UNITARY BORDISM OF CLASSIFYING SPACES OF QUATERNION GROUPS

Abdeslam Mesnaoui
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ABDESLAM MESNAOUI

Let $\Gamma_k$ be the generalized quaternion group of order $2^k$. In this article we determine a set of generators for the $U_*(pt)$-module $\tilde{U}_*(B\Gamma_k)$ and give all linear relations between them. Moreover their orders are calculated.

0. Introduction. In this article we first study the case $\Gamma_k = \Gamma$ the quaternion group of order 8. We recall that

$$\Gamma = \{\pm 1, \pm i, \pm j, \pm k\}, \quad i^2 = j^2 = k^2 = -1, \quad ij = k, \quad jk = i, \quad ki = ij.$$ 

$\Gamma$ acts on $S^{4n-3}$ by using $(n+1)\eta$ where $\eta$ denotes the following unitary irreducible representation of $\Gamma$: $i \rightarrow \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $j \rightarrow \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and we get the element $w_{4n+3} = [S^{4n+3}/\Gamma, q] \in \tilde{U}_{4n+3}(B\Gamma)$, $q$ being the natural embedding: $S^{4n+3}/\Gamma \subset B\Gamma$. In [6] we have defined three elements of $\tilde{U}^2(B\Gamma)$ denoted by $A$, $B$, $C$ as Euler classes for $MU$ of irreducible representations of $\Gamma$ of dimension 1 over $\mathbb{C}$. Let $u_{4n+1} \in \tilde{U}_{4n+1}(B\Gamma)$, $v_{4n+1} \in \tilde{U}_{4n+1}(B\Gamma)$ be respectively $A \cap w_{4n+3}$ and $B \cap w_{4n+3}$. Our first result is:

**Theorem 2.2.** The set $\{u_{4n+1}, v_{4n+1}, w_{4n+3}\}_{n \geq 0}$ is a system of generators for the $U_*(pt)$-module $\tilde{U}_*(B\Gamma)$.

Their orders are given by:

**Theorem 2.6.** We have: $\text{ord } w_{4n+3} = 2^{2n+3}$.

**Theorem 2.8.** We have: $\text{ord } u_{4n+1} = \text{ord } v_{4n+1} = 2^{n+1}$.

Now let $\Omega_*$ be $U^*(pt)[[Z]]$ graded by taking $\dim Z = 4$. If $P(Z) = \sum_{i \geq r} \alpha_i Z^i \in \Omega_4$ and $\alpha_r \neq 0$ then we denote $\nu(P) = 4r$. Let $W_1, V_1, V_2$ be the submodules of $\tilde{U}_*(B\Gamma)$ generated respectively by $\{w_{4n+3}\}_{n \geq 0}$, $\{u_{4n+1}\}_{n \geq 0}$, $\{v_{4n+1}\}_{n \geq 0}$. The following result gives the $U_*(pt)$-module structure of $\tilde{U}_*(B\Gamma)$ and uses the elements $T(Z) \in \Omega_4$, $J(Z) \in \Omega_0$ as defined in [6], Section II.
THEOREM 2.4. 
(a) \( \tilde{U}_*(B \Gamma) = W \oplus V_1 \oplus V_2. \)
(b) In \( \tilde{U}_{2p+1}(B \Gamma) \) we have
\[
0 = a_0 w_3 + a_1 w_7 + \cdots + a_n w_{4n+3} = b_0 u_1 + \cdots + b_m u_{4m+1}
\]
iff there are homogeneous polynomials \( M(Z), M_2(Z) \) and homogeneous formal power series \( N(Z), N_1(Z) \) of \( \Omega * \) satisfying:
\[
b_m Z + b_{m-1} Z^2 + \cdots + b_0 Z^{m+1} = M(Z)(2 + J(Z)) + N(Z), \quad a_n Z + a_{n-1} Z^2 + \cdots + a_0 Z^{n+1} = M_1(Z)T(Z) + N_1(Z),
\]
\( \nu(N) > 4(n+1), \quad \nu(N_1) > 4(n+1). \) Moreover \( b_0 u_1 + \cdots + b_m u_{4m+1} = 0 \) iff \( b_0 v_1 + \cdots + b_m v_{4m+1} = 0. \)

In Section III we consider \( \tilde{U}_*(B \Gamma_k), k \geq 4. \) The generalized quaternion group \( \Gamma_k \) is generated by \( u, v \) with \( u^* = v^2, t = 2^{k-2}, uvu = v. \) \( \Gamma_k \) acts on \( S^{4n+3} \) by means of the irreducible unitary representation \( \eta_1 \) of \( \Gamma_k: \)
\[
u \rightarrow \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \quad u \rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]
\( \omega \) being a primitive \( 2^{k-1} \)th root of unity. We get:
\[
w_{4n+3}' = [S^{4n+3}/\Gamma_k, q'] \in \tilde{U}_{4n+3}(B \Gamma_k), \quad q': S^{4n+3}/\Gamma_k \subset B \Gamma_k.
\]
Now we use the elements \( B_k' = B_k + G_k(D_k) \in \tilde{U}^2(B \Gamma_k), \quad C'_k = C_k + G_k(D_k) \in \tilde{U}^2(B \Gamma) \) (see [6], Theorem 3.14) to define \( u_{4n+1}' = B_k' \cap w_{4n+3}' \in \tilde{U}_{4n+1}(B \Gamma_k), \quad v_{4n+1}' = C_k' \cap w_{4n+3}' \in \tilde{U}_{4n+1}(B \Gamma_k). \) Then we have Theorems 3.1, 3.2 identical respectively to the above Theorems 2.2, 2.4 where \( w_{4n+3}, u_{4n+1}, v_{4n+1} \) are replaced by \( w_{4n+3}', u_{4n+1}', v_{4n+1}'. \) However:

**THEOREM 3.4.** We have: \( \text{ord } w_{4n+3}' = 2^{2n+k}, \quad n \geq 0. \)

**THEOREM 3.5.** We have: \( \text{ord } u_{4n+1}' = \text{ord } v_{4n+1}' = 2^{n+1}, \quad n \geq 0, \) which are therefore independent of \( k. \)

The layout is as follows:

I. Preliminaries and notations.
II. Calculations in \( \tilde{U}_*(B \Gamma): \) generators, orders and relations.
III. \( \tilde{U}_*(B \Gamma_k), k \geq 4: \) generators, orders and relations.

We assume that the reader is acquainted with the notations and results of [6].

I. Preliminaries and notations. The notation \( U_*-\text{AHSS} \) will be used for the Atiyah-Hirzebruch spectral sequence corresponding to the homology theory determined by \( MU; \mu \) and \( \mu' \) denote the edge homomorphisms \( U^*(X) \rightarrow H^*(X) \) and \( U_*(X) \rightarrow H_*(X) \) obtained from the
$U_\ast$-AHSS for a CW complex $X$. We have the following well-known result:

**Theorem 1.1.** Suppose $X$ a CW-complex such that:
(a) The $U_\ast$-AHSS for $X$ collapses.
(b) For each $n \geq 0$ there is a system $(a_{in})$ generating the group $H_n(X)$.

Then for each $n \geq 0$ there is a system $(A_{in})$ such that:
(a) $A_{in} \in U_n(X)$, $\mu'(A_{in}) = a_{in}$ for every $(i, n)$.
(b) The system $(A_{in})$ generates $U_\ast(X)$ as a $U_\ast(pt)$-module.

Moreover, (b) is valid for every system $(A_{in})$ such that $\mu'(A_{in}) = a_{in}$. □

Consider the map of ring spectra $f: MU \to H$ (see [1]); by naturality of spectral sequences it follows that if $X$ is a CW-complex then $f^\#(X) = \mu$ and $f_\#(X) = \mu'$ where $f^\#(X): U^\ast(X) \to H^\ast(X)$, $f_\#(X): U_\ast(X) \to H_\ast(X)$ denote the maps induced by $f$.

**Proposition 1.2.** If $X$ is a CW-complex then the following diagram commutes:

$$
\begin{array}{ccc}
U^m(X) \otimes U_n(X) & \xrightarrow{\cap} & U_{n-m}(X) \\
\mu \otimes \mu' \downarrow & & \downarrow \mu' \\
H^m(X) \otimes H_n(X) & \xrightarrow{\cap} & H_{n-m}(X)
\end{array}
$$

**Proof.** Take $E = MU$. The cap product is the composite:

$$
\tilde{E}_m(X^+) \otimes \tilde{E}_n(X^+) \xrightarrow{1 \otimes \Delta} \tilde{E}_m(X^+) \otimes \tilde{E}_n(X^+ \land X^+) \xrightarrow{\mathbf{\mu}} \tilde{E}_{n-m}(X^+),
$$

\$\Delta$ being the slant product and $\Delta(x) = [x, x]$. Since $\Delta_*$ commutes with $f_\#(-)$ we have to prove that the diagram:

$$
\begin{array}{ccc}
\tilde{E}^m(X^+) \otimes \tilde{E}_n(X^+ \land X^+) & \xrightarrow{\mathbf{\mu}} & \tilde{E}_{n-m}(X^+) \\
\downarrow \mathbf{\mu}^*(-) \otimes f_\#(-) & & \downarrow f_\#(-) \\
\tilde{H}^m(X^+) \otimes \tilde{H}_n(X^+ \land X^+) & \xrightarrow{\mathbf{\mu}} & \tilde{H}_{n-m}(X^+)
\end{array}
$$

commutes. More generally the diagram

$$
\begin{array}{ccc}
\tilde{E}^m(Y) \otimes \tilde{E}_n(Y \land Z) & \xrightarrow{\mathbf{\mu}} & \tilde{E}_{n-m}(Z) \\
\downarrow \mathbf{\mu}^*(-) \otimes f_\#(-) & & \downarrow f_\#(-) \\
\tilde{H}^m(Y) \otimes \tilde{H}_n(Y \land Z) & \xrightarrow{\mathbf{\mu}} & \tilde{H}_{n-m}(Z)
\end{array}
$$

commutes if $Y, Z$
are pointed CW-complexes: indeed let \( x \) and \( y \) be any elements of \( \tilde{E}^m(Y) \) and \( \tilde{E}^n(Y \wedge Z) \) respectively represented by \( g: Y \to \Sigma^m E, \)
\( h: S^n \to E \wedge Y \wedge Z. \) Then \( f^\#(-)(x) \) is represented by the composite
\[
\begin{align*}
g_1: Y \xrightarrow{g} \Sigma^m E \xrightarrow{\Sigma^m f} \Sigma^m H \quad \text{and} \quad f^\#(-)(y)
\end{align*}
\]
by the composite:
\[
\begin{align*}
h_1: S^n \xrightarrow{h} E \wedge Y \wedge Z \xrightarrow{f^\wedge_{1,1}} H \wedge Y \wedge Z.
\end{align*}
\]
If we denote by \( T \) the transposition and \( k, k' \) the ring-spectra products then the diagram pictured on the next page commutes. Since the top line represents \( x \)\( \setminus y \) and the bottom line
\[
f^\#(-)(x) \setminus f^\#(-)(y)
\]
we have \( f^\#(-)(x \setminus y) = f^\#(-)(x) \setminus f^\#(-)(y). \) \hfill \( \square \)

Let \( X \) be any CW-complex and \( \xi \) a complex vector bundle of \( \mathbb{C} \)-dimension \( n \) over \( X. \) If \( h \) denotes a map: \( X \to BU(n) \) classifying \( \xi \)
and \( M(\xi) \) the Thom space of \( \xi, \) then \( M(h): M(\xi) \to MU(n) \) determines an element \( t_0(\xi) \in U^{2n}(M(\xi)) \) which is a particular Thom class
for \( \xi \) called the canonical Thom class for \( \xi. \) Moreover if \( j: X \to M(\xi) \)
is the zero section we have \( j^*(t_0(\xi)) = cf_n(\xi), \) the highest Conner-Floyd characteristic class of \( \xi; \) \( j^*(t_0(\xi)) \) is also called the Euler class \( e(\xi) \) of \( \xi. \)

Fundamental classes for a \( U \)-manifold \( M^n \) for \( E = MU \) or \( H \)
may be obtained in the following manner: \( M^n \) can be embedded in \( S^{n+2k} \) for some large \( k \) and the normal bundle \( \tau \) can be given a \( U(k) \)-structure; let \( N \) be a tubular neighbourhood of \( M^n, \) which we identify
with the total space of the normal disk bundle \( D(\tau); \) we have the map \( \pi: S^{n+2k} \to M(\tau) \) defined as follows: if \( x \in N \) then \( \pi(x) \) is the image of \( x \) by the projection \( D(\tau) \to M(\tau) \) and if \( x \in S^{n+2k} - \overset{\circ}{N}, \) then \( \pi(x) = * \) the base point of \( M(\tau); \) let \( t \) be a Thom class of \( \xi \) for \( E; \)
we have the Thom-isomorphism \( \phi_t: E_{2k+r}(M(\tau)) \to E_r(M^n) \) such that \( \phi_t(x) = p_*(t \cap x), \) \( p \) being the projection \( D(\tau) \to M^n; \) let \( u: S^0 \to E \)
be the unit of \( E; \) the map \( u \) is a map of spectra and is therefore a collection of maps \( u_m: S^m \to E_m \) satisfying well-known axioms; then by [8], page 333, if \( [u_{n+2k}] \) is the element of \( \tilde{E}^n(S^{n+2k}) \) corresponding to \( u_{n+2k}, \) then the element \( c(M) = \phi_t(\pi_*([u_{n+2k}])) \in E_n(M^n) \) is a fundamental class for \( M^n. \) Evidently the same method produces fundamental classes for the homology theory defined by the spectrum \( H. \)
\[ S^{n-m} \xrightarrow{\Sigma^{-m} h} (\Sigma^{-m} E) \land Y \land Z \xrightarrow{T_{\land 1}} Y \land \Sigma^{-m} E \land Z \xrightarrow{g_{\land 1 \land 1}} \Sigma^{m} E \land \Sigma^{-m} E \land Z \cong E \land E \land Z \xrightarrow{k_{\land 1}} E \land Z \]

\[ s^{n-m} \xrightarrow{\Sigma^{-m} h_1} (\Sigma^{-m} H) \land Y \land Z \xrightarrow{T_{\land 1}} Y \land \Sigma^{-m} H \land Z \xrightarrow{g_{\land 1 \land 1}} \Sigma^{m} H \land \Sigma^{-m} H \land Z \cong H \land H \land Z \xrightarrow{k'_{\land 1}} H \land Z \]
From [8], page 335, §14-45, we have:

**Proposition 1.3.** If $M^n$ is a closed $U$-manifold then $[M^n, 1] \in U_n(M^n) = E_n(M^n)$ is a fundamental class for $M^n$ deduced from the canonical Thom class $t_0(\tau)$, $\tau$ being the normal bundle of an embedding $M^n \subset S^{n+2k}$, $k$ large. \hfill $\square$

**Proposition 1.4.** Let $M^n$ be a closed $U$-manifold; then

$$f_#(-)([M^n, 1]) \in H_n(M^n)$$

is a fundamental class for $M^n$.

**Proof.** From 1.3 we have

$$[M^n, 1] = \phi_{t_0}(\pi_*[u_{n+2k}]) = c(M);$$

then

$$f_#(-)(c(M)) = f_#(-)[\phi_{t_0}(\pi_*([u_{n+2k}]))] = f_#(-)[p_*(t_0 \cap \pi_*([u_{n+2k}]))]$$

$$= p_*[f_#(-)(t_0 \cap \pi_*([u_{n+2k}]))]$$

$$= p_*[f^#(-)(t_0) \cap f_#(-)(\pi_*([u_{n+2k}]))]$$

$$= p_*[f^#(-)(t_0) \cap \pi_*(f(-)([u_{n+2k}])))].$$

Since $f$ is a map of spectra the unit of $H$ is the composite $\nu: S^0 \xrightarrow{\mu} MU \xrightarrow{f} H$ and hence $f_#(-)([u_{n+2k}]) = [v_{n+2k}]$. Now $f^#(-)(t_0)$ is a Thom class $t_1$ for $H$ and therefore

$$f_#(-)(c(M)) = p_*[t_1 \cap \pi_*([v_{n+2k}]])$$

$$= \phi_{t_1}(\pi_*([v_{n+2k}])) = c_1(M^n) \in H_n(M^n)$$

is a fundamental class for $M^n$. \hfill $\square$

The notation $c(M^n)$ will be for the fundamental class $[M^n, 1] \in U_n(M^n)$ and $c_1(M^n) \in H_n(M^n)$ will be the fundamental class $\mu'(c(M^n))$.

If PD or PD$_1$ denotes the Poincaré duality then we have:

**Proposition 1.5.** The following diagram commutes

$$\begin{align*}
U^m(M^n) & \xrightarrow{\text{PD}} U_{n-m}(M^n) \\
\downarrow \mu & \quad \downarrow \mu' \\
H^m(M^n) & \xrightarrow{\text{PD}_1} H_{n-m}(M^n)
\end{align*}$$
Proof. We have
\[ \mu'(\text{PD}(x)) = \mu'(x \cap c(M^n)) = \mu(x) \cap \mu'(c(M^n)) = \mu(x) \cap c_1(M^n) = (\text{PD})_1(\mu(x)) \]
by 1.2. \( \square \)

Let \( N^m \) be a closed \( U \)-submanifold of a closed \( U \)-manifold \( M^n \), and \( i \) the inclusion \( N^m \subset M^n \); then the normal bundle \( \tau \) of \( N^m \) in \( M^n \) is a complex-vector-bundle if \( (n - m) \) is even and we have:

**Proposition 1.6.** If \( (n - m) \) is even then \( (\text{PD})^{-1}([N^m, i]) \) is represented by:

\[ M^n \rightarrow M^n/(M^n - \overset{\circ}{N}) \simeq D(\tau)/S(\tau) = M(\tau) \xrightarrow{M(h)} MU(\frac{1}{2}(n - m)), \]

where \( h \) is a map classifying \( \tau \) and \( N \) a tubular neighborhood of \( N^m \) homeomorphic to \( D(\tau) \) (see [3], [7]). \( \square \)

The generalized quaternion group \( \Gamma_k, k \geq 4 \), is generated by \( u, v \) subject to the relations \( u^k = v^2, t = 2^{k-2}, uvu = v \). Consider the irreducible unitary representation \( \eta_1 \) of \( \Gamma_k : u \rightarrow (\omega 0, 0), v \rightarrow (0 \omega^{-1}), \) \( \omega \) being a primitive \( 2^{k-1} \)-th-root of unity. The group \( \Gamma_k \) acts on \( S^{4n+3} \) by means of \( (n + 1)\eta_1 \) as a group of \( U \)-diffeomorphisms and we get a canonical \( U \)-structure on \( S^{4n+3}/\Gamma_k \) and a natural injection \( S^{4n+3}/\Gamma_k \subset \mathcal{B} \Gamma_k = \bigcup_{n \geq 0} S^{4n+3}/\Gamma_k \) (see [3], [10], page 508).

Let \( \alpha \) be the complex vector bundle: \( S^{4n+3} \times \Gamma_k \mathbb{C}^2 \rightarrow S^{4n+3}/\Gamma_k \) where \( \Gamma_k \) acts on \( S^{4n+3} \) and \( \mathbb{C}^2 \) respectively by means of \( (n + 1)\eta_1 \) and \( \eta_1 \): if \( a \in \Gamma_k \) and \( (x, v) \in S^{4n+3} \times \mathbb{C}^2 \) we have \( a(s, w) = (as, aw) = (sa^{-1}, aw) \) and \( S^{4n+3} \times \Gamma_k \mathbb{C}^2 = (S^{4n+3} \times \mathbb{C}^2)/\Gamma_k \). Then by a result of R. H. Szczarba ([9]) we have \( T(S^{4n+3}/\Gamma_k) + 1 = (n + 1)\alpha \) where \( T(S^{n+3}/\Gamma_k) \) denotes the tangent bundle of \( S^{4n+3}/\Gamma_k \). As an easy consequence we have:

**Proposition 1.7.** If \( i \) denotes the embedding \( S^{4n+3}/\Gamma_k \subset S^{4n+7}/\Gamma_k \) such that

\[ i([z_1, z_2, \ldots, z_{2n+2}]) = [z_1, z_2, \ldots, z_{2n+2}, 0, 0], \]

then the normal bundle of \( S^{4n+3}/\Gamma_k \) in \( S^{4n+7}/\Gamma_k \) is isomorphic to the complex vector bundle \( \alpha \). \( \square \)
We shall give a proof of the next result which can be found in [7]:

**Proposition 1.8.** If \( i \) denotes the embedding \( S^{4n+3}/\Gamma_k \subset S^{4n+7}/\Gamma_k \) then \( i^* \circ (PD)^{-1}([S^{4n+3}/\Gamma_k, i]) = e(\alpha) \).

**Proof.** Denote by \( \tau \) the normal bundle of \( S^{4n+3}/\Gamma_k \) in \( S^{4n+7}/\Gamma_k \) and by \( h \) a classifying map: \( S^{4n+3}/\Gamma_k \to BU(2) \) for \( \tau \). Then by 1.6, 
\[
(PD)^{-1}([S^{4n+3}/\Gamma_k, i]) \text{ is represented by the composite:} 
\]
\[
\begin{align*}
S^{4n+7}/\Gamma_k & \to (S^{4n+7}/\Gamma_k) / (S^{4n+7}/\Gamma_k - \tilde{N}) \\
& \simeq \frac{D(\tau)}{S(\tau)} = M(\tau) \xrightarrow{M(h)} MU(2),
\end{align*}
\]
\( N \) being a tubular neighbourhood of \( S^{4n+3}/\Gamma_k \) homeomorphic to \( D(\tau) \). Since the composite:
\[
S^{4n+3}/\Gamma_k \xrightarrow{i} S^{4n+7}/\Gamma_k \to S^{4n+7}/\Gamma_k / (S^{4n+7}/\Gamma_k - \tilde{N}) \\
\simeq \frac{D(\tau)}{S(\tau)} = M(\tau)
\]
is the zero section: \( S^{4n+3}/\Gamma_k \to M(\tau) \), it follows that
\[
i^* \circ (P(D)^{-1})([S^{4n+3}/\Gamma_k, i]) = e(\tau).
\]
Since \( \tau \) and \( \alpha \) are isomorphic as complex vector bundles by 1.7 the proposition is proved. \( \square \)

In Section III we shall use the following Euler classes for \( MU \) (see [6]):

\[
A_k = e(\xi_1) \in \widetilde{U}^2(B\Gamma_k), \quad B_k = e(\xi_2) \in \widetilde{U}^2(B\Gamma_k), \\
C_k = e(\xi_3) \in \widetilde{U}^2(B\Gamma_k), \quad D_k = e(\eta_1) \in \widetilde{U}^4(B\Gamma_k)
\]
where \( \xi_1, \xi_2, \xi_3, \eta_1 \) are the complex vector bundles corresponding to the irreducible unitary representations \( \xi_1: u \to 1, v \to -1, \xi_2: u \to -1, v \to 1, \xi_3: k \to -1, v \to -1 \) and \( \eta_1 \) as defined above.

In order to calculate \( U_n(B\Gamma_k) \) we first consider the case \( k = 3: \Gamma_3 = \Gamma \), the quaternion group of order 8. We recall that \( \Gamma = \{ \pm 1, \pm i, \pm j, \pm k \} \) subject to the relations \( i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j \). The irreducible unitary representations of \( \Gamma \) are \( 1: i \to 1, j \to 1, \xi_i: i \to 1, j \to -1, \xi_j: i \to -1, j \to 1, \xi_k: i \to -1, j \to -1 \) and \( \eta: i \to \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), j \to \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \). The character table of \( \Gamma \) is drawn on the next page.

The group \( \Gamma \) acts on \( S^{4n+3} \) by means of \( (n + 1)\eta \) as a group of \( U \)-diffeomorphisms; as with \( \Gamma_k \) we get a \( U \)-manifold \( S^{4n+3}/\Gamma \subset B\Gamma = \bigcup_{n \geq 0} S^{4n+3}/\Gamma \). There will be no ambiguity if we use the same notation.
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conjugacy classes

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\(\alpha\) as for \(\Gamma_k\) for the complex vector bundle \(S^{4n+3} \times \mathbb{C}^2 \to S^{4n+3}/\Gamma\). Evidently the Propositions 1.6 and 1.7 are valid if \(\Gamma_k\) is replaced by \(\Gamma\).

In Section II the following Euler class for \(MU\) will be of fundamental importance (see [6]):

\[
A = e(\xi_i) \in \tilde{U}^2(B\Gamma), \quad B = e(\xi_j) \in \tilde{U}^2(B\Gamma),
\]

\[
C = e(\xi_k) \in \tilde{U}^2(B\Gamma) \quad \text{and} \quad D = e(\eta) \in \tilde{U}^4(B\Gamma).
\]

II. Calculation of \(\tilde{U}_*(B\Gamma)\): generators, orders and relations. The reduced homology groups \(\tilde{H}_*(B\Gamma)\) are such that:

\[
\tilde{H}_{2n}(B\Gamma) = 0, \quad \tilde{H}_{4n+1}(B\Gamma) = \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad \tilde{H}_{4n+3}(B\Gamma) = \mathbb{Z}_8, \quad n \geq 0.
\]

The \(\tilde{U}_*\)-AHSS of \(B\Gamma\) collapses and we have a filtration of \(\tilde{U}_n(B\Gamma)\):

\[
J_{-1,n+1} = 0 \subset J_{0,n} \subset \cdots \subset J_{p,n-p} \subset \cdots \subset J_{n,0} = \tilde{U}_n(B\Gamma)
\]

with \(J_{p,q} = \text{Im}(\tilde{U}_{p+q}(X^p) \to \tilde{U}_{p+q}(B\Gamma))\), \(X^p\) being the \(p\)-skeleton of \(B\Gamma\). Moreover \(J_{p,q}/J_{p-1,q+1} = H_p(B\Gamma, U_q(pt))\).

**Proposition 2.1.** (a) \(\tilde{U}_{2n}(B\Gamma) = 0\), \(\tilde{U}_{2n+1}(B\Gamma) = U_{2n+1}(B\Gamma)\), \(U_{2n}(B\Gamma) = U_{2n}(pt)\).

(b) \(\text{Ord}(\tilde{U}_{4n+3}(B\Gamma)) = 2^r\), \(\text{Ord}(\tilde{U}_{4n+1}(B\Gamma)) = 2^s\),

\[
r = 3 \left( \sum_{i=0}^{n} \text{Rank} U_{4i}(pt) \right)
\]

\[
+ 2 \left( \sum_{i=0}^{n} \text{Rank} U_{4i+2}(pt) \right); \quad \text{Ord}(\tilde{U}_{4n+1}(B\Gamma)) = 2^s,
\]
\[ s = 3 \left( \sum_{i=0}^{n-1} \text{Rank } U_{4i+2}(pt) \right) + 2 \left( \sum_{i=0}^{n} \text{Rank } U_{4i}(pt) \right). \]

**Proof.** (a) From the filtration \( J_{-1,2n+1} = 0 \subset J_{0,2n} \subset \cdots \subset J_{2n,0}, \) and \( J_{p,2n-p}/J_{p-1,2n-p+1} = H_p(B\Gamma, U_{2n-p}(pt)) = 0 \) it follows that \( \tilde{U}_{2n}(B\Gamma) = 0. \) Hence \( U_{2n}(B\Gamma) = U_{2n}(pt) \) and \( \tilde{U}_{2n+1}(B\Gamma) = U_{2n+1}(B\Gamma) \) because \( U_{2n+1}(pt) = 0. \)

(b) The orders are easy consequences of:

\[
\begin{align*}
J_{4p+3,2q}/J_{4p+2,2q+1} &= H_{4p+3}(B\Gamma, U_{2q}(pt)) = \mathbb{Z}_8 \otimes U_{2q}(pt) = U_{2q}(pt)/8U_{2q}(pt), \\
J_{4p+2,2q+1}/J_{4p+1,2q+2} &= 0, \\
J_{4p+1,2q+2}/J_{4p,2q+3} &= U_{2q+2}(pt)/2U_{2q+2}(pt) \oplus U_{2q+2}(pt)/2U_{2q+2}(pt), \\
J_{4p,2q+3}/J_{4p-1,2q+4} &= 0. 
\end{align*}
\]

Let \( w_{4n+3} \in \tilde{U}_{4n+3}(B\Gamma) \) be \([S^{4n+3}/\Gamma, q], \) \( q \) being the inclusion \( S^{4n+3}/\Gamma \subset B\Gamma, \) \( u_{4n+1} = A \cap w_{4n+3} \in \tilde{U}_{4n+1}(B\Gamma), \) \( v_{4n+1} = B \cap w_{4n+3} \in \tilde{U}_{4n+1}(B\Gamma). \)

**Theorem 2.2.** The set \( \{u_{4n+1}, v_{4n+1}, w_{4n+3}\}_{n \geq 0} \) is a system of generators for the \( U_*(pt)\)-module \( \tilde{U}_*(B\Gamma). \)

**Proof.** Since the \( U_*\)-AHSS for \( B\Gamma \) collapses we can use 1.1. If \( \mu' \) denotes the edge homomorphism it is enough to prove that \( \mu'(w_{4n+3}), \) \( \{\mu'(u_{4n+1}), \mu'(v_{4n+1})\} \) are systems of generators respectively for \( \tilde{H}_{4n+3}(B\Gamma) \) and \( \tilde{H}_{4n+1}(B\Gamma). \)

(a) Consider the following commutative diagram:

\[
\begin{align*}
\tilde{U}_{4n+3}(S^{4n+3}/\Gamma) &\xrightarrow{\mu'} \tilde{U}_{4n+3}(B\Gamma) \\
\tilde{H}_{4n+3}(S^{4n+3}/\Gamma) &\xrightarrow{\mu'} \tilde{H}_{4n+3}(B\Gamma).
\end{align*}
\]

We have \( \mu'([S^{4n+3}/\Gamma, 1]) = c_1(S^{4n+3}/\Gamma), \) where \( c_1(S^{4n+3}/\Gamma) \) denotes the fundamental class of \( S^{4n+3}/\Gamma \) (for \( H \)). Since \( c_1(S^{4n+3}/\Gamma) \) is a generator of \( \tilde{H}_{4n+3}(B\Gamma) \) it follows that \( q_*c_1(S^{4n+3}/\Gamma) \) is a generator of \( \tilde{H}_{4n+3}(B\Gamma) \) because \( S^{4n+3}/\Gamma \) is the \((4n+3)\)-skeleton of \( B\Gamma. \) Now \( q_*([S^{4n+3}/\Gamma, 1]) = [S^{4n+3}/\Gamma, q] \) and then \( \mu'(S^{4n+3}/\Gamma, q) \) is a generator of \( \tilde{H}_{4n+3}(B\Gamma). \)
(b) By [6], Section II, \( \mu(A) \) and \( \mu(B) \) generate the group \( H^2(B\Gamma) \) and then if \( A_1 = q^*(A) \in U^2(S^{4n+3}/\Gamma), B_1 = q^*(B) \in U^2(S^{4n+3}/\Gamma) \), then the elements \( \mu(A_1), \mu(B_1) \) generate \( H^2(S^{4n+3}/\Gamma) \) because the following diagram commutes:

\[
\begin{array}{ccc}
U^2(B\Gamma) & \xrightarrow{q^*} & U^2(S^{4n+3}/\Gamma) \\
\mu & \downarrow & \mu \\
H^2(B\Gamma) & \xrightarrow{q^*} & H^2(S^{4n+3}/\Gamma)
\end{array}
\]

and the bottom line is an isomorphism. Consider \( t_{4n+3} = [S^{4n+3}/\Gamma, 1] \in U_{4n+3}(S^{4n+3}/\Gamma) \); then \( \mu'(t_{4n+3}) = c_1(S^{4n+3}/\Gamma) \). Since the diagram:

\[
\begin{array}{ccc}
U^2(S^{4n+3}/\Gamma) & \xrightarrow{\cap t_{4n+3}} & U_{4n+1}(S^{4n+3}/\Gamma) \\
\mu & \downarrow & \mu' \\
H^2(S^{4n+3}/\Gamma) & \xrightarrow{\cap c_1(S^{4n+3}/\Gamma)} & H_{4n+1}(S^{4n+3}/\Gamma)
\end{array}
\]

commutes by 1.5 and since the bottom line is an isomorphism it follows that \( \mu'(A_1 \cap t_{4n+3}) \) and \( \mu'(B_1 \cap t_{4n+3}) \) generate the group \( H_{4n+1}(S^{4n+3}/\Gamma) \). Now by using the commutative diagram:

\[
\begin{array}{ccc}
U_{4n+1}(S^{4n+3}/\Gamma) & \xrightarrow{q^*} & U_{4n+1}(B\Gamma) \\
\mu' & \downarrow & \mu' \\
H_{4n+1}(S^{4n+3}/\Gamma) & \xrightarrow{q^*} & H_{4n+1}(B\Gamma)
\end{array}
\]

we see that \( q_*(A_1 \cap t_{4n+3}) \) and \( q_*(B_1 \cap t_{4n+3}) \) generate the group \( H_{4n+1}(B\Gamma) \). Since \( q_*(A_1 \cap t_{4n+3}) = q_*(q^*(A) \cap t_{4n+3}) = A \cap q_*(t_{4n+3}) = A \cap w_{4n+3} \) and \( q_*(B_1 \cap t_{4n+3}) = B \cap w_{4n+3} \) the assertion (b) has been proved. \( \square \)

(1) \textbf{Relations between the generators.} We first recall the definition of the pull back transfer. Let \( M^n \) be a closed \( U \)-manifold, \( N^m \) a closed \( U \)-submanifold of \( M^n \) with \( (n - m) \) even and \( i \) the inclusion \( N^m \subset M^n \). If \( [V', f] \in U_r(M^n) \), then there is a weakly complex representative map \( g: V' \to M^n \) transversal to \( N^m \). Hence \( g^{-1}(N^m) \) is a smooth closed submanifold of \( V' \) and \( \dim g^{-1}(N^m) = r + m - n \). Since \( N^m \) is a \( U \)-submanifold of \( M^n \) the normal vector bundle \( \tau \) of \( N^m \) is in fact a complex vector bundle and by transversality we have \( T(W^{r+m-n} + g_1^*(\tau) = j^*(T(V')) \) (1) where \( W^{r+m-n} = g^{-1}(N^m), g_1 = g|g^{-1}(N^m), j: W^{r+m-n} \subset V' \) and \( T(-) \) being the tangent vector.
bundle. Since $V^r$ is a $U$-manifold the stable tangent bundle of $V^r$ has a complex structure and the above relation (1) determines a unique complex structure on the stable tangent bundle of $W^{r+m-n}$ (see [5], page 16). Then we define $i!: U_r(M^n) \to U_{r+m-n}(N^m)$ by $i!([V^r, f]) = [W^{r+m-n}, g_1]$. Moreover, the following diagram is commutative:

$$
\begin{array}{ccc}
U^k(M^n) & \xrightarrow{i^*} & U^k(N^m) \\
\downarrow \text{PD} & & \downarrow \text{PD} \\
U_{n-k}(M^n) & \xrightarrow{i_!} & U_{m-k}(N^m)
\end{array}
$$

PD being the Poincaré duality (see [2], [7]).

Now, there is a map $\Delta: \tilde{U}_*(B\Gamma) \to \tilde{U}_*(B\Gamma)$ defined by $\Delta(x) = D \cap x$, with $D = e(\eta)$, the Euler class of $\eta$. The map $\Delta$ is a homomorphism of graded $U_*(pt)$-modules of degree $-4$.

**Proposition 2.3.** We have

$$
\Delta(w_{4n+3}) = w_{4(n-1)+3}, \quad \Delta(u_{4n+1}) = u_{4(n-1)+1},
\Delta(v_{4n+1}) = v_{4(n-1)+1}, \quad n \geq 0.
$$

**Proof.** Let $p, r, s$ be respectively the inclusions $S^{4(n-1)+3}/\Gamma \subset S^{4n+3}/\Gamma$, $S^{4n+3}/\Gamma \subset S^{4n+7}/\Gamma$, $S^{4n+7}/\Gamma \subset B\Gamma$. Then

$$[S^{4n+3}/\Gamma, r] \in U_{4n+3}(S^{4n+7}/\Gamma).$$

We have the pull back transfer

$$r! : U_{4n+3}(S^{4n+7}/\Gamma) \to U_{4(n-1)+3}(S^{4n+3}/\Gamma)$$

and the commutative diagram:

$$
\begin{array}{ccc}
U^4(S^{4n+7}/\Gamma) & \xrightarrow{r^*} & U^4(S^{4n+3}/\Gamma) \\
\downarrow \text{PD} & & \downarrow \text{PD} \\
U_{4n+3}(S^{4n+7}/\Gamma) & \xrightarrow{r!} & U_{4(n-1)+3}(S^{4n+3}/\Gamma).
\end{array}
$$

The element $r!([S^{4n+3}/\Gamma, i]) = [g^{-1}(S^{4n+3}/\Gamma), g|g^{-1}(S^{4n+3}/\Gamma)]$ where $g$ is the map: $S^{4n+3}/\Gamma \to S^{4n+7}/\Gamma$ defined by $g([z_1, z_2, \ldots, z_{2n+2}]) = [z_1, z_2, \ldots, z_{2n}, 0, 0, z_{2n+2}]$ because $g$ is homotopic to $r$ and transversal to $S^{4n+3}/\Gamma$. But $g^{-1}(S^{4n+3}/\Gamma) = S^{4(n-1)+3}/\Gamma$ and $g|g^{-1}(S^{4n+3}/\Gamma) = p$. It is easily seen that

$$r!([S^{4n+3}/\Gamma, r]) = [S^{4(n-1)+3}/\Gamma, p] \in U_{4(n-1)+3}(S^{4n+3}/\Gamma),$$
the $U$-structure on $S^{4(n-1)+3}/\Gamma$ being the canonical one (this result can be found in [7], Lemma 2.5, page 145). Now by 1.8 we have $r^* \circ (PD)^{-1}([S^{4n+3}/\Gamma, r]) = e(\alpha)$, $\alpha$ being $\mathbb{C}$-vector bundle $S^{4n+3} \times_{\Gamma} \mathbb{C}^2 \to S^{4n+3}/\Gamma$, $\Gamma$ acting on $S^{4n+3}$ and $\mathbb{C}^2$ respectively by using $(n+1)\eta$ and $\eta$ (see Section I). Since $\alpha = (s \circ r)^*(\eta)(\eta: E \times_{\Gamma} \mathbb{C}^2 \to B\Gamma)$, we have $e(\alpha) = (s \circ r)^*(D)$ and then $r^* \circ (PD)^{-1}([S^{4n+3}/\Gamma, r]) = (s \circ r)^*(D)$. From the above diagram it follows that $(s \circ r)^*(D) = (PD)^{-1}([S^{4(n-1)+3}/\Gamma, p])$. The fundamental class of $S^{4n+3}/\Gamma$ for $MU$ involved in the Poincaré duality being $[S^{4n+3}/\Gamma, 1] \in U_{4n+3}(S^{4n+3}/\Gamma)$ (see 1.3) we have:

$$(s \circ r)^*(D) \cap [S^{4n+3}/\Gamma, 1] = [S^{4(n-1)+3}/\Gamma, p]$$

and consequently

$$w_{4(n-1)+3} = (s \circ r)_*([S^{4(n-1)+3}/\Gamma, p]) = (s \circ r)_*[(s \circ r)^*(D) \cap [S^{4n+3}/\Gamma, 1]]$$

$$= D \cap (s \circ r)_*([S^{4n+3}/\Gamma, 1])$$

$$= D \cap [S^{4n+3}/\Gamma, s \circ r] = D \cap w_{4n+3} = \Delta(w_{3n+3}).$$

We have

$$\Delta(u_{4n+1}) = \Delta(A \cap w_{4n+3}) = (D \cdot A) \cap (w_{4n+3})$$

$$= A \cap [D \cap w_{4n+3}] = A \cap w_{(n-1)+3} = u_{4(n-1)+1}.$$ 

Similarly $\Delta(v_{4n-1}) = v_{4(n-1)+1}$. 

\[\Box\]

**Remark.** The homomorphism $\Delta$ is sometimes called the Smith-homomorphism.

We recall from [6], Lemma 2.11 and Theorem 2.12, that if $\Lambda_*$ denotes the $U^*(pt)$-graded algebra $U^*(pt)[[X, Y, Z]]$, $\dim X = \dim Y = 2$, $\dim Z = 4$ and $\Omega_*$ the sub-$U^*(pt)$-algebra $U^*(pt)[[Z]]$ then there is $T(Z) = 8Z + 2\lambda_2 Z^2 + \sum_{i \geq 3} \lambda_i Z^i \in \Omega_4$, $\lambda_2 \notin 2U^*(pt)$, such that:

$M(D) = 0$ ($M(Z) \in \Omega_4$) iff $M(Z) \in T(Z)\Omega_*$. Moreover by [6], Lemmas 2.13, 2.15, there is

$$J(Z) = \mu_1 Z + \sum_{i \geq 2} \mu_i Z^i \in \Omega_0, \quad \mu_1 \notin 2U^*(pt),$$

such that: $E(D) + AM(D) + BN(D) = 0$ iff $M(Z), N(Z)$ belong to $(2 + J(Z))\Omega_*$ and $E(Z)$ to $T(Z)\Omega_*$. ($M(Z), N(Z), E(Z)$ are elements of $\Omega_*$). We also recall the following notation: if $M(Z) = \sum_{i \geq r} a_i Z^i \in \Omega_{2n}$ with $a_i \neq 0$ then $\nu(M) = 4r$. Let $W, V_1, V_2$ be the $U_*(pt)$-submodules of $\bar{U}_*(B\Gamma)$ generated respectively by $\{W_{4n+3}\}_{n \geq 0}$, $\{u_{4n+1}\}_{n \geq 0}$, $\{v_{4n+1}\}_{n \geq 0}$.
THEOREM 2.4. (a) \( \tilde{U}_*(B\Gamma) = W \oplus V_1 \oplus V_2 \).

(b) In \( \tilde{U}_{2p+1}(B\Gamma) \) we have \( 0 = a_0w_3 + a_1w_7 + \cdots + a_nw_{4n+3} = b_0u_1 + \cdots + b_mu_{4m+1} \) iff there are homogeneous polynomials \( M(Z), M_1(Z) \) and homogeneous formal power series \( N(Z), N_1(Z) \) of \( \Omega_* \) satisfying:

\[
b_mZ + b_{m-1}Z^2 + \cdots + b_0Z^{m+1} = M(Z)(2 + J(Z)) + N(Z), \quad a_nZ + a_{n-1}Z^2 + \cdots + a_0Z^{n+1} = M_1(Z)T(Z) + N_1(Z),
\]

\( \nu(N) > 4(m + 1) \), \( \nu(N_1) > 4(n + 1) \). Moreover, \( b_0u_1 + \cdots + b_mu_{4m+1} = 0 \) iff \( b_0v_1 + \cdots + b_mv_{4m+1} = 0 \).

Proof. (a) Suppose that \( (a_0w_3 + \cdots + a_nw_{4n+3}) = (b_0u_1 + \cdots + b_mu_{4m+1}) = 0 \). Then a proof similar to that of Lemma 2.14 of [6] shows that \( b_m = 2d_m, d_m \in U_*(pt) \). Consider \( H(Z) = b_mZ + b_{m-1}Z^2 + \cdots + b_0Z^{m+1} \); we have: \( H(Z) = b_mZ + b_{m-1}Z^2 + \cdots + b_0Z^{m+1} + F(Z), \nu(F) > 4(m + 1) \).

Then \( AH(D) = A[b_{m-1}D^2 + \cdots + b_0D^{m+1}] + AF(D) \) and by taking the cup product by \( w_{4m+7} \) we obtain \( b_0u_1 + \cdots + b_mu_{4m+1} = b_0'v_1 + \cdots + b_{m-1}'u_{4(m-1)+1} \). As seen before, we have: \( b_{m-1}' = 2d_{m-1}', \quad d_{m-1}' \in U_*(pt) \). We repeat the same process and after a finite number of operations we get \( b_mZ + b_{m-1}Z^2 + \cdots + b_0Z^{m+1} = M(D)(2 + J(D)) + N(D), M(D) \) being a homogeneous polynomial and \( N(D) \) a homogeneous formal power series such that \( \nu(N) > 4(m + 1) \). Hence \( b_0u_1 + \cdots + b_mu_{4m+1} = M(D)A(2 + J(D)) \cap w_{4m+7} = 0 \). Similarly \( c_0v_1 + \cdots + c_rv_{4r+1} = 0 \) which ends the proof of part (a).

(b) Suppose that \( a_0w_3 + \cdots + a_nw_{4n+3} = 0 \). As in Proposition 2.6 of [6] we have \( a_n = 8e_n, e_n \in U_*(pt) \). We form \( a_nZ + \cdots + a_0Z^{n+1} = e_nT(Z) = a_n'Z^2 + \cdots + a_0'Z^{n+1} + F_1(Z), \nu(F_1) > 4(n + 1) \) and by taking the cup-product by \( w_{4n+7} \) we obtain: \( a_0w_3 + \cdots + a_nw_{4n+3} = a_n'w_3 + a_{n-1}'w_{4(n-1)+3} \). As before, we have \( a_{n-1}' = 8e_{n-1}', e_{n-1}' \in U_*(pt) \).

We repeat the same process with \( a_n'Z^2 + a_{n-2}'Z^3 + \cdots + a_0'Z^{n+1} \) and after a finite number of operations we get: \( a_nZ + \cdots + a_0Z^{n+1} = M_1(Z)T(Z) + N_1(Z), \nu(N_1) > 4(n + 1), M_1(Z) \) being a homogeneous polynomial and \( N_1(Z) \) a homogeneous formal power series. The proof of part (a) shows that \( b_mZ + b_{m-1}Z^2 + \cdots + b_0Z^{m+1} = M(Z)(2 + J(Z)) + N(Z), \nu(N) > 4(m + 1) \). The remaining part of (b) is evident. \( \square \)

(2) Orders of the Generators.

LEMMA 2.5. Suppose \( t \in \tilde{U}_{2n+1}(B\Gamma) \), \( t \neq 0 \). If \( a \in U_4(pt) \) is such that \( a \notin 2U_4(pt) \) then \( a \cdot t \neq 0 \).

Proof. Since \( t \neq 0 \) there is an integer \( q \geq 0 \) such that \( t \in J_{q,2n+1-q} \) and \( t \notin J_{q-1,2n+1-(q-1)} \). We have either \( q = 4s + 3 \) or \( q = 4s + 1 \).
Suppose \( q = 4s + 3 \). We have the following commutative diagram:

\[
\begin{array}{ccc}
U_4(pt) \otimes J_{4s+3,2n+1-(4s+3)} & \xrightarrow{\times} & J_{4s+3,2(n-2s+1)} \\
1 \otimes h & \downarrow h & \\
U_4(pt) \otimes U_{2(n-2s-1)}(pt) \otimes \mathbb{Z}_8 & \xrightarrow{\times \otimes 1} & U_{2(n-2s+1)} \otimes \mathbb{Z}_8
\end{array}
\]

where \( h \) is the canonical map: \( J_* \rightarrow E_* = H_*(B\Gamma, U_*(pt) = U_*(pt))^* \rightarrow H_*(B\Gamma) \). It is enough to prove that in \( U_*(pt) = \mathbb{Z}[x_1, x_2, \ldots, x_4, \ldots] \) if \( a \in U_4(pt), a \not\in 2U_4(pt), b \in U_{2k}(pt), b \not\in 8U_{2k}(pt) \) then \( ab \not\in 8U_{2(k+2)}(pt) \); we may suppose that \( a \) and \( b \) are monomials and then the assertion is clear. The case \( q = 4s + 1 \) is similar.

**Theorem 2.6.** We have \( \text{ord } w_{4n+3} = 2^{2n+3} \).

**Proof.** (a) \( \text{ord } w_3 = 2^3 \).
We have \( 0 = T(D) = 2^3D + H(D)D^2 \) and \( 0 = T(D) \cap w_7 = 2^2w_3 + H(D) \cap (D^2 \cap w_7) = 2^3w_3 \) because \( D^2 \cap w_7 \in U_{-1}(B\Gamma) = 0 \). Then by using the edge homomorphism \( \mu': \tilde{U}_3(B\Gamma) \rightarrow \tilde{H}_3(B\Gamma) = \mathbb{Z}_8 \) we see that \( 2^2w_3 \neq 0 \). Hence \( \text{ord } w_3 = 2^3 \).

(b) Suppose \( \text{ord } w_{4i+3} = 2^{2i+3}, 0 \leq i \leq n - 1 \).
We have \( 0 = T(D) = 2^3D + 2\lambda_2D^2 + \lambda_3D^3 + \cdots + \lambda_{n+1}D^{n+1} + H(D)D^{n+2}, \lambda_2 \in U^{-4}(pt) = U_4(pt), \lambda_2 \not\in 2U_4(pt) \). Take the cup-product by \( w_{4n+7}: 2^3w_{4n+3} + 2\lambda_2w_{4(n-1)+3} + \lambda_3w_{4(n-2)+3} + \cdots + \lambda_{n+1}w_3 = 0 \) and after multiplication by \( 2^{2n-1} \) we get: \( 2^{2n+2}w_{4n+3} + \lambda_22^{2n}w_{4(n-1)+3} = 0 \); since \( \text{ord } w_{4(n-1)+3} = 2^{2n+1} \) we have \( 2^{2n}w_{4(n-1)+3} \neq 0 \) and by 2.5 \( \lambda_22^{2n}w_{4(n-1)+3} \neq 0 \) because \( \lambda_2 \not\in 2U_4(pt) \). Hence \( 2^{2n+2}w_{4n+3} \neq 0 \). Now we have: \( 2^{2n+3}w_{4n+3} = -\lambda_22^{2n+1}w_{4(n-1)+3} = 0 \). It follows that \( \text{ord } w_{4n+3} = 2^{2n+3} \) which ends the proof of 2.6.

**Lemma 2.7.** If \( G_n = U_{4n-2}(pt)w_3 + U_{4n}(pt)u_1 + U_{4n}(pt)v_1, \quad G'_n = U_{4n}(pt)w_3 + U_{4n+2}(pt)u_1 + U_{4n+2}(pt)v_1 \) then we have the exact sequences:

\[
0 \rightarrow G_n \rightarrow \tilde{U}_{4n+1}(B\Gamma) \xrightarrow{\Delta} \tilde{U}_{4(n-1)+1}(B\Gamma) \rightarrow 0
\]

\[
0 \rightarrow G'_n \rightarrow \tilde{U}_{4n+3}(B\Gamma) \xrightarrow{\Delta} \tilde{U}_{4(n-1)+3}(B\Gamma) \rightarrow 0
\]

**Proof.** We wish to show that the sequence:

\[
0 \rightarrow G_n \rightarrow \tilde{U}_{4n+1}(B\Gamma) \xrightarrow{\Delta} \tilde{U}_{4(n-1)+1}(B\Gamma) \rightarrow 0
\]
is exact. It follows by 2.3 that Δ is surjective and \( G_n \subset \ker \Delta \). Suppose
\( 0 = aw_3 + bu_1 + cv_1, a \in U_{4n-2}(pt), b \in U_{4n}(pt), c \in U_{4n}(pt) \). Then
\( a \cdot w_3 \in J_{3,4n-2} \) and since \( bu_1 + cv_1 \in J_{1,4n} \) we have \( w_3 \in J_{2,4n-1} \supset J_{1,4n} \). If \( h \) denotes the quotient map:
\( J_{3,4n-2} \to J_{3,4n-2}/J_{2,4n-1} = H^3(B\Gamma, U_{4n-2}(pt)) = U_{4n-2}(pt)/8U_{4n-2}(pt) \), it follows that \( h(aw_3) = 0 \) and consequently \( a = 2^3 a' \). Hence \( aw_3 = a'2^3 w_3 = 0 \) and then
\( bu_1 + cv_1 = 0 \). Similarly we have \( b = 2b' \), \( c = 2c' \) which means that
\( 0 = aw_3 + bu_1 + cv_1 \) if and only if \( a = 2^3 a', b = 2b', c = 2c' \). Hence \( \text{ord } G_n = 2^k \),
\( k = 3 \text{ Rank } U_{4n-2}(pt) + 2 \text{ Rank } U_{4n}(pt) \). Now, we have \( \text{ord } \ker \Delta = \text{ord } \tilde{U}_{4n+1}(B\Gamma)/\text{ord } \tilde{U}_{4(n-1)+1}(B\Gamma) = 2^k \) by 2.1. From \( G_n \subset \ker \Delta \) and
\( \text{ord } G_n = \text{ord } \ker \Delta \) we see that the sequence \( 0 \to G_n \to \tilde{U}_{4n+1}(B\Gamma) \to \tilde{U}_{4(n-1)+1}(B\Gamma) \to 0 \) is exact. A similar proof shows that the sequence
\( 0 \to G'_n \to \tilde{U}_{4n+3}(B\Gamma) \to \tilde{U}_{4(n-1)+3}(B\Gamma) \to 0 \) is exact.

\[ \text{THEOREM 2.8.} \text{ We have } \text{ord } u_{4n+1} = \text{ord } v_{4n+1} = 2^{n+1}. \]

\[ \text{Proof.} \text{ If } n = 0 \text{ the assertion is clear. Suppose } \text{ord } u_{4i+1} = 2^{i+1}, \]
\( 0 \leq i \leq n - 1 \). Then \( \Delta(2^n u_{4n+1}) = 2^n u_{4(n-1)+1} = 0 \) and since the
sequence \( 0 \to G_n \to U_{4n+1}(B\Gamma) \to U_{4(n-1)+1}(B\Gamma) \to 0 \) is exact, (see
2.7), there are \( a \in U_{4n-2}(pt), b \in U_{4n}(pt), c \in U_{4n}(pt) \) such that
\( 2^n u_{4n+1} = aw_3 + bu_1 + cv_1 \). It follows that \( -bu_1 + 2^n \cdot u_{4n+1} = 0 \) and
\( 2^n \cdot u_{4n+1} = 0 \); hence \( \text{ord } u_{4n+1} \leq 2^{n+1} \). By Theorem 2.4 there are
\( M(Z), N(Z) \) in \( \Omega_* \) such that: \( 2^n Z - bZ^{n+1} = M(Z)(2+J(Z))+N(Z), \)
\( v(N) > 4(n + 1) \). If \( M(Z) = h_1 Z + h_2 Z^2 + \cdots \), then we have:

\[ 2^n Z - bZ^{n+1} = (2 + \mu_1 Z + \mu_2 Z^2 + \cdots)(h_1 Z + h_2 Z^2 + \cdots) \]
\[ + e_{n+2} Z^{n+2} + e_{n+3} Z^{n+3} + \cdots, \quad \mu_1 \notin 2U_*(pt). \]

A straightforward calculation shows that \( 2^{n-j} | h_j \) and \( 2^{n-j+1} \nmid h_j, 1 \leq j \leq n \). We have:
\(-b = 2h_{n+1} + \mu_1 h_n + \mu_2 h_{n-1} + \cdots + \mu_n h_1; \) as \( 2|h_j, 1 \leq j \leq n - 1, 2 \nmid h_n, 2 \nmid \mu_1 \) we have \( 2 \nmid b \). As a consequence we get
\( 2^n u_{4n+1} \neq 0 \) and \( \text{ord } u_{4n+1} = 2^{n+1} \). Similarly \( \text{ord } v_{4n+1} = 2^{n+1}. \)

\[ \text{III. } \tilde{U}^*(B\Gamma_k), k \geq 4: \text{ generators, orders and relations.} \text{ We have seen in \cite{6}, Section III, that there are elements } D_k \in \tilde{U}^4(B\Gamma_k), B_k \in \tilde{U}^2(B\Gamma_k), C_k \in \tilde{U}^2(B\Gamma_k) \text{ defined as Euler classes of irreducible unitary representations } \eta_1, \xi_2, \xi_3 \text{ of } \Gamma_k. \text{ Moreover in the same article} \]
(Sec. III) we have determined three homogeneous formal power series $T_k(Z) \in \Omega_4$, $J_k(Z) \in \Omega_0$, $G_k(Z) \in \Omega_2$ such that $B_k(2+J(D_k)) + G_k(D_k) = C_k(2+J(D_k)) + G_k(D_k) = 0$ and there is $G_k'(Z) \in \Omega_2$ satisfying $G_k(Z) = (2+J(Z))G_k'(Z)$. Then with $B_k' = B_k + G_k'(D_k)$, $C_k' = C_k + G_k'(D_k)$ and $\mu$ being the edge homomorphism: $U^2(B\Gamma_k) \to H^2(B\Gamma_k)$ we see that $\mu(B_k') = \mu(B_k)$ and $\mu(C_k') = \mu(C_k)$ are generators of the group $H^2(B\Gamma_k)$; $\mu(D_k)$ is obviously a generator of $H^4(B\Gamma_k)$.

Moreover $B_k'(2+J(D_k)) = C_k'(2+J(D_k)) = 0$.

Now let $w_{4n+3}' \in \tilde{U}_{4n+3}(B\Gamma_k)$ be $[S^{4n+3}/\Gamma_k, q']$, $q'$ being the inclusion $S^{4n+3}/\Gamma_k \subset B\Gamma_k$, $u_{4n+1}' = B_k' \cap w_{4n+3}' \in \tilde{U}_{4n+1}$, $v_{4n+1}' = C_k' \cap w_{4n+1}' \in \tilde{U}_{4n+1}(B\Gamma_k)$. Then we have the following theorems whose proofs are identical respectively to Theorem 2.2 and Theorem 2.4 and therefore will be omitted.

**Theorem 3.1.** The set $\{u_{4n+1}', v_{4n+1}', w_{4n+3}'\}_{n \geq 0}$ is a system of generators for the $U(pt)$-module $\tilde{U}_*(B\Gamma_k)$.

Now let $W', V_1', V_2'$ be the $U_*(pt)$-submodules of $\tilde{U}_*(B\Gamma_k)$ generated respectively by $\{w_{4n+3}'\}_{n \geq 0}$, $\{u_{4n+1}'\}_{n \geq 0}$, $\{v_{4n+1}'\}_{n \geq 0}$.

**Theorem 3.2.** (a) $\tilde{U}_*(B\Gamma_k) = W' \oplus V_1' \oplus V_2'$.

(b) In $\tilde{U}_{2p+1}(B\Gamma_k)$ we have $0 = a_0w_3' + a_1w_7' + \cdots + a_nw_{4n+3}' = b_0u_1' + \cdots + b_mu_{4m+1}'$ iff there are homogeneous polynomials $M(Z), M_1(Z)$ and homogeneous formal power series $N(Z), N_1(Z)$ of $\Omega_*$ satisfying:

\[
b_mZ + b_{m-1}Z^2 + \cdots + b_0Z^{m+1} = M(Z)(2+J(Z)) + N(Z), a_nZ + a_{n-1}Z^2 + \cdots + a_0Z^{n+1} = M_1(Z)T_k(Z) + N_1(Z), \nu(N) > 4(m+1), \nu(N_1) > 4(n+1).
\]

Moreover $b_0u_1' + \cdots + b_mu_{4m+1}' = 0$ iff $b_0v_1' + \cdots + b_mv_{4m+1}' = 0$.

There is a Smith homomorphism $\Delta: \tilde{U}_*(B\Gamma_k) \to \tilde{U}_*(B\Gamma_k)$ of degree $-4$ such that

\[
\Delta(w_{4n+3}') = D_k \cap w_{4n+3}' = w_{4(n-1)+3}',
\]

\[
\Delta(u_{4n+1}') = D_k \cap u_{4n+1}' = D_k \cap (B_k' \cap w_{4n+3}') = B_k' \cap (D_k \cap w_{4n+3}') = B_k' \cap w_{4(n-1)+3}' = u_{4(n-1)+1}', \Delta(v_{4n+1}') = v_{4(n-1)+1}'.
\]

If

\[
F_n = U_{4n}(pt)w_3' + U_{4n+2}(pt)u_1' + U_{4n+2}(pt)v_1',
\]

\[
F_n' = U_{4n-2}(pt)w_3' + U_{4n}(pt)u_1' + U_{4n}(pt)v_1'.
\]
then we have:

**Lemma 3.3.** The following sequences are exact:

\[
0 \rightarrow F_n \rightarrow U_{4n+3}(B\Gamma_k) \xrightarrow{\Delta} U_{4(n-1)+3}(B\Gamma_k) \rightarrow 0,
\]

\[
0 \rightarrow F'_n \rightarrow U_{4n+1}(B\Gamma_k) \xrightarrow{\Delta} U_{4(n-1)+1}(B\Gamma_k) \rightarrow 0.
\]

**Proof.** The proof is similar to that of Lemma 2.7. \(\Box\)

It remains to calculate the orders of the generators.

**Theorem 3.4.** We have: \(\text{ord } w'_{4n+3} = 2^{2n+k}, \; n \geq 0.\)

**Proof.** We have \(0 = T_k(D_k) = 2^k D_k + H(D_k)D_k^2\) and then \(0 = (2^k D_k + H(D_k)D_k^2) \cap w_7 = 2^k w_3\) because: \(D_k^2 \cap w_7 \in \tilde{U}_2(B\Gamma_k) = 0.\) Now if \(\mu\) is the edge homomorphism: \(U_3(B\Gamma_k) \rightarrow H_3(B\Gamma_k) = \mathbb{Z}2k\) then we have \(\mu'(w'_{3}) = 1 \in \mathbb{Z}2k\) and consequently \(2^{k-1} w_3 \neq 0.\) Then \(\text{ord } w_3 = 2^k.\)

Suppose that \(\text{ord } w'_{4i+3} = 2^{2i+k}, \; 0 \leq i \leq n - 1.\) Then

\[
0 = T_k(D_k) \cap w'_{4n+7} = 2^k w'_{4n+3} + 2^{k-2} \lambda_2 w'_{4(n-1)+3} + \cdots + 2^{k-i} \lambda_i w'_{4(n-i)+3} + \cdots + 2^{k-k-1} w'_{4(n-k+2)+3} + \cdots + \lambda_k w'_{4(n-k+1)+3} + \cdots + \lambda_m w'_{4(n-m+1)+3} + \cdots
\]

the number of non-zero elements in this sum being finite. If \(3 \leq i \leq k - 1\) we have \(2^{2n-1+k-1} w'_{4(n-i+1)+3} = 0\) because \(\text{ord } w'_{4(n-i+1)+3} = 2^{2(n-i+1)+k}\) and \(2(n-i+1)+k \leq 2n-1+k-i\) since \(i \geq 3.\) If \(m \geq k (\geq 3)\) we have \(2^{2n-1} w'_{4(n-m+1)+3} = 0\) because \(\text{ord } w'_{4(n-m+1)+3} = 2^{2(n-m+1)+k}\) and \(2(n-m+1) + 1 \leq 2n-1\) since \(k \leq m \leq 2m - 3.\) It follows that \(2^{2n-1+k} w'_{4n+3} + 2^{2n-3+k} \lambda_2 w'_{4n-1} = 0.\) Now \(2^{2n-3+k} w'_{4n-1} \neq 0\) because \(\text{ord } w'_{4n-1} = 2^{2n-k};\) since \(\lambda_2 \notin 2U^{-4}(pt)\) we have \(2^{2n-k} \lambda_2 w'_{4n-1} \neq 0\) (see 2.5). Hence

\[
2^{2n-1+k} w'_{4n+3} \neq 0 \quad \text{and} \quad 2^{2n+k} w'_{4n+3} = -2^{2(n-1)+k} \lambda_2 w'_{4n-1} = 0.
\]

We have proved that \(\text{ord } w_{4n+3} = 2^{2n+k}.\) \(\Box\)

**Theorem 3.5.** We have: \(\text{ord } w'_{4n+1} = \text{ord } w'_{4n+1} = 2^{n+1}, \; n \geq 0,\) which are therefore independent of \(k.\)

**Proof.** The proof of 3.5 is based on Theorem 3.2 and Lemma 3.3 and is exactly the same as the one of Theorem 2.8. \(\Box\)
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