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**SOME FUNDAMENTAL PROPERTIES OF CONTINUOUS  
FUNCTIONS AND TOPOLOGICAL ENTROPY**

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## SOME FUNDAMENTAL PROPERTIES OF CONTINUOUS FUNCTIONS AND TOPOLOGICAL ENTROPY

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**The purpose of this paper is to clarify some properties and results related to continuous functions on compact spaces and topological entropy.**

**1. Definitions and Propositions.** Note that in this paper we assume that the spaces are compact metric spaces unless otherwise stated.

If  $\alpha, \beta$  are open covers of  $X$  their join

$$\alpha \vee \beta = \{A \cap B : A \in \alpha \text{ and } B \in \beta\}.$$

We define  $H(\alpha) = \log N(\alpha)$ , where  $N(\alpha)$  is the number of sets in a finite subcover of  $\alpha$  for  $X$  with smallest cardinality. Note that  $H(\alpha) \geq 0$  and  $H(\alpha \vee \beta) \leq H(\alpha) + H(\beta)$  [10]. Define

$$\text{diam}(\alpha) = \max\{\text{diam}(U) : U \in \alpha\}.$$

Let  $(X, \phi)$  denote a continuous real flow [i.e.,  $\phi: X \times \mathbb{R} \rightarrow X$  continuous and  $\phi(x, t + s) = \phi(\phi(x, t), s)$ ] on a compact metric space  $X$ . The topological entropy of  $\phi$  is denoted by  $h(\phi)$  and defined to be  $h(\phi) = h(\phi_1)$ , where  $\phi_t: X \rightarrow X$  is a homeomorphism defined by  $\phi_t(x) = \phi(x, t)$ .

We recall that the flows  $(X, \phi)$  and  $(Y, \varphi)$  are conjugate (topologically conjugate) if there is a homeomorphism  $\gamma$  from  $X$  onto  $Y$  mapping orbits of  $\phi$  onto orbits of  $\varphi$  with preserved orientation. For more details see [2, 8, 9].

**PROPOSITION 1.1** (cf. [5]). *If  $(X, \phi)$  and  $(Y, \psi)$  are conjugate flows and they have no fixed points, then*

$$h(\phi) = ch(\psi),$$

where  $c$  is a finite positive constant.

Let  $T: X \rightarrow X$  be a homeomorphism and let  $f: X \rightarrow \mathbb{R}$  be any positive real valued continuous function. The *suspension of  $T$  under*

$f$  [2, 8] is defined to be the flow  $\phi_f$  on the space

$$X_f = \bigcup_{0 \leq t \leq fx} \{(x, t) : (x, fx) \sim (Tx, 0)\}$$

defined for small non-negative time by  $\phi_f(x, t) = \phi_f(x, t + s)$  with  $0 \leq t + s \leq fx$ .

It is well known that the suspension flows  $(X_f, \phi_f)$  and  $(X_g, \phi_g)$  of  $T: X \rightarrow X$  under  $f$  and  $g$  respectively are conjugate and a homeomorphism from  $X_f$  onto  $X_g$  that conjugates the flows is given by  $(x, t) \rightarrow (x, tg(x)/f(x))$ .

Let  $d_1$  and  $d_2$  be metrics defined on  $X$ . These metrics are *Lipschitz-equivalent* (*L-equivalent*) if there exist positive constants  $c_1$  and  $c_2$  such that

$$c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y)$$

for every  $x, y \in X$ . A map  $f$  from a metric space  $(X, d)$  into a metric space  $(Y, \sigma)$  is *Lipschitz* (*L-map*) if there exists a positive constant  $c$  such that

$$\sigma(fx, fy) \leq cd(x, y)$$

for every  $x, y \in X$ . A Lipschitz bijective map  $f: X \rightarrow Y$  such that  $f^{-1}$  is also Lipschitz will be called *L-homeomorphism* and denoted by  $f: X \cong Y$ . A metric space  $(X, d)$  is *L-embedded* in a metric space  $(Y, \sigma)$  if there exists an injective *L-map*  $i: X \rightarrow Y$  such that  $X \cong i(X) \subseteq Y$ .

It is obvious that any compact differentiable manifold  $M$  with the Riemannian metric is *L-embedded* in the Euclidean space  $R^m$  for some positive integer  $m$ .

**2. Continuous functions.** In this section we will introduce our basic proposition.

**PROPOSITION 2.1.** *Let  $f: X \rightarrow R$  be a positive real valued continuous function on  $X$  (compact metric space). Given  $\varepsilon > 0$ , then for every positive integer  $n$ , there exists an open cover  $\alpha_n$  of  $X$  such that  $\text{diam}(fU) \leq \varepsilon/n$  for all  $U$  in  $\alpha_n$  and*

$$\lim_{n \rightarrow \infty} \frac{1}{n} H(\alpha_n) = 0.$$

*Proof.* Let  $(Y = Y_f, \phi = \phi_f)$  be the suspension flow of the identity map  $I: X \rightarrow X$  under the given function  $f$ . Since  $(Y, \phi)$  is conjugate to the suspension flow  $(X_1, \phi_1)$  of  $I: X \rightarrow X$  under the constant 1 and  $h(\phi_1) = h(I) = 0$ , Proposition 1.1 implies that  $h(\phi) = 0$ . Now given

$n > 0$ , let  $t_n = n \sup_{x \in X} (fx)$  and take  $E_n$  to be  $(t_n, \varepsilon/2)$ -spanning set of  $X \times \{0\}$  with respect to  $\phi$  and with minimum cardinality. For  $e \in E_n$ , let

$$U_e = \{x \in X : d(\phi_s x, \phi_s e) < \varepsilon/2 \text{ for } 0 \leq s \leq t_n\}.$$

Then  $U_e$  is an open neighborhood of  $e$ . Suppose  $\text{diam}(fU_e) = \lambda_e$ . Then  $\text{diam}(U_e) + m \cdot \lambda_e \leq \varepsilon$  for some  $m \geq n$ . Hence  $\lambda_e \leq \varepsilon/m \leq \varepsilon/n$ . Let  $\alpha_n = \{U_e : e \in E_n\}$ . It is clear that  $\alpha_n$  is an open cover of  $X$  and  $\text{card}(\alpha_n) \leq \text{card}(E_n)$ . Since

$$\lim_{n \rightarrow \infty} \frac{1}{t_n} \log \text{card}(E_n) \leq h(\phi)$$

and  $h(\phi) = 0$ , therefore

$$\lim_{n \rightarrow \infty} \frac{1}{n} H(\alpha_n) \leq \lim_{n \rightarrow \infty} \left( \sup_{x \in X} f(x)/t_n \right) \log \text{card}(E_n) = 0,$$

and the proof is finished.

*Claim 1.* Let  $f: X \rightarrow R$  be a continuous real valued function on  $X$ . Given  $\varepsilon > 0$ , then for every positive integer  $n$ , there exists an open cover  $\alpha_n$  of  $X$  such that  $\text{diam}(fU) \leq \varepsilon/n$  for all  $U$  in  $\alpha_n$  and

$$\lim_{n \rightarrow \infty} \frac{1}{n} H(\alpha_n) = 0.$$

*Proof.* Let  $g = f + \alpha$  where  $\alpha > |\inf_{x \in X} (fx)|$ . Proposition 2.1 and the fact that  $\text{diam}(fU) = \text{diam}(gU)$  for every subset  $U$  of  $X$  finish the proof.

*Claim 2.* Let  $f: X \rightarrow R^m$  be a continuous function from a metric space  $X$  into  $(R^m, d_\infty)$ , where  $d_\infty(X, Y) = \max\{|x_i - y_i| : i = 1, 2, 3, \dots, m\}$ . Given  $\varepsilon > 0$ , then for every positive integer  $n$ , there exists an open cover  $\alpha_n$  of  $X$  such that  $\text{diam}(fU) \leq \varepsilon/n$  for all  $U$  in  $\alpha_n$  and

$$\lim_{n \rightarrow \infty} \frac{1}{n} H(\alpha_n) = 0.$$

*Proof.* Let  $\Pi_i: R^n \rightarrow R$  be the natural projection over  $R$  (i.e.,  $\Pi_i(x_1, x_2, \dots, x_m) = x_i$ ). For an integer  $n > 0$  let  $\alpha_n$  be an open cover for  $X$  satisfying Claim 2 with respect to  $\Pi_i f$  for  $i = 1, 2, \dots, m$ . Take

$\alpha_n = \bigvee_{i=1}^m \alpha_{n_i}$ . Then  $\alpha_n$  is an open cover for  $X$  and  $\text{diam}(fU) \leq \varepsilon/n$  for every  $U \in \alpha_n$ . Hence

$$\lim_{n \rightarrow \infty} \frac{1}{n} H(\alpha_n) \leq \sum_{i=1}^m \lim_{n \rightarrow \infty} \frac{1}{n} H(\alpha_{n_i}) = 0.$$

*Claim 3.* Let  $f: X \rightarrow R^m$  be a continuous function from  $X$  into  $(R^m, d)$  where  $d$  is a metric on  $R^m$  which is  $L$ -equivalent to  $d_\infty$ . Given  $\varepsilon > 0$ , then for every integer  $n > 0$ , there exists an open cover  $\alpha_n$  of  $X$  such that  $\text{diam}(fU) \leq \varepsilon/n$  for all  $U$  in  $\alpha_n$  and

$$\lim_{n \rightarrow \infty} \frac{1}{n} H(\alpha_n) = 0.$$

*Proof.* Is an easy exercise for the reader.

From Claim 3 we obtain immediately:

**THEOREM 1.** Let  $f: X \rightarrow Y$  be a continuous map from a metric space  $X$  into a metric space  $Y$  and suppose that  $Y$  is  $L$ -embedded in the Euclidean space  $R^m$  for some positive integer  $m$ . Given  $\varepsilon > 0$ . Then for every integer  $n > 0$ , there exists an open cover  $\alpha_n$  of  $X$  such that  $\text{diam}(fU) \leq \varepsilon/n$  for all  $U \in \alpha_n$  and

$$\lim_{n \rightarrow \infty} \frac{1}{n} H(\alpha_n) = 0.$$

**COROLLARY.** Let  $f: X \rightarrow X$  be a continuous map from  $X$  into itself and suppose that  $X$  is  $L$ -embedded in the Euclidean space  $R^m$  for some  $m > 0$ . Given  $\varepsilon > 0$ , then for every integer  $n > 0$ , there exists an open cover  $\alpha_n$  of  $X$  such that  $\text{diam}(U) \leq \varepsilon/n$  and  $\text{diam}(fU) \leq \varepsilon/n$  for all  $U \in \alpha_n$  and

$$\lim_{n \rightarrow \infty} \frac{1}{n} H(\alpha_n) = 0.$$

*Proof.* If  $f: X \rightarrow X$  is continuous and  $X$  is  $L$ -embedded in  $R^m$  for some  $m > 0$ , then without loss of generality we can consider  $f$  as a continuous function from  $X$  into  $R^m$ . This can be done also for an identity map  $I: X \rightarrow X$ . Given  $\varepsilon > 0$  and a positive integer  $n$ , Theorem 1 implies that there exist open covers  $\beta_n$  and  $\gamma_n$  of  $X$  such that  $\text{diam}(fU) \leq \varepsilon/n$  for every  $U \in \beta_n$  and  $\text{diam}(W) \leq \varepsilon/n$  for every  $W \in \gamma_n$  with

$$\lim_{n \rightarrow \infty} \frac{1}{n} H(\beta_n) = 0, \quad \lim_{n \rightarrow \infty} \frac{1}{n} H(\gamma_n) = 0.$$

Let  $\alpha_n = \beta_n \vee \gamma_n$ . Then  $\alpha_n$  is an open cover for  $X$  with

$$\max\{\text{diam}(U), \text{diam}(fU)\} \leq \varepsilon/n$$

for every  $U \in \alpha_n$  and

$$\lim_{n \rightarrow \infty} \frac{1}{n} H(\alpha_n) \leq \lim_{n \rightarrow \infty} \frac{1}{n} H(\beta_n) + \lim_{n \rightarrow \infty} \frac{1}{n} H(\gamma_n) = 0.$$

This finishes the proof.

**3. Topological entropy and examples.** Let  $f: X \rightarrow X$  be continuous. For  $E \subseteq X$  we say  $E$   $(n, \varepsilon)$ -spans  $X$  [1, 10], if for each  $x \in X$  there is an  $e \in E$  so that  $d(f^i x, f^i e) \leq \varepsilon$  for all  $0 \leq i \leq n$ . We let  $r_n(X, \varepsilon) = r_n(X, \varepsilon, f)$  denote the minimum cardinality of a set which  $(n, \varepsilon)$ -spans  $X$ . We define

$$h(f, \varepsilon) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log r_n(X, \varepsilon).$$

Notice that  $h(f, \varepsilon)$  increases as  $\varepsilon$  decreases. Finally, we define the topological entropy  $h(f)$  by

$$h(f) = \lim_{\varepsilon \rightarrow 0} h(f, \varepsilon).$$

In order to show that  $L$ -embedded condition is necessary in Theorem 1 we need to rewrite the corollary of Theorem 1 as follows:

**PROPOSITION 3.1.** *If  $f: X \rightarrow X$  is continuous and the metric space  $X$  is  $L$ -embedded in Euclidean space  $R^m$  for some  $m > 0$ , then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log r_1(X, \varepsilon/n) = 0.$$

*Proof.* By the corollary of Theorem 1 we let  $\alpha_n$  be an open cover of  $X$  with

$$\max\{\text{diam}(U), \text{diam}(fU)\} \leq \varepsilon/n$$

for every  $U \in \alpha_n$  and  $\lim_{n \rightarrow \infty} \frac{1}{n} H(\alpha_n) = 0$ . Pick a point in  $U$  and let  $E_n$  be the set of all such points. It is obvious that  $E_n$  is  $(1, \varepsilon/n)$ -spanning set of  $X$  and  $\text{card}(E_n) \leq \text{card}(\alpha_n)$ . This finishes the proof.

The following example [10] shows that  $L$ -embedded condition is necessary in Theorem 1.

**EXAMPLE 3.2.** Let  $k$  be a fixed positive integer and let  $C = \{0, 1, 2, \dots, k - 1\}$  with the discrete topology. Consider the product space  $\Sigma = \prod_{-\infty}^{\infty} C$  with the product topology and the shift homeomorphism  $\sigma: \Sigma \rightarrow \Sigma$  defined by  $\sigma(\{w_n\}_{-\infty}^{\infty}) = \{w_{n+1}\}_{-\infty}^{\infty}$ . A metric on  $\Sigma$  can be

defined by  $d(\{x_i\}, \{y_i\}) = 1/(m + 2)$  if  $m$  is the largest positive integer with  $x_i = y_i$  for all  $|i| \leq m$  and  $d(\{x_i\}, \{y_i\}) = 1$  if  $x_0 \neq y_0$ . Now it is an easy exercise to show that  $r_1(\Sigma, 1) = 1$  and  $r_1(\Sigma, \frac{1}{n}) \geq k^n$  for every positive integer  $n$ . This means that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log r_1(\Sigma, \frac{1}{n}) \geq \log k > 0$$

which contradicts Proposition 3.1 and shows that  $(\Sigma, d)$  is not  $L$ -embedded in any Euclidean space.

Now we consider the following example:

**EXAMPLE 3.3.** One considers Smale’s horseshoe [7], i.e., a diffeomorphism  $f: D \rightarrow D$ , where  $D$  is a 2-dimensional disk. We may assume that  $f/\partial D$  is the identity map. For more details, see the book of Nitecki [4]. This example has the property that for every positive integer  $n$ , there is  $\varepsilon > 0$  such that

$$h(f^n, \varepsilon) = n \log 2.$$

Now we consider a sequence of disks  $D_n$  on the plane of radii  $2^{-n}$ , disjoint and converging to a point. Let us also fix a sequence of natural numbers  $\{n_i\}_{i=1}^\infty$ . We define a map  $g: R^2 \rightarrow R^2$  as follows:

$$g(x) = \begin{cases} f_i^{n_i}(x), & \text{if } x \in D_i, \\ x, & \text{if } x \in R^2 \setminus \bigcup_{i=1}^\infty D_i. \end{cases}$$

Here  $f_i: D_i \rightarrow D_i$  is a homothetic copy of  $f$ . Obviously,  $g$  extends to the one-point compactification  $S^2$  and we can say

$$h(g, \varepsilon_i) \geq h(f_i^{n_i}, \varepsilon_i) \geq n_i \log 2.$$

Here also  $\varepsilon_i$  is a homothetic copy of  $\varepsilon$ .

The question we want to discuss here is whether it is possible to choose a sequence of natural numbers  $\{n_i\}_{i=1}^\infty$  and a sequence of  $\{\varepsilon_i\}_{i=1}^\infty$  such that  $\varepsilon_i \cdot n_i \rightarrow \infty$  (i.e., is it possible to construct a  $g: S^2 \rightarrow S^2$  with the property that  $\varepsilon_i h(g, \varepsilon_i) \rightarrow \infty$  as  $i \rightarrow \infty$ ). Note that  $\varepsilon_i$  is not independent of  $n_i$ ; otherwise such a question is trivially true. In fact the answer for this question is not true. Moreover, we show later in Theorem 2 that  $\varepsilon_i h(g, \varepsilon_i)$  must always vanish (i.e.,  $\varepsilon_i h(g, \varepsilon_i) \rightarrow 0$  as  $i \rightarrow \infty$ ).

**LEMMA 3.4.** *If  $E$  is  $(1, \varepsilon)$ -spanning set of a metric space  $X$ , then for every positive integer  $k$ , there exists a set  $W$  which is  $(k, 2\varepsilon)$ -spanning set of  $X$  and  $\text{card}(W) \leq (\text{card}(E))^k$ .*

*Proof.* Special case of Lemma 2.1 in [1].

**THEOREM 2.** *If  $f: X \rightarrow X$  is a homeomorphism on  $X$  and if  $X$  is  $L$ -embedded in  $R^n$  for some positive integer  $n$ , then  $eh(f, \varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .*

*Proof.* Let  $E_n$  be any  $(1, \varepsilon/2n)$ -spanning set of  $X$  with minimum cardinality. Using Lemma 3.4 there exists a set  $W_n$  which  $(p, \varepsilon/n)$ -spans  $X$  and  $\text{card}(W_n) \leq (\text{card}(E_n))^p$  for every positive integer  $p$ . Hence

$$\frac{1}{p} \log r_p(X, \varepsilon/n) \leq \log r_1(X, \varepsilon/2n).$$

Therefore

$$h(f, \varepsilon/n) \leq \log r_1(X, \varepsilon/2n).$$

This means that

$$\lim_{n \rightarrow \infty} \frac{1}{n} h(f, \varepsilon/n) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \log r_1(X, \varepsilon/2n).$$

Proposition 3.1 finishes the proof of this theorem.

**4. Topological entropy of expansive maps.** During the remainder of this section we assume that  $f$  is an expansive homeomorphism of a compact metric space  $(X, d)$  onto itself with expansive constant  $e > 0$  (i.e.,  $x \neq y$  implies  $d(f^n x, f^n y) \geq e$  for some integer  $n$ ).

In this section we will use an adaptation of work by Reddy [6] to show that we can find a metric compatible with the topology of  $X$  and a positive real number  $\lambda$ ,  $0 < \lambda < 1$ , such that  $\lim_{m \rightarrow \infty} \frac{1}{m} \log r_0(X, \lambda^m)$  is the topological entropy of  $f$  (i.e., the present tells us about the past and the future).

For an integer  $n \geq 0$  define,

$$W_n = \{(x, y) \in X \times X : d(f^i x, f^i y) \leq e \text{ for } -n \leq i \leq n\}.$$

It is obvious that  $\bigcap W_n = \Delta$  where  $\Delta = \{(x, x) : x \in X\}$ .

**LEMMA 4.1.**  *$\forall \varepsilon > 0, \exists N > 0$ , such that  $d(f^i x, f^i y) \leq e$  for all  $i$  with  $|i| \leq N$ , implies  $d(x, y) \leq \varepsilon$ .*

*Proof.* Walters [11].

Take  $\varepsilon$  small enough such that  $3\varepsilon \leq e$ . Choose  $N$  using the above lemma with respect to  $\varepsilon$ . Define  $V_n = W_{nN}$  for  $n = 0, 1, 2, 3, \dots$

The following was proved by W. Reddy [6]. Because  $V_n$  is defined slightly differently in [6] we will give another proof.

LEMMA 4.2 (cf. [6, Lemma 2]). *The sequence  $\{V_n\}$  is a nested sequence of symmetric neighborhoods of  $\Delta$  whose intersection is  $\Delta$  and such that  $V_{n+1} \circ V_{n+1} \circ V_{n+1} \subseteq V_n$  for each  $n \geq 0$ .*

*Proof.* Let  $(x, y) \in V_{n+1} \circ V_{n+1} \circ V_{n+1}$ . There exist  $a, b$  elements in  $X$  such that  $(x, a) \in V_{n+1}$ ,  $(a, b) \in V_{n+1}$ , and  $(b, y) \in V_{n+1}$ . Hence

$$\begin{aligned} d(f^i x, f^i a) &\leq e \quad \text{for } -(n+1)N \leq i \leq (n+1)N, \\ d(f^i a, f^i b) &\leq e \quad \text{for } -(n+1)N \leq i \leq (n+1)N, \end{aligned}$$

and

$$d(f^i b, f^i y) \leq e \quad \text{for } -(n+1)N \leq i \leq (n+1)N.$$

Lemma 4.1 implies that

$$\begin{aligned} d(f^i x, f^i a) &\leq \varepsilon \quad \text{for } -nN \leq i \leq nN, \\ d(f^i a, f^i b) &\leq \varepsilon \quad \text{for } -nN \leq i \leq nN, \end{aligned}$$

and

$$d(f^i b, f^i y) \leq \varepsilon \quad \text{for } -nN \leq i \leq nN.$$

The triangle inequality implies

$$d(f^i x, f^i y) \leq 3\varepsilon \leq e \quad \text{for } -nN \leq i \leq nN.$$

This means that  $(x, y) \in V_n$ .

The following is an immediate consequence of Lemma 4.2 and the Metrization lemma [3].

LEMMA 4.3. *There is a metric  $\rho$  compatible with the topology of  $X$  such that*

$$N(\Delta; 1/2^{n+1}) \subseteq V_n \subseteq N(\Delta; 1/2^n)$$

for  $n \geq 1$ .

LEMMA 4.4. *There is a metric  $\rho$  compatible with the topology of  $X$  and there is  $\lambda$ ,  $0 < \lambda < 1$ , such that*

$$N(\Delta; \lambda^{m+2N}) \subseteq W_m \subseteq N(\Delta; \lambda^{m-N})$$

for all  $m \geq 0$ .

*Proof.* Suppose  $m = nN + j$  where  $0 \leq j < N$ . It is obvious that

$$V_{n+1} = W_{(n+1)N} = W_{nN+N} \subseteq W_{nN+j} = W_m \subseteq W_{nN} = V_n.$$

Therefore  $V_{n+1} \subseteq W_m \subseteq V_n$ . Using Lemma 4.3 we have  $N(\Delta; 1/2^{n+2}) \subseteq V_{n+1}$  and  $V_n \subseteq N(\Delta; 1/2^n)$ . Take  $\lambda = (1/2)^{1/N}$ . It is clear that

$$\begin{aligned} N(\Delta; \lambda^{m+2N}) &\subseteq N(\Delta; \lambda^{m+2N-j}) = N(\Delta; \lambda^{nN+2N}) \\ &= N(\Delta; (1/2)^{n+2}) \subseteq V_{n+1} \subseteq W_m \subseteq V_n \\ &\subseteq N(\Delta; 1/2^n) = N(\Delta; (1/2)^{nN/N}) \\ &= N(\Delta; \lambda^{nN}) = N(\Delta; \lambda^{m-j}) \subseteq N(\Delta; \lambda^{m-N}). \end{aligned}$$

This finishes the proof of this lemma.

**LEMMA 4.5** (cf. [1, Theorem 2.4 and Corollary 2.5]).  $\exists \varepsilon > 0$  such that  $h(f) = h(f, \varepsilon)$  and  $\frac{1}{n} \log r_n(X, \varepsilon) \rightarrow h(f)$ .

**THEOREM 3.** *There is a metric  $\rho$  compatible with the topology of  $X$  and there is  $\lambda, 0 < \lambda < 1$ , such that*

$$h(\phi) = \lim_{m \rightarrow \infty} \frac{1}{m} \log r_0(X, \lambda^m).$$

*Proof.* Let  $E$  be  $(2m, e)$ -spanning set of  $X$  with minimum cardinality. It is obvious that for every  $x \in X$ , there exists  $w \in f^m(E)$  so that  $d(f^i x, f^i w) \leq e$  for  $-m \leq i \leq m$ . So  $(x, w) \in W_m$ . Using Lemma 4.4 we can find a metric  $\rho$  on  $X$  and  $\beta, 0 < \beta < 1$ , such that  $(x, w) \in N(\Delta; \beta^{m-N})$ . This means that there exists an  $F$  which is  $(0, \beta^{m-N})$ -spanning set of  $X$  (i.e.,  $\beta^{m-N}$ -net with respect to  $\rho$ ) and  $\text{card}(F) \leq \text{card}(f^m E) = \text{card}(E)$ . Therefore  $r_0(X, \beta^{m-N}) \leq r_{2m}(X, e)$ .

Now suppose  $F$  is  $(0, \beta^{m+2N})$ -spanning set of  $X$  with respect to the metric  $\rho$  and with minimum cardinality. Thus for every  $x \in X$ , there exists  $w \in F$  such that  $(x, w) \in N(\Delta; \beta^{m+2N})$ . So  $(x, w) \in W_m$  (i.e.,  $d(f^i x, f^i w) \leq e$  for  $-m \leq i \leq m$ ). This implies that  $d(f^i f^{-m} x, f^i f^{-m} w) \leq e$  for all  $0 \leq i \leq 2m$ . Thus we can find  $E$  which is  $(2m, e)$ -spanning set of  $X$  and  $\text{card}(E) \leq \text{card}(F)$ . Therefore  $r_{2m}(X, e) \leq r_0(X, \beta^{m+2N})$ . Take  $\lambda = \beta^{1/2}$ . Therefore  $r_0(X, \lambda^{2m-2N}) \leq r_{2m}(X, e)$  and  $r_{2m}(X, e) \leq r_0(X, \lambda^{2m+4N})$ . Using Lemma 4.5 and taking  $e \leq \varepsilon$  we have

$$h(\phi) = \lim_{m \rightarrow \infty} \frac{1}{2m} \log r_0(X, \lambda^{2m}).$$

But

$$\frac{1}{2m+1} \log r_0(X, \lambda^{2m+1}) \geq \frac{1}{2m+1} \log r_0(X, \lambda^{2m}) \rightarrow h(\phi),$$

and

$$\frac{1}{2m-1} \log r_0(X, \lambda^{2m-1}) \leq \frac{1}{2m-1} \log r_0(X, \lambda^{2m}) \rightarrow h(\phi).$$

Therefore

$$h(\phi) = \lim_{m \rightarrow \infty} \frac{1}{m} \log r_0(X, \lambda^m)$$

and the proof is finished.

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