OUTER CONJUGACY OF SHIFTS ON THE HYPERFINITE $\text{II}_1$-FACTOR

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For a shift $\sigma$ on the hyperfinite II$_1$ factor $R$, we define the derived shift $\sigma_\infty$ to be the restriction of $\sigma$ to the von Neumann algebra generated by the $(\sigma^k(R))' \cap R$. Outer conjugacy of shifts implies conjugacy of derived shifts. In the case of $n$-shifts with $n$ prime, we calculate $\sigma_\infty$ explicitly. Combining this with the known classification of $n$-shifts up to conjugacy, we obtain useful outer-conjugacy invariants for $n$-shifts.

Following Powers [5], we define a shift $\sigma$ on a von Neumann algebra $M$ to be a unit-preserving *-endomorphism of $M$ such that $\bigcap_{k=1}^\infty \sigma^k(M) = \mathbb{C}$, the complex numbers. We define the derived shift $\sigma_\infty$ to be the restriction of $\sigma$ to the von Neumann algebra $M_\infty$ generated by all the $(\sigma^k(M))' \cap M$. When two shifts on a factor of type II$_1$ are outer conjugate, their derived shifts are conjugate (Theorem 1.2, below). This gives us a useful outer-conjugacy invariant. In particular, for shifts $\sigma$ such that $\sigma_\infty = \sigma$, this shows that outer-conjugacy implies conjugacy (when specialized to binary shifts, this is the affirmative answer to a conjecture of Enomoto and Watatani [3]).

In §2, we compute $\sigma_\infty$ explicitly when $\sigma$ is an $n$-shift on the hyperfinite II$_1$ factor $R$ and $n$ is prime. 2-shifts, called binary shifts in [5], were introduced by R. Powers in [5]. $n$-shifts have been studied in [1], [2] and [7]. In the notation of [1], every $n$-shift can be associated with a doubly-infinite sequence $(a(k))_{k \in \mathbb{Z}}$ in $\mathbb{Z}$ which is odd and fails to be periodic mod $p$ for all primes $p$ dividing $n$. Furthermore, every such sequence occurs. In case $n$ is square-free, two shifts with sequences $(a_1(k))$ and $(a_2(k))$ are conjugate if and only if there exists an $m$ in $\mathbb{Z}$ such that $a_2(k) = m^2(a_1(k))$ for all $k$. Thus, up to multiplication by a square, the sequence associated with $\sigma_\infty$ is an outer conjugacy invariant for $\sigma$.

The computation of $\sigma_\infty$ breaks down into three cases. First, if $(a(k))$ fails to be ultimately periodic then $R_\infty = \mathbb{C}$; in this case $\sigma_\infty$ is trivial and contains no information. Secondly, at the opposite extreme, if $a(k) = 0$ for all but finitely many $k$ then $R_\infty = R$ and $\sigma_\infty = \sigma$; in
this case outer conjugacy is equivalent to conjugacy. Finally, the most interesting case occurs when \((a(k))\) is ultimately periodic but doesn’t end in 0’s: here \(R_\infty\) is a factor not equal to \(C\) or \(R\) and \(\sigma_\infty\) is an \(n\)-shift; we are able (Theorem 2.1) to calculate explicitly the sequence associated with \(\sigma_\infty\) from \((a(k))\).

**Problem.** If \(\sigma_1\) and \(\sigma_2\) are \(n\)-shifts with \(R_\infty \neq C\), does conjugacy of the derived shifts \((\sigma_1)_\infty\) and \((\sigma_2)_\infty\) imply outer conjugacy of \(\sigma_1\) and \(\sigma_2\)? Equivalently, if \(\sigma\) is an \(n\)-shift with \(R_\infty \neq C\), are \(\sigma\) and \(\sigma_\infty\) outer conjugate?

In attempting to answer this problem, we present in §3 a method for producing many shifts outer conjugate to a given shift. This yields many interesting examples. But even in simple specific cases, given that \((\sigma_1)_\infty = (\sigma_2)_\infty\) it is still not clear whether \(\sigma_1\) and \(\sigma_2\) are outer conjugate.

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1. **Definition and properties of \(\sigma_\infty\).** As in [5], a shift \(\sigma\) on a von Neumann algebra \(M\) is defined to be a unital \(*\)-endomorphism of \(M\) such that \(\bigcap_{k=1}^\infty \sigma^k(M) = C\). Two shifts \(\sigma_1\) and \(\sigma_2\), on \(M_1\) and \(M_2\) respectively, are said to be conjugate when there exists a \(*\)-isomorphism \(\phi\) of \(M_2\) onto \(M_1\) such that \(\sigma_1 \circ \phi = \phi \circ \sigma_2\), and outer conjugate when there exists a unitary \(u\) in \(A\) such that \((adu) \circ \sigma_1\) and \(\sigma_2\) are conjugate.

Let \(\sigma\) be a shift on \(M\). Define

\[ M_k = (\sigma^k(M))' \cap M \quad \text{for } k = 0, 1, 2, \ldots. \]

Evidently \(M_0\) is the center of \(M\) and \(M_0 \subset M_1 \subset M_2 \subset \cdots\). Let \(M_\infty\) be the von Neumann subalgebra of \(M\) generated by the \(M_k\) and let \(\sigma_\infty\) be the restriction of \(\sigma\) to \(M_\infty\). We call \(\sigma_\infty\) the derived shift of \(\sigma\).

**Lemma 1.1.** \(\sigma_\infty\) is a shift on \(M_\infty\).

**Proof.** First note that \(\sigma_\infty(M_\infty) \subset M_\infty\), since \(x \in M_k\) implies that for all \(y \in M\),

\[ \sigma(x)\sigma^{k+1}(y) = \sigma(x\sigma^k(y)) = \sigma(\sigma^k(y)x) = \sigma^{k+1}(y)\sigma(x), \]

which shows that \(\sigma(x) \in M_{k+1} \subset M_\infty\).

Then \(\sigma_\infty\) is a shift because \(\bigcap_{k=1}^\infty \sigma^k(M_\infty) \subset \bigcap_{k=1}^\infty \sigma^k(M) = C\).

**Theorem 1.2.** Let \(\sigma_1\) and \(\sigma_2\) be shifts on the type II\(_1\)-factors \(M_1\) and \(M_2\) respectively. If \(\sigma_1\) and \(\sigma_2\) are outer conjugate then their derived shifts \((\sigma_1)_\infty\) and \((\sigma_2)_\infty\) are conjugate.
Proof. Evidently if \( \sigma_1 \) and \( \sigma_2 \) are conjugate then so are \((\sigma_1)_\infty\) and \((\sigma_2)_\infty\). Hence given that \( \sigma_1 \) and \( \sigma_2 \) are outer conjugate we may assume without loss of generality that \( M_1 = M_2 = M \) and that \( \sigma_2 = (\text{Ad } w)\sigma_1 \) for some unitary \( w \) in \( M \). Set \( w_1 = w \) and for \( k = 2, 3, \ldots \) set \( w_k = w\sigma_1(w)\sigma_2^k(w) \cdots \sigma_1^{k-1}(w) \). Then we can see that:

\[
(1.1) \quad (\text{Ad } w_k) \circ \sigma_1^k = \sigma_2^k \quad \text{for } k = 1, 2, \ldots .
\]

For \( (1.1) \) holds for \( k = 1 \), and, for all \( y \in M \),

\[
[(\text{Ad } w_k) \circ \sigma_1^k]y = (\text{Ad } w_{k-1}) \circ \sigma_1^{k-1}(w)\sigma_1^k(y)(\sigma_1^{k-1}(w))^* = (\text{Ad } w_{k-1}) \circ \sigma_1^{k-1}(w_1(y)w^*) = [(\text{Ad } w_{k-1}) \circ \sigma_1^{k-1}]\sigma_2(y).
\]

Thus \( (1.1) \) follows by induction.

From \( (1.1) \), \( \text{Ad } w_k \) maps \( \sigma_1^k(M) \) isomorphically onto \( \sigma_2^k(M) \); therefore \( \text{Ad } w_k \) maps \( M_1^{(1)} = (\sigma_1^k(M))^\prime \cap M \) isomorphically onto \( M_2^{(2)} = (\sigma_2^k(M))^\prime \cap M \). For all \( x \in M_1^{(1)} \),

\[
(\text{Ad } w_{k+1})(x) = (\text{Ad } w_k)(\sigma_1^k(w)x(\sigma_1^k(w)^*)) = (\text{Ad } w_k)(x).
\]

Hence the isomorphisms \( \text{Ad } w_k \) are compatible with the inclusions \( M_1^{(1)} \subset M_2^{(1)} \) and \( M_2^{(2)} \subset M_1^{(2)} \), the following diagram is commutative:

\[
\begin{array}{ccc}
\cdots & \rightarrow & M_1^{(1)} \rightarrow M_2^{(1)} \rightarrow \cdots \\
\text{Ad } w_k & \downarrow & \text{Ad } w_{k+1} \\
\cdots & \rightarrow & M_2^{(2)} \rightarrow M_1^{(2)} \rightarrow \cdots
\end{array}
\]

Thus there exists a unique *-isomorphism \( \phi \) from the \( C^* \)-algebra generated by the \( M_1^{(1)} \) onto the \( C^* \)-algebra generated by the \( M_2^{(2)} \) such that

\[
\phi(x) = (\text{Ad } w_k)(x) \quad \text{for all } x \in M_1^{(1)}.
\]

Because \( \text{Ad } w_k \) preserves the trace \( \tau \) on \( M \), so does \( \phi \). Hence \( \phi \) extends to an isomorphism \( \overline{\phi} \) of von Neumann algebras from \((M_1)_\infty\) onto \((M_2)_\infty\).

Finally we check that \( \overline{\phi} \circ (\sigma_1)_\infty = (\sigma_2)_\infty \circ \overline{\phi} \). For \( x \in M_1^{(1)} \):

\[
\overline{\phi} \circ (\sigma_1)_\infty = \phi(\sigma_1(x)) = (\text{Ad } w_{k+1})(\sigma_1(x)) = (\text{Ad } w)(\sigma_1(w_k x w_k^*)) = \sigma_2(w_k x w_k^*) = ((\sigma_2)_\infty \circ \phi)(x).
\]

**Corollary 1.3.** Suppose that \( \sigma_1 \) and \( \sigma_2 \) are shifts on the type II\(_1\) factors \( M_1 \) and \( M_2 \) respectively. Suppose that \( (M_1)_\infty = M_1 \) and...
Then \( \sigma_1 \) and \( \sigma_2 \) are outer conjugate if and only if they are conjugate.

The following are examples of shifts \( \sigma \) such that \( M_\infty = M \) so that \( \sigma_\infty = \sigma \) and Corollary 1.3 applies.

**Example 1.** Let \( \sigma \) be an \( n \)-shift with determining sequence \( (a(k))_{k \in \mathbb{Z}} \) such that \( a(k) = 0 \) for all but finitely many \( k \) (see §2 for details). Corollary 1.3 applied in this case demonstrates a conjecture of [3].

**Example 2.** Let \( \sigma \) be the canonical shift of the hyperfinite II\(_1\) -factor \( R \) realized as the von Neumann algebra of the GNS-representation associated with the unique tracial state on a UHF-algebra of type \( n^\infty \).

**Example 3.** Let \( R \) be realized as the von Neumann algebra generated by a sequence of projections \( p_1, p_2, \ldots \) satisfying the Jones relations

\[
\begin{align*}
(i) & \quad p_ip_jp_i = \tau p_i \text{ for } |i - j| = 1. \\
(ii) & \quad p_ip_j = p_jp_i \text{ for } |i - j| \geq 2. \\
(iii) & \quad \text{There is a trace on } R \text{ for which the conditional expectation } E_n \text{ onto the } *\text{-algebra generated by } p_1, \ldots, p_n \text{ and } 1 \text{ satisfies: } E_n(p_{n+1}) = \tau.
\end{align*}
\]

Let \( \sigma \) be the shift \( \sigma(p_i) = p_i+1 \) (see [4] and [1, §5]).

The common feature of these examples is the existence of \( a \in R \) such that the \( a^k = \sigma^k(a) \) generate \( R \) and that each \( a_j \) commutes with all \( a_k \) for all \( k \geq k_0(j) \). Then \( a_j \in R_{k_0(j)} \subset R_\infty \), so \( R_\infty = R \) and \( \sigma_\infty = \sigma \). We have shown:

**Lemma 1.4.** Suppose that \( \sigma \) is a shift on \( M \) and that there exists an \( a \) in \( M \) such that:

(i) \( a, \sigma(a), \sigma^2(a), \ldots \) generate \( M \), and 

(ii) there is a \( k_0 \) such that \( a \) commutes with \( \sigma^k(a) \) for all \( k \geq k_0 \).

Then \( M_\infty = M \) and \( \sigma_\infty = \sigma \).

**Lemma 1.5.** \( (M_\infty)_\infty = M_\infty \), \( (\sigma_\infty)_\infty = \sigma_\infty \).

**Proof.** Let \( S_k = (\sigma^k(R_\infty))' \cap R_\infty \). Then

\[
S_k \supset (\sigma^k(R))' \cap R_\infty = ((\sigma^k(R))' \cap R) \cap R_\infty = R_k \cap R_\infty = R_k.
\]

Thus \( (R_\infty)_\infty \), the \( W^* \)-algebra generated by the \( S_k \), contains \( R_\infty \). Since the opposite inclusion is evident, \( (R_\infty)_\infty = R_\infty \).

**Lemma 1.6.** Suppose that \( \sigma \) is a group shift, \( \sigma = \sigma(G, s, \omega) \) in the notation of [1], where \( s \) is a shift on the abelian group \( G \), and \( \omega \) is
an s-invariant cocycle on G. Define \( \rho(g \wedge h) = \omega(g, h) \overline{\omega(h, g)} \) for all \( h, g \in G \). Let, for \( k = 0, 1, 2, \ldots, \)
\[
D_k = \{ g \in G | \rho(g \wedge s^k(G)) = 1 \}
\]
and let \( D_\infty = \bigcup_{k=0}^{\infty} D_k \). Let \( \bar{s} \) and \( \bar{\omega} \) be the restrictions of \( s \) and \( \omega \) to \( D_\infty \). Then \( \sigma_\infty \) is the group shift \( \sigma(D_\infty, \bar{s}, \bar{\omega}) \).

**Proof.** Use Proposition 1.2 of [1].

**Corollary 1.7.** There exist shifts on the hyperfinite II_1-factor \( R \) which fail to be outer conjugate to any group shift.

**Proof.** By Lemma 1.6 and Theorem 1.2, it suffices to display a shift \( \sigma \) on \( R \) which is not a group shift and for which \( \sigma_\infty = \sigma \). In Example 3 above, take \( \tau = 1/p \) where \( p \) is a prime \( > 4 \). Then \( \sigma_\infty = \sigma \) and \( \sigma \) is not conjugate to a group shift by Proposition 5.4 of [1].

2. \( n \)-shifts on the hyperfinite factor: calculation of \( \sigma_\infty \). Fix an integer \( n \geq 2 \). For the main results of this section \( n \) will be assumed prime. Fix \( \gamma = \exp(2\pi i/n) \).

An \( n \)-shift \( \sigma \) on the hyperfinite factor \( R \) may be characterized (see [1], [7], [2]) by the existence of a unitary \( u \) in \( R \) such that:

(i) \( u^n = 1, u^m \notin \mathbb{C} \) for \( m = 1, 2, \ldots, n-1 \),
(ii) \( R \) is generated by the \( \sigma^k(u) \) for \( k = 0, 1, 2, \ldots, \) and
(iii) \( u \) and \( \sigma^k(u) \) commute up to scalars:
\[
u(\sigma^k(u)) u^* (\sigma^k(u))^* \in \mathbb{C} \quad \text{for} \quad k = 1, 2, \ldots.
\]
We write:
\[
u_k = \sigma^k(u), \quad u_j u_k u_j^* u_k^* = \gamma^{a(k-j)} \quad \text{for all} \quad j, k = 0, 1, \ldots
\]
where \( a(k) \in \mathbb{Z}_n \). Then we call \((a(k))_{k \in \mathbb{Z}}\) a determining sequence for \( \sigma \). The sequence \((a(k))\) is odd and fails to be periodic mod \( p \) for every prime \( p \) dividing \( n \); furthermore all such sequences occur as the determining sequence of an \( n \)-shift \( \sigma \) on \( R \) (see [1]). When \( n \) is square-free, two sequences \((a_1(k))\) and \((a_2(k))\) determine conjugate shifts if and only if there is an \( m \in \mathbb{Z}_n \) such that \( a_2(k) = m^2(a_1(k)) \) for all \( k \) (see [1]).

Here we are concerned with the calculation of \( \sigma_\infty \) and \( R_\infty \). \( \sigma \) is a group shift \( \sigma(G, s, \rho) \) with \( G = \bigoplus_{k=0}^{\infty} (\mathbb{Z}_n)^{(k)} \), \( s \) the canonical shift \( s : e_k \rightarrow e_{k+1} \) on \( G \), and \( \rho(e_j \wedge e_k) = \gamma^{a(k-j)} \) for \( j, k = 0, 1, 2, \ldots \). From Lemma 1.6 we know that \( \sigma_\infty \) is a group shift, namely \( \sigma(D_\infty, \bar{s}, \bar{\rho}) \) where
\( \hat{s} \) and \( \hat{p} \) are the restrictions of \( s \) and \( p \) to \( D_\infty \) and \( D_\infty = \bigcup_{k=0}^\infty D_k \). As in Lemma 1.6,

\[
D_k = \{ g \in G | p(g \wedge s^k(G)) = 1 \}.
\]

\( \sigma_\infty \) is not always an \( m \)-shift (see Example 7 at the end of \( \S 2 \)). If, however, \( n \) is a prime, then \( \sigma_\infty \) is an \( n \)-shift. Theorem 2.1 summarizes the calculation of \( \sigma_\infty \) in this case.

**Theorem 2.1.** Let \( n \) be a prime and let \( \sigma \) be an \( n \)-shift on the hyperfinite II\(_1\)-factor \( R \) with determining sequence \( (a(k)) \). Let \( \sigma_\infty \) on \( R_\infty \) be the derived shift of \( \sigma \).

**Part A.** (i) \( R_\infty = R \) if and only if \( a(k) = 0 \) for all but finitely many \( k \).

(ii) \( R_\infty \neq \mathbb{C} \) if and only if \( (a(k)) \) is ultimately periodic; i.e. there exist \( T > 0 \) and \( K \) such that \( a(k + T) = a(k) \) for all \( k \geq K \).

(iii) In all cases \( R_\infty \) is a factor. If \( R_\infty \neq \mathbb{C} \) then \( \sigma_\infty \) is an \( n \)-shift and \( R_\infty \) is isomorphic to \( R \).

**Part B.** Suppose now that \( (a(k)) \) is ultimately periodic so that \( R_\infty \neq \mathbb{C} \). Let \( q_0 \) be the smallest integer such that \( R_{q_0} \neq \mathbb{C} \). Define the length of a nonzero \( v \) in \( G \) to be \( L \) when \( v = \sum_{j=0}^L v_je_j \) with \( v_L \neq 0 \). Then we have:

(iv) Let \( v \neq 0 \) be in \( D_{q_0} \). Then \( v \) spans \( D_{q_0} \) and \( v, s(v), s^2(v), \ldots, s^k(v) \) is a basis for \( D_{q_0+k} \). Hence \( D_\infty \) is isomorphic to \( G = \bigoplus_{k=0}^\infty (\mathbb{Z}_n)^{(k)} \) by the mapping \( s^k(v) \to e_k \).

(v) \( g \) has minimal length in \( D_\infty - \{0\} \) if and only if \( g \) spans \( D_{q_0} \).

**Part C.** Let \( v \) be a vector of minimal length \( L \) in \( D_\infty - \{0\} \). Suppose that \( a(k) \) commences its ultimate periodicity at \( k_0 \) so that \( a(k + T) = a(k) \) for all \( k \geq k_0 \) and \( a(k_0 - 1 + T) \neq a(k_0 - 1) \).

Then

(vi) \( q_0 = k_0 + L \).

(vii) \( k_0 \) is the smallest integer such that \( \hat{v} \perp A^k \) for all \( k \geq k_0 \), where \( \hat{v} = [v_L, v_{L-1}, \ldots, v_0] \) and \( A^k = [a(k), a(k + 1), \ldots, a(k + L)] \) are in \( (\mathbb{Z}_n)^{L+1} \) with the usual inner product.

(viii) \( L \) is the rank of the \( T \times T \) matrix \( A \) with \( j \)-th row \( A_j = [a(k_0 + j - 1), a(k_0 + j), \ldots, a(k_0 + j + T - 2)] \).

(ix) \( \sigma_\infty \) has determining sequence \( (b(k)) \) given by \( \gamma^{b(k)} = p(v \wedge s^k(v)) \). Then \( b(q_0 - 1) \neq 0 \) and \( b(k) = 0 \) for all \( k \geq q_0 \).

(x) The Jones index \([R:R_\infty]\) is \( n^L \).
Proof. (i) $R_\infty = R$ if and only if $D_\infty = G$ if and only if $e_0 \in D_\infty$.
That happens if and only if, for some $m$, $\rho(e_0 \land e_k) = 1$ for all $k \geq m$, i.e. $a(k) = 0$ for $k \geq m$.
(ii) Suppose that $a(k + T) = a(k)$ for all $k \geq k_0$. Then $g = e_0 - e_T$ is in $D_{k_0} \subset D_\infty$ and $R_\infty \neq \mathbb{C}$.
Conversely, suppose that $R_\infty \neq \mathbb{C}$. Then $D_{k_0} \neq 0$ for some $k_0$.
Taking $g = \sum g_j e_j \neq 0$ in $D_{k_0}$, we get (Lemma 3.2 of [1])
$$
\sum_{j=0}^{\infty} g_j a(k - j) = 0 \quad \text{for all } k \geq k_0.
$$
From here, as in the proof of Lemma 3.4 of [1], we easily see that $a(k)$ is ultimately periodic.
(iii) See the proof of (ix).
(iv) Lemma. If $g = \sum_{j=0}^{\infty} g_j e_j$ is in $D_{q_0+k}$ and if $g_0 = g_1 = \cdots = g_k = 0$ then $g = 0$.
Proof of the Lemma. Assume that $g_0 = g_1 = \cdots = g_k = 0$ and $g \in D_{q_0+k}$. Then $g = s^{k+1} g'$ for some $g' \in G$, so $\rho(g' \land e_j) = \rho(g \land e_{j+k+1}) = 0$ for all $j$ with $j + k + 1 \geq q_0 + k$ or for all $j$ with $j \geq q_0 - 1$. Hence $g'$ is in $D_{q_0-1} = 0$ so $g' = 0$ and $g = 0$.
Proof of (iv). Suppose $v, w \in D_{q_0}$ with $v \neq 0$. Then $v_0 \neq 0$ and there exists $\lambda \in \mathbb{Z}_n$ such that $(w - \lambda v)_0 = 0$. Then $w = \lambda v$ by the lemma. We have shown that $v$ spans $D_{q_0}$.
Evidently $v, s(v), \ldots, s^k(v)$ are linearly independent (they are in row echelon form) in $D_{q_0+k}$. For $w \in D_{q_0+k}$ we can successively find $\lambda_0, \lambda_1, \ldots, \lambda_k$ such that $w' = w - \sum_{j=0}^{k} \lambda_j s^j v$ has $w'_0 = w'_1 = \cdots = w'_k = 0$. Then the lemma shows that $w' = 0$, and we have shown that $v, s(v), \ldots, s^k v$ span $D_{q_0+k}$.
(v) By (iv), every non-zero $g$ in $D_\infty$ can be written in the form
$$
g = \sum_{j=0}^{k} \lambda_j s^j v \quad \text{with } \lambda_k \neq 0.
$$
Evidently the length of $g$ is equal to $k + L$ where $L$ is the length of $v$.
Hence $g$ is of minimal length in $D_\infty - \{0\}$ if and only if $g = \lambda v$ for $\lambda \neq 0$.
(vi) Write $v = \sum_{k=0}^{L} v_k e_k$ with $v_0, v_L \neq 0$. Then because $v$ is in $D_{q_0}$,
$$
\sum_{j=0}^{L} v_j a(k - j) = 0 \quad \text{for all } k \geq q_0.
$$
As in the proof of Lemma 3.4 of [1], that implies periodicity of \( a(k) \) commencing at \( q_0 - L \). Hence \( k_0 \leq q_0 - L \) or \( k_0 + L \leq q_0 \).

To prove the opposite inequality use \( a(k + T) = a(k) \) for all \( k \geq k_0 \). Combining that with \( \sum_{j=0}^{L} v_j a(k - j) = 0 \) for \( k \) large enough we obtain \( \sum_{j=0}^{L} v_j a(k - j) = 0 \) for all \( k \) such that \( k - L \geq k_0 \) or \( k \geq k_0 + L \). That shows \( v \) is in \( D_{k_0 + L} \) and therefore that \( k_0 + L \geq q_0 \).

(vii) \( q_0 \) is the smallest integer such that, for all \( k \geq q_0 \), \( \rho(v \wedge e_k) = 1 \). This is equivalent to

\[
0 = \sum_{j=0}^{L} v_j a(k - j) = \sum_{j=0}^{L} \tilde{v}_j a(k - L + j) = (\tilde{v} | A^{k-L}).
\]

Hence \( q_0 \) is the smallest integer such that \( \tilde{v} \perp A^{k-L} \) for all \( k \geq q_0 \), and \( k_0 = q_0 - L \) is the smallest integer such that \( \tilde{v} \perp A^k \) for all \( k \geq k_0 \).

(viii) From \( a(k + T) = a(k) \) for all \( k \geq k_0 \) it follows that \( e_0 - e_T \) is in \( D_\infty \) so \( L \leq T \). If \( r = \text{rank } A < T \) choose \( T - r \) linearly independent vectors \( \tilde{v}(1), \tilde{v}(2), \ldots, \tilde{v}(T - r) \) in \( (\mathbb{Z}_n)^T \) perpendicular to \( A_1, A_2, \ldots, A_T \). Taking a suitable linear combination of the \( \tilde{v}(k) \) we can find a vector \( \tilde{g} \) of the form \([g_T, g_{T-1}, \ldots, g_1, g_0, 0, \ldots, 0]\). Then \( g = \sum_{k=0}^{T} g_k e_k \) is in \( D_\infty \) so \( L \leq r \). In all cases, then, we have proved \( L \leq r \). If \( L = T \) then \( L = r = T \), so to complete the proof we need only show that \( r \leq L \) provided \( L < T \).

Suppose then that \( L < T \). Let \( \tilde{\omega} = [v_L, v_{L-1}, \ldots, v_0, 0, \ldots, 0] \) in \( (\mathbb{Z}_n)^T \) where \( v \) has minimal length in \( D_\infty \). Then \( \tilde{v}, s\tilde{v}, \ldots, s^{T-(L+1)} \tilde{v} \) are \( T-L \) linearly independent vectors perpendicular to \( A_1, A_2, \ldots, A_T \). Hence \( r = \text{rank } A \leq T - (T - L) = L \).

(ix) \( D_\infty \) is isomorphic to \( G \) by \( s^k v \to e_k \). Under this isomorphism the restriction of \( s \) to \( D_\infty \) corresponds to \( s \) and the restriction of \( \rho \) to \( D_\infty \) corresponds to \( \tilde{\rho}(e_0 \wedge e_k) = \rho(v \wedge e_k) \). Hence \( \sigma_\infty \) has defining sequence \( (b(k)) \) given by:

\[
\gamma^{b(k)} = \rho(v \wedge s^kv).
\]

Because \( v \in D_{q_0} \) and \( D_{q_0-1} = 0 \), \( \rho(v \wedge e_k) = 1 \) for all \( k \geq q_0 \) and \( \rho(v \wedge e_{q_0-1}) \neq 1 \). That implies \( \rho(v \wedge s^k v) = 1 \) for all \( k \geq q_0 \) and \( \rho(v \wedge s^{k-1} v) \neq 1 \), where we use the fact that \( v_0 \neq 0 \). Thus \( b(k) = 0 \) for \( k \geq q_0 \) and \( b(q_0 - 1) \neq 0 \).

Then \( (b(k)) \) is not periodic; therefore \( R_\infty \) is a factor and is in fact isomorphic to \( R \) by [1]. This also proves (iii).

(x) The span of \( e_0, e_1, \ldots, e_{L-1} \) is a complement for \( D_\infty \) in \( G \). Hence \( G/G_\infty \) is isomorphic to \( (\mathbb{Z}_n)^L \), and, by Proposition 1.4 of [1], \( [R: R_\infty] = n^L \).
EXAMPLES. In each case we specify \( \sigma \) by giving the determining sequence \( (a(k))_{k \in \mathbb{Z}} \): we write \( a = a(0), a(1), a(2) \ldots \). Similarly we specify \( \sigma_\infty \) by giving its determining sequence \( (b(k)) \). \( n \) can be taken to be an arbitrary prime with the noted exceptions: it is understood that integers are to be reduced mod \( n \). The first repeating period is underlined.

1. \( a = 0, 1, 1, 1, 1, \ldots \)
   \[ k_0 = 1, L = T = 1, q_0 = 2, v = e_0 - e_1, \]
   \[ b = 0, 1, 0, 0, \ldots \]

2. \( a = 0, 0, 1, 2, 1, 2, \ldots, n \neq 2, 3. \)
   \[ k_0 = 2, T = 2, A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \text{ has rank } 2, \]
   \[ L = r = 2, q_0 = 4. \]
   Then \( v = e_0 - e_2, b(k) = 2a(k) - [a(k + 2) + a(k - 2)]. \)
   \[ b = 0, -2, 1, 2, 0, 0, \ldots \]

3. \( a = 0, 0, 1, -2, 1, 2, \ldots \) with \( n = 3. \)
   As in Example 2, \( k_0 = 2 \) and \( T = 2 \) but now \( A \) has rank 1, so
   \[ L = r = 1 \text{ and } q_0 = 3, \]
   \[ v = e_0 - 2e_1, \]
   \[ b(k) = 2a(k) + a(k - 1) + a(k + 1), \]
   \[ b = 0, 1, 1, 0, 0, \ldots \]

4. \( a = 0, 0, 1, -1, 1, -1, \ldots \)
   \[ k_0 = 2, v = e_0 + e_1, q_0 = 3, \]
   \[ b(k) = 2a(k) + (a(k + 1) + a(k - 1)), \]
   \[ b = 0, 1, 1, 0, 0, \ldots \]

5. \( a = 0, 0, 1, 2, 3, 4, \ldots \)
   \[ T = n, k_0 = 1, v = e_0 - 2e_1 + e_2 \text{ is of minimal length in } D_\infty \]
   because
   \[ \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \text{ has rank } 2. \]
   \[ L = 2, q_0 = 3, \]
   \[ b(k) = 6a(k) - 4[a(k + 1) + a(k - 1)] + [a(k + 2) + a(k - 2)], \]
   \[ b = 0, -2, 1, 0, 0, \ldots \]

6. \( a_1 = 0, 0, 1, 0, 0, 1, \ldots \)
   \[ a_2 = 0, 1, 0, 0, 1, 0, \ldots \text{ for } n \neq 2 \]
   \[ a_3 = 0, 2, 2, 0, 2, 2, \ldots \text{ for } n \neq 2 \]
   all have \( L = T = 3, k_0 = 0, q_0 = 3, v = e_0 - e_3. \)
   \[ b = 0, 1, 1, 0, 0, \ldots \]

In the calculation of \( b_3 \) we use the fact that multiplying a determining sequence by a square does not change its conjugacy class (see [1]).
7. \( a = 0, 3, 0, 0, \ldots, 0, 6, 18, \ldots \) for \( n \neq 3, N \) arbitrary \( \geq 3 \) where 
\( a(0) = 0, a(1) = 3, a(k) = 0 \) for \( 2 \leq k \leq N - 1 \),
and for \( k \geq N \):

\[
(2.1) \quad a(k) = 2 \sum_{i=k-N}^{k-1} a(i).
\]

Then (2.1) holds for all \( k \geq 2 \) but not for \( k = 1 \) since \( 2 \sum_{i=1-N}^{0} a(i) = 2a(-1) = -6 \) and \( n \neq 3 \). Hence \( a(k) \) is not periodic, but is ultimately periodic commencing with \( k_0 = -N + 2 \). A minimal \( v \) in \( D_\infty \) is given by \( v = e_0 - 2 \sum_{i=1}^{N} e_i \).

Therefore \( L = N \) and \( q_0 = 2 \). A direct calculation of \( b(1) \) gives \( 9 = 3^2 \) so

\[
9 = b = 0, 1, 0, 0, 0, \ldots.
\]

8. A 4-shift \( \sigma \) on \( R \) such that \( \sigma_{\infty} \) is not an \( m \)-shift for any \( m \):

\[
a = 0, 1, 2, 2, \ldots, \quad n = 4.
\]

Since \( (a(k)) \) fails to be periodic mod 2 the factor condition is satisfied
and \( \sigma \) is a shift on \( R \) by [1]. In \( G = \bigoplus_{k=0}^{\infty}(Z_4)^{(k)} \) take \( v_0 = 2e_0, v_k = e_{k-1} + e_k \) for \( k \geq 1 \). Then \( s(v_0) = v_0 + 2v_1, s(v_k) = v_{k+1} \) for \( k \geq 1 \). We see easily (as in the proof of Theorem 2.1) that \( D_2 = Z_2v_0, D_3 = Z_2v_0 \oplus Z_4v_1 \) and finally that

\[
D_\infty = Z_2v_0 \oplus Z_4v_1 \oplus Z_4v_2 \oplus \ldots.
\]

Hence \( \sigma_{\infty} \) is the group shift \( \sigma(D_\infty, \hat{s}, \hat{\rho}) \) where \( \hat{s} \) and \( \hat{\rho} \) are the restrictions to \( D_\infty \) of \( s \) and \( \rho \) on \( G \). If \( \sigma_{\infty} \) were an \( m \)-shift, there would exist a \( g \in D_\infty \) such that \( g, s(g), s^2(g), \ldots \) generate \( D_\infty \) (see Proposition 5.2 of [1]). It is easy to check that this is impossible. It is also easy to check that \( \hat{\rho} \) is non-degenerate on \( D_\infty \) so that \( R_\infty \) is a factor.

3. **Outer conjugacies.** Given an \( n \)-shift \( \sigma \) with determining sequence \((a(k))\) we give one method for calculating determining sequences of \( n \)-shifts outer conjugate to \( \sigma \). Although this method produces some interesting examples we are unable to exploit it to the extent of showing when \( \sigma \) and \( \sigma_{\infty} \) are outer conjugate in general.

A basic lemma from operator theory follows.

**Lemma 3.1.** Suppose that \( n \) is an integer \( \geq 2 \) and that \( u \) is a unitary operator with \( u^n = 1 \). Then there exists a unitary \( y \) in the *-algebra generated by \( u \) with the following properties:

1. \( y^n = 1 \) in case \( n \) is odd; \( y^{2n} = 1 \) in case \( n \) is even.
2. Let $\gamma = \exp(2\pi i/n)$. For all unitaries $v$ such that $uvu^*v^* = \gamma^a$ where $a \in \mathbb{Z}_n$,

\[ yvy^* = u^av \quad \text{for } n \text{ odd}, \]

\[ yvy^* (u^a v)^* \in \mathbb{C} \quad \text{for } n \text{ even}. \]

**Proof.** Suppose first that $n$ is odd. Let $T_n = \{ \lambda \in \mathbb{C} | \lambda^n = 1 \}$. It suffices to produce a function $f: T_n \rightarrow T_n$ such that

\[ f(\gamma z) = zf(z) \quad \text{for all } z \in T_n. \tag{3.1} \]

For given such a function, let $y = f(u)$. Then $y$ is unitary and $y^n = 1$. If $uvu^*v^* = \gamma^a$ then $uvu^* = \gamma^{-a}u$ so $vf(u)v^* = f(\gamma^{-a}u) = F(u)$ where $F(z) = f(\gamma^{-a}z) = \overline{\gamma}^a f(z)$ by (3.1). Then $F(u) = (u^a) f(u)$ so $vyv^* = u^{-a}y$ or $yvy^* = u^a v$.

To show that a function $f$ satisfying (3.1) exists, let

\[ f(\gamma^s) = \gamma^{[s(s-1)/2]} \quad \text{for } s = 0, 1, \ldots, n - 1. \tag{3.2} \]

We confirm that (3.2) holds for $s = n$ also, since $(n-1)/2$ is an integer, and then easily check that $f$ satisfies (3.1).

Suppose now that $n$ is even. (Then of course a function $f$ satisfying (3.1) cannot exist.) Let $\delta = \exp(\pi i/n)$ and define $f(\gamma^s) = \delta^s \gamma^{[s(s-1)/2]}$ for $s = 0, 1, \ldots, n - 1$. Then $f(\gamma z) = \delta z f(z)$ for all $z \in T_n$ and, as in the case when $n$ is odd, $y = f(u)$ has the required properties.

**Corollary 3.2.** Suppose that $\sigma$ is an $n$-shift on $M$, $\sigma = \sigma(G, s, \rho)$ where $G = \bigoplus_{k=0}^{\infty} (\mathbb{Z}_n)^{(k)}$. Let $g \rightarrow u_g$ be the canonical twisted representation of $G$ in $M$, and define a bilinear map $[ , ]$ from $G \times G$ to $\mathbb{Z}_n$ by:

\[ \gamma^{[g,h]} = \rho(g \wedge h) = u_g u_h u_g^* u_h^* \quad \text{for } g, h \in G. \]

Fix $g \in G$ and define $\phi_g: G \rightarrow G$ by: $\phi_g(h) = h + [g, h]g$ for all $h \in G$. Then there exists a unitary $y_g$ in $M$ such that

\[ y_g u_h y_g^* = \lambda(g, h) u_{\phi_g(h)} \quad \text{for all } h \in G \]

where $\lambda(g, h) \in \mathbb{C}$.

**Proposition 3.3.** Suppose that $n$ is a prime and that the $n$-shift $\sigma$ on the hyperfinite factor $R$ has determining sequence $(a(k))$. Let $G = \bigoplus_{k=0}^{\infty} (\mathbb{Z}_n)^{(k)}$, let $s$ be the shift $e_k \rightarrow e_{k+1}$ on $G$, let $\rho$ on $G$ be defined by $(a(k))$, and let $[ , ]$ and $\phi_g$ be defined as in Corollary 3.2, so that

\[ [e_i, e_j] = a(j - i) \quad \text{for all } i, j = 0, 1, 2, \ldots. \]
Suppose that \( g(1), g(2), \ldots, g(m) \) are in \( G \) and let \( \phi \) be \( \phi_{g(1)} \circ \phi_{g(2)} \circ \phi_{g(3)} \circ \cdots \circ \phi_{g(m)} \). Suppose that \( v(0) \) in \( G \) is such that \( G \) is generated by \( v(0), v(1), v(2), \ldots \) where \( v(k) = \phi(s(v(k - 1))) \). Then \( b(k) = [v(0), v(k)] \) defines a determining sequence \((b(k))\) of an \( n\)-shift \( \sigma' \) on \( R \) which is outer conjugate to \( \sigma \).

**Proof.** We may assume that \( \sigma = \sigma(G, s, \rho) \) and that \( R = W^*(G, \rho) \).

Let \( y = y_{g(1)}y_{g(2)} \cdots y_{g(n)} \) where \( y_{g(k)} \) is given by Corollary 3.2. Then \( yu_h y^* = \lambda(h)u_{\phi(h)} \) for all \( h \in G \), where \( \lambda(h) \in \mathbb{C} \). Hence

\[
[(\text{Ad}y) \circ \sigma](u_{v(k)}) = \lambda_k u_{v(k+1)}
\]

for \( \lambda_k \in \mathbb{C} \). Now let \( \sigma' = (\text{Ad}y) \circ \sigma \) and let \( w_0 = u_{v(0)} \). Then

1. \( w_0^n = 1 \) and \( w_k^k \neq 1 \) for \( k = 1, \ldots, n - 1 \);
2. the \( w_k = (\sigma')^k w_0 \) generate \( R \);
3. \( w_0 w_k w_0^* w_k^* = \gamma^{v(0), v(k)} \).

Therefore (Proposition 4.1 of [1]), \( \sigma' \) is an \( n \)-shift on \( R \) with determining sequence \( b(k) = [v(0), v(k)] \).

**Examples.** 1. Take \( \sigma_0 \) given by the sequence \( 0, 1, 0, 0, \ldots \) (i.e. \( a(0) = 0, a(1) = 1, a(2) = 0, \ldots \)). Then the shifts given by each of the following sequences are outer conjugate to \( \sigma_0 \), and hence, for each, the derived shift is \( \sigma_0 \) and \( q_0 = 2 \).

(a) \( 0, 1, 1, 1, \ldots \)
(b) \( 0, 2, 0, 2, 0, \ldots, \) for \( n \neq 2 \),
(c) \( 0, 1, a, a^2, \ldots, \)
(d) \( 0, \lambda + 1, \lambda^2 - 1, \lambda^3 + 1, \ldots, \) for \( \lambda \neq -1, n \neq \lambda + 1 \),
(e) \( 0, 1 - \lambda \mu, (1 - \lambda \mu)(\lambda^2 - \mu^2)/\lambda - \mu, \ldots, (1 - \lambda \mu)(\lambda^n - \mu^n)/\lambda - \mu, \ldots, \) for \( \lambda \neq \mu, \lambda \mu \neq 1 \).

The \( g(i) \)'s in Proposition 3.3 which demonstrate the above outer conjugacies are

(a) \( g_1 = e_0 \),
(b) \( g_1 = -e_1, g_2 = e_0 \),
(c) \( g_1 = (1 + a)e_0, g_2 = -e_1 \),
(d) \( \mu = -1 \) in (e),
(e) \( g_1 = \mu e_1, g_2 = \lambda e_0 \).

In each case we can take \( v(0) = e_0 \).

**Remarks.** Given a shift \( \sigma \) of forms (c), (d) or (e) for example, the calculation of \( \sigma_\infty \) or \( q_0 \) by the methods of §2 might be very difficult even for one prime \( n \). There are, however, shifts which have derived
shift \( \sigma_0 \) which are not obviously outer conjugate to \( \sigma \) (see Example 7 of §2).

2. Take \( \sigma_0 \) given by \( b = 0, 0, 1, 0, 0, \ldots \). Then the shifts given by the following defining sequences are outer conjugate to \( \sigma_0 \):

(a) \( 0, 0, 1, 0, 1, \ldots \),
(b) \( 0, 0, 2, 0, 0, 2, 0, \ldots \), for \( n \neq 2 \) (note \( k_0 = -1 \)),
(c) \( 0, 0, 1, 0, \lambda, 0, \lambda^2, 0, \ldots \).

The \( g(i) \)'s in Proposition 3.3 which demonstrate the above outer conjugacies are as follows: (a) \( g(0) = e_0 \); (b) \( g(0) = -e_2, g(1) = e_0 \); (c) \( g(0) = \lambda e_0 \).

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