = -d_j} c_{\mathbf{O}} \hat{\mu}_{\mathbf{O}}.$$

Retain the above notation. Then Theorem 1.19 implies the following

THEOREM 2.2. *Let π be an admissible representation of G of finite length. Then*

$$\mathrm{WF}_{\Lambda}^0(T) = \bigcup_{j=1}^r \mathrm{supp} \hat{T}_j.$$

The left hand side of the first equation may be thought of as the wave front set of the representation π (see [H], [He]) and the right hand side as the asymptotic support (see [B-V]) of π . Recall also [He, Theorem 3.4] that for π unitary $\mathrm{WF}_{\Lambda}^0(T)$ coincides with the wave front set of π defined by the trace class operators. A statement analogous to Theorem 2.2 for the real reductive Lie groups was conjectured in [B-V] (and should hold via the inverse of the Lefschetz principle). Theorem 1.19 is true in the real case and its proof is equally easy.

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