Pacific Journal of Mathematics

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Vol. 142, No. 2 February 1990

PACIFIC JOURNAL OF MATHEMATICS
Vol. 142, No. 2, 1990

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The *resultant* $R(f, g)$ of two polynomials f and g is an irreducible polynomial such that $R(f, g) = 0$ if and only if the equations $f = 0$ and $g = 0$ have one common root.

When $g = f'/p$, then $D(f) = R(f, g)$ is called the *discriminant* of f and the *discriminant hypersurface* $D_p = \{f \in \mathbf{C}^p, D(f) = 0\}$ can be identified to the discriminant of a versal deformation of the simple hypersurface singularity $A_{p-1}: x^p = 0$. In particular, the fundamental group $\pi = \pi_1(\mathbf{C}^p \backslash D_p)$ is the famous *braid group* and $\mathbf{C}^p \backslash D_p$ in fact a $K(\pi, 1)$ space.

Here we prove the following.

THEOREM. $\pi_1(\mathbf{C}^{p+q} \backslash R_{p,q}) = Z$.

As $\mathbf{C}^p \backslash D_p$ can be regarded as a linear section of $\mathbf{C}^{p+q} \backslash R_{p,q}$, this theorem shows that by a nongeneric linear section the fundamental group may change drastically, in contrast with the case of generic section.

Let $f = x^p + a_1 x^{p-1} + \cdots + a_p$ and $g = x^q + b_1 x^{q-1} + \cdots + b_q$ be two monic polynomials with complex coefficients of degree p and q respectively.

The *resultant* of them $R(f, g)$ is an irreducible polynomial in the coefficients a_i, b_j such that $R(f, g) = 0$ if and only if the equations $f = 0$ and $g = 0$ have at least one common root. Explicitly, the resultant is given by the next formula (see for instance **[5]**, p. 136):

$$R(f,g) = R(a,b) = \begin{vmatrix} 1 & a_1 & \cdots & \cdots & a_p & \cdots & 0 & \cdots & 0 \\ & 1 & a_1 & \cdots & \cdots & \cdots & a_p & \cdots & 0 \\ & & & & & 1 & \cdots & \cdots & a_p \\ 1 & b_1 & & \cdots & b_q & & 0 & \cdots & 0 \\ & 1 & b_1 & \cdots & & \cdots & b_q & \cdots & 0 \\ & & & & & & 1 & \cdots & b_q \end{vmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} q \text{ lines} \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} p \text{ lines} \end{matrix}$$

When $g = f'/p$, then $D(f) = (f, g)$ is called the *discriminant* of the polynomial f and the *discriminant hypersurface* $D_p = \{f \in \mathbf{C}^p, D(f) = 0\}$ has occurred several times in Singularity Theory, since it can be identified to the discriminant of a versal deformatioin of the simple hypersurface singularity $A_{p-1}: x^p = 0$, see for instance **[1]**, **[3]**, **[9]**. In particular, the fundamental group $\pi = \pi_1(\mathbf{C}^p \backslash D_p)$ is the famous *braid group* [**1**] (with p strings) and $\mathbf{C}^p \backslash D_p$ is in fact a $K(\pi, 1)$ space.

In this note we consider the analogous *resultant hypersurface*

$$R_{p,q} = \{(f, g) \in \mathbf{C}^{p+q}; R(f, g) = 0\}$$

and prove the following.

THEOREM. $\pi_1(\mathbf{C}^{p+q} \backslash R_{p,q}) = Z$.

Since $\mathbf{C}^p \backslash D_p$ can be regarded as a linear section of $\mathbf{C}^{p+q} \backslash R_{p,q}$, this theorem shows that by a nongeneric linear section the fundamental group may change drastically, in contrast with the case of generic section [**4**].

It is also interesting to note that the complements $F_{p,q} = \mathbf{C}^{p+q} \backslash R_{p,q}$ have already occurred in an important topological problem [**7**], going back to certain questions in Control Theory [**2**]. In short, consider the space of rational *real* functions of the form

$$\phi = \frac{x^n + \alpha_1 x^{n-1} + \cdots + \alpha_n}{x^n + \beta_1 x^{n-1} + \cdots + \beta_n}$$

with $\alpha_i, \beta_j \in R$ and the numerator and the denominator having no common root. Then ϕ induces a continuous map $P^1(\mathbf{C}) = \mathbf{C} \cup \{\infty\} \to \mathbf{C} \cup \{\infty\} = P^1(\mathbf{C})$ of degree n and its restriction to the equator $R \cup \{\infty\} = S^1 \subset S^2 = P^1(\mathbf{C})$ gives a map $S^1 \to S^1$ having degree r such that $-n \leq r \leq n$ and $n - r \equiv 0 \bmod 2$. Let E_{n-r} denote the space of these mappings with n and r fixed, with the obvious topology. Then Segal has shown in [**7**] that $E_{n,r}$ is homeomorphic to $F_{p,q}$ with $p+q = n$ and $p - q = r$. He has also proved our Theorem in the special case $p = q$, by a method completely different from ours.

We derive our Theorem from some basic properties of the resultant hypersurface (which are also interesting in themselves) combined with a deep result of Lê-Saito [**6**] on the connectivity of the Milnor fiber of non-isolated singularity.

LEMMA 1. *$R \in \mathbf{C}[a, b]$ is a weighted homogeneous polynomial of degree pq with respect to the weights* $\mathrm{wt}(a_i) = \mathrm{wt}(b_i) = i$.

Proof. Note that the polynomial $t \cdot f = x^p + ta_1 x^{p-1} + \cdots + t^p a_p$ has as roots the elements tx_i, where x_i are the roots of f, for any $t \in \mathbf{C}^*$. Then, using [**5**], p.137, we get $R(t \cdot f, t \cdot g) = \prod_{i,j}(tx_i - ty_j) = t^{pq} \prod_{i,j}(x_i - y_j) = t^{pq} R(f, g)$, where y_j are the roots of g. □

The key remark in the proof is that the resultant hypersurface has a *smooth normalization* ν which can be described explicitly as follows:

$$\nu = \mathbf{C} \times \mathbf{C}^{p-1} \times \mathbf{C}^{q-1} \to R_{p,q} \subset \mathbf{C}^{p+q}$$

$\nu(t, \alpha, \beta) = ((x-t)f_\alpha, (x-t)g_\beta)$, where $f_\alpha = x^{p-1} + \alpha_1 x^{p-2} + \cdots + \alpha_{p-1}$, $g_\beta = x^{q-1} + \beta_1 x^{q-2} + \beta_1 x^{q-2} + \cdots + \beta_{q-1}$. Then ν is clearly surjective onto $R_{p,q}$ and the cardinal of a fiber $\nu^{-1}(f, g)$ is equal to the number of common roots of the equations $f = 0$, $g = 0$, counted without taking their multiplicities into account. Hence ν is a finite morphism which is generically one-to-one so that ν is indeed a normalization for $R_{p,q}$.

We use ν to investigate the singularities of the hypersurface $R_{p,q}$. To do this, we first compute the differential of ν at a point (t_0, α_0, β_0):

$$\begin{aligned} &d\nu(t_0, \alpha_0, \beta_0)(t, \alpha, \beta) \\ &\qquad = ((x - t_0)(f_\alpha - x^{p-1}) - t f_{\alpha_0}, (x - t_0)(g_\beta - x^{q-1}) - t g_{\beta_0}). \end{aligned}$$

Assume that t_0 is not a root for f_{α_0} and g_{β_0} simultaneously. Then it follows that $d\nu(t_0, \alpha_0, \beta_0)$ is an injective linear map and its image (which is a hyperplane in the vector space V of all the pairs (A, B), with $A, B \in \mathbf{C}[x]$, $\deg A \leq p-1$, $\deg B \leq q-1$) is given by the equation

$$f_{\alpha_0}(t_0)B(t_0) - g_{\beta_0}(t_0)A(t_0) = 0.$$

Let $d(f, g)$ be the greatest common divisor of the polynomials f and g. The above computation gives us the next

COROLLARY 2. *The point (f, g) is nonsingular on the hypersurface $R_{p,q}$ if and only if* $\deg d(f, g) = 1$.

Proof. Use the fact that a point $(f, g) \in R_{p,q}$ is nonsingular if and only if $\nu^{-1}(f, g)$ consists of one point, say y, and the corresponding germ $\nu: (\mathbf{C}^{p+q}, y) \to (R_{p,q}, (f, g))$ is an isomorphism. □

We have also the more general result.

PROPOSITION 3. *Assume that $d(f, g) = (x - t_1) \ldots (x - t_s)$ is a product of s linear distinct factors. Then the germ $(R_{p,q}, (f, g))$ consists of s smooth hypersurface germs passing through (f, g) with normal crossings.*

Proof. In this case the fiber $\nu^{-1}(f, g)$ consists of s points, say y_k with $k = 1, \ldots, s$. Moreover, the germs $\nu_i: (\mathbf{C}^{p+q-1}, y_i) \to (R_{p,q}, (f, g)) \subset (\mathbf{C}^{p+q}, (f, g))$ induced by ν are all imbeddings and $H_i = \operatorname{im}(\nu_i)$ are precisely the (smooth) irreducible components of the germ $(R_{p,q},(f,g))$. The corresponding tangent spaces are $T_k = T_{(f,g)}H_k\colon \overline{f}(t_k)B(t_k) - \overline{g}(t_k)A(t_k) = 0$ for $K-1,\dots,s$ and $\overline{f} = f/d(f,g)$, $\overline{g} = g/d(f,g)$. The condition of normal crossing in this case means that $\operatorname{codim}(\bigcap_{k=1,s} T_k) = s$.

But this intersection corresponds to the kernel of the following linear map. $T\colon V \simeq \mathbf{C}^{p+q} \to \mathbf{C}[x]/(d(f,g)) \simeq \mathbf{C}^s$ such that the kth component of $T(A,B)$ is just the evaluation on t_k of $(\overline{f}\cdot B - \overline{g}\cdot A)$, for $k = 1,\dots,s$. It is easy to check that T is a surjective map and hence $\operatorname{codim}(\bigcap_{k=1,s} T_k) = \operatorname{codim}(\ker T) = s$.

COROLLARY 4. *The hypersurface $R_{p,q}$ has only normal crossings singularities in codimension* 1 *and hence $\pi_1(\mathbf{C}^{p+q}\backslash R_{p,q}) = Z$.*

Proof. The singularities of $R_{p,q}$ which are not normal crossings (as described in Proposition 3) lie in the image of the map

$$\tau\colon \mathbf{C}\times\mathbf{C}^{p-2}\times\mathbf{C}^{q-2} \to R_{p,q},$$

$$\tau(t,\alpha,\beta) = ((x-t)^2\widetilde{f}_\alpha,(x-t)^2\widetilde{g}_\beta)$$

with $\widetilde{f}_\alpha$, $\widetilde{g}_\beta$ having a meaning similar to f_α, g_β. But $\dim(\operatorname{im}\tau) \le p+q-3 = \dim R_{p,q} - 2$ which proves the first assertion above. Next consider the fibration $F \to \mathbf{C}^{p+q}\backslash R_{p,q} \to \mathbf{C}^*$ with $F = F^{-1}(1) = \{(f,g) \in \mathbf{C}^{p+q}; R(f,g) = 1\}$. Using the weighted homogeneity of R given by Lemma 1, we can identify this fibration with the Milnor fibration of the hypersurface singularity $(R_{p,q},(x^p,y^q))$. It follows by [6] that $\prod_1(F) = 0$ and hence we get an isomorphism

$$R_\# = \prod_1(\mathbf{C}^{p+q}\backslash R_{p,q}) \to \prod_1(\mathbf{C}^*) = Z.$$

This ends the proof of this corollary as well as giving a more precise version of our Theorem above.

REMARK 5. There is a natural $\mathbf{C}$-action on $\mathbf{C}^{p+q}$ leaving the resultant hypersurface $R_{p,q}$ invariant. Namely we define the translation of an element (f,g) by the complex number λ to be the element (f^λ, g^λ) where

$$f^\lambda = \prod_{i=1,p}(x-x_i-\lambda), \qquad g^\lambda = \prod_{j=1,q}(x-y_j-\lambda)$$

with x_i (resp. y_j) being the roots of f (resp. g). Since the hyperplane $a_1 = 0$ is clearly transversal to all the C-orbits, it follows that

$$R_{p,q} = \overline{R}_{p,q} \times \mathbf{C} \quad \text{with } \overline{R}_{p,q} = R_{p,q} \cap \{a_1 = 0\}.$$

The first non-trivial case of a resultant hypersurface is for $p = q = 2$. Then $\overline{R}_{2,2}$ is just the Whitney umbrella $W: \overline{b}_2^2 - b_1^2 a_2 = s$, with $\overline{b}_2 = b_2 - a_2$, called also a D_∞-surface singularity for a pinch point. It follows that $\mathbf{C}^4 \backslash R_{2,2} = (\mathbf{C}^3 \backslash W) \times \mathbf{C}$ and the homotopy groups of $\mathbf{C}^3 \backslash W$ can be derived from the Milnor fibration $F_\infty \to \mathbf{C}^3 \backslash W \to \mathbf{C}^*$ associated to the D_∞-singularity **[8]**. It is known that F_∞ has the homotopy type of the 2-sphere S^2 and hence

$$\prod_k (\mathbf{C}^4 \backslash R_{2,2}) = \prod_k (S^2) \quad \text{for } k \geq 2.$$

In particular $\mathbf{C}^4 \backslash R_{2,2}$ is not a $K(Z, 1)$ space, since $\Pi_2(\mathbf{C}^4 \backslash R_{2,2}) = Z$.

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Received April 29, 1988.

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Pacific Journal of Mathematics

Vol. 142, No. 2 February, 1990