ON THE RESULTANT HYPERSURFACE

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The resultant $R(f, g)$ of two polynomials $f$ and $g$ is an irreducible polynomial such that $R(f, g) = 0$ if and only if the equations $f = 0$ and $g = 0$ have one common root.

When $g = f'/p$, then $D(f) = R(f, g)$ is called the discriminant of $f$ and the discriminant hypersurface $D_p = \{ f \in \mathbb{C}^p, D(f) = 0 \}$ can be identified to the discriminant of a versal deformation of the simple hypersurface singularity $A_{p-1}: x^p = 0$. In particular, the fundamental group $\pi = \pi_1(\mathbb{C}^p \setminus D_p)$ is the famous braid group and $\mathbb{C}^p \setminus D_p$ in fact a $K(\pi, 1)$ space.

Here we prove the following.

**Theorem.** $\pi_1(\mathbb{C}^{p+q} \setminus R_{p,q}) = \mathbb{Z}$.

As $\mathbb{C}^p \setminus D_p$ can be regarded as a linear section of $\mathbb{C}^{p+q} \setminus R_{p,q}$, this theorem shows that by a nongeneric linear section the fundamental group may change drastically, in contrast with the case of generic section.

Let $f = x^p + a_1 x^{p-1} + \cdots + a_p$ and $g = x^q + b_1 x^{q-1} + \cdots + b_q$ be two monic polynomials with complex coefficients of degree $p$ and $q$ respectively.

The resultant of them $R(f, g)$ is an irreducible polynomial in the coefficients $a_i, b_j$ such that $R(f, g) = 0$ if and only if the equations $f = 0$ and $g = 0$ have at least one common root. Explicitly, the resultant is given by the next formula (see for instance [5], p. 136):

$$R(f, g) = R(a, b) = \begin{vmatrix} 1 & a_1 & \cdots & a_p & 0 & \cdots & 0 \\ 1 & a_1 & \cdots & \cdots & a_p & \cdots & 0 \\ & & & 1 & \cdots & a_p & \cdots \\ 1 & b_1 & \cdots & b_q & 0 & \cdots & 0 \\ 1 & b_1 & \cdots & \cdots & b_q & \cdots & 0 \\ & & & & 1 & \cdots & b_q \end{vmatrix}$$

$q$ lines

$p$ lines

When $g = f'/p$, then $D(f) = (f, g)$ is called the discriminant of the polynomial $f$ and the discriminant hypersurface $D_p = \{ f \in \mathbb{C}^p, D(f) = 0 \}$ has occurred several times in Singularity Theory, since it can be identified to the discriminant of a versal deformation of the simple hypersurface singularity $A_{p-1}: x^p = 0$, see for instance [1], [3], [9]. In
particular, the fundamental group $\pi = \pi_1(C^p \setminus D_p)$ is the famous braid group [1] (with $p$ strings) and $C^p \setminus D_p$ is in fact a $K(\pi, 1)$ space.

In this note we consider the analogous resultant hypersurface

$$R_{p,q} = \{(f, g) \in C^{p+q}; R(f, g) = 0\}$$

and prove the following.

**Theorem.** $\pi_1(C^{p+q} \setminus R_{p,q}) = \mathbb{Z}$.

Since $C^p \setminus D_p$ can be regarded as a linear section of $C^{p+q} \setminus R_{p,q}$, this theorem shows that by a nongeneric linear section the fundamental group may change drastically, in contrast with the case of generic section [4].

It is also interesting to note that the complements $F_{p,q} = C^{p+q} \setminus R_{p,q}$ have already occurred in an important topological problem [7], going back to certain questions in Control Theory [2]. In short, consider the space of rational real functions of the form

$$\phi = \frac{x^n + \alpha_1 x^{n-1} + \cdots + \alpha_n}{x^n + \beta_1 x^{n-1} + \cdots + \beta_n}$$

with $\alpha_i, \beta_j \in \mathbb{R}$ and the numerator and the denominator having no common root. Then $\phi$ induces a continuous map $P^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\} = P^1(\mathbb{C})$ of degree $n$ and its restriction to the equator $R \cup \{\infty\} = S^1 \subset S^2 = P^1(\mathbb{C})$ gives a map $S^1 \to S^1$ having degree $r$ such that $-n \leq r \leq n$ and $n - r \equiv 0 \mod 2$. Let $E_{n-r}$ denote the space of these mappings with $n$ and $r$ fixed, with the obvious topology. Then Segal has shown in [7] that $E_{n,r}$ is homeomorphic to $F_{p,q}$ with $p + q = n$ and $p - q = r$. He has also proved our Theorem in the special case $p = q$, by a method completely different from ours.

We derive our Theorem from some basic properties of the resultant hypersurface (which are also interesting in themselves) combined with a deep result of Lê-Saito [6] on the connectivity of the Milnor fiber of non-isolated singularity.

**Lemma 1.** $R \in \mathbb{C}[a, b]$ is a weighted homogeneous polynomial of degree $pq$ with respect to the weights $\text{wt}(a_i) = \text{wt}(b_i) = i$.

**Proof.** Note that the polynomial $t \cdot f = x^p + t a_1 x^{p-1} + \cdots + t^p a_p$

has as roots the elements $t x_i$, where $x_i$ are the roots of $f$, for any $t \in \mathbb{C}^*$. Then, using [5], p.137, we get $R(t \cdot f, t \cdot g) = \prod_i (tx_i - ty_j) = t^{pq} \prod_{i,j} (x_i - y_j) = t^{pq} R(f, g)$, where $y_j$ are the roots of $g$. \qed
The key remark in the proof is that the resultant hypersurface has a smooth normalization \( \nu \) which can be described explicitly as follows:

\[
\nu(t, \alpha, \beta) = ((x-t)f_\alpha, (x-t)g_\beta), \quad \text{where} \quad f_\alpha = x^{p-1} + \alpha_1 x^{p-2} + \cdots + \alpha_{p-1}, \quad g_\beta = x^{q-1} + \beta_1 x^{q-2} + \cdots + \beta_{q-1}. \]

Then \( \nu \) is clearly surjective onto \( R_{p,q} \) and the cardinal of a fiber \( \nu^{-1}(f, g) \) is equal to the number of common roots of the equations \( f = 0, g = 0 \), counted without taking their multiplicities into account. Hence \( \nu \) is a finite morphism which is generically one-to-one so that \( \nu \) is indeed a normalization for \( R_{p,q} \).

We use \( \nu \) to investigate the singularities of the hypersurface \( R_{p,q} \). To do this, we first compute the differential of \( \nu \) at a point \((t_0, \alpha_0, \beta_0)\):

\[
d\nu(t_0, \alpha_0, \beta_0)(t, \alpha, \beta) = ((x-t_0)(f_\alpha - x^{p-1}) - tf_{\alpha_0}, (x-t_0)(g_\beta - x^{q-1}) - tg_{\beta_0}).
\]

Assume that \( t_0 \) is not a root for \( f_{\alpha_0} \) and \( g_{\beta_0} \) simultaneously. Then it follows that \( d\nu(t_0, \alpha_0, \beta_0) \) is an injective linear map and its image (which is a hyperplane in the vector space \( V \) of all the pairs \( (A, B) \), with \( A, B \in \mathbb{C}[x] \), \( \deg A \leq p-1, \deg B \leq q-1 \)) is given by the equation

\[
f_{\alpha_0}(t_0)B(t_0) - g_{\beta_0}(t_0)A(t_0) = 0.
\]

Let \( d(f, g) \) be the greatest common divisor of the polynomials \( f \) and \( g \). The above computation gives us the next

**Corollary 2.** The point \((f, g)\) is nonsingular on the hypersurface \( R_{p,q} \) if and only if \( \deg d(f, g) = 1 \).

**Proof.** Use the fact that a point \((f, g)\) is nonsingular if and only if \( \nu^{-1}(f, g) \) consists of one point, say \( y \), and the corresponding germ \( \nu: (C^{p+q}, y) \to (R_{p,q}, (f, g)) \) is an isomorphism. \( \square \)

We have also the more general result.

**Proposition 3.** Assume that \( d(f, g) = (x-t_1) \cdots (x-t_s) \) is a product of \( s \) linear distinct factors. Then the germ \( (R_{p,q}, (f, g)) \) consists of \( s \) smooth hypersurface germs passing through \((f, g)\) with normal crossings.

**Proof.** In this case the fiber \( \nu^{-1}(f, g) \) consists of \( s \) points, say \( y_k \) with \( k = 1, \ldots, s \). Moreover, the germs \( \nu_i: (C^{p+q-1}, y_i) \to (R_{p,q}, (f, g)) \subset (C^{p+q}, (f, g)) \) induced by \( \nu \) are all imbeddings and \( H_i = \text{im}(\nu_i) \) are pre-
closely the (smooth) irreducible components of the germ \((R_{p,q},(f, g))\).

The corresponding tangent spaces are

\[ T_k = T_{(f,g)}H_k : \bar{f}(t_k)B(t_k) - \bar{g}(t_k)A(t_k) = 0 \]

for \(K = 1, \ldots, s\) and \(\bar{f} = f/d(f, g), \bar{g} = g/d(f, g)\). The condition of normal crossing in this case means that \(\text{codim}(\bigcap_{k=1,s} T_k) = s\).

But this intersection corresponds to the kernel of the following linear map.

\[ T : V \simeq \mathbb{C}^{p+q} \to \mathbb{C}[x]/(d(f, g)) \simeq \mathbb{C}^s \]

such that the \(k\)th component of \(T(A, B)\) is just the evaluation on \(t_k\) of \((\bar{f} \cdot B - \bar{g} \cdot A)\), for \(k = 1, \ldots, s\). It is easy to check that \(T\) is a surjective map and hence \(\text{codim}(\bigcap_{k=1,s} T_k) = \text{codim}(\ker T) = s\).

**Corollary 4.** The hypersurface \(R_{p,q}\) has only normal crossings singularities in codimension 1 and hence \(\pi_1(\mathbb{C}^{p+q}\setminus R_{p,q}) = \mathbb{Z}\).

**Proof.** The singularities of \(R_{p,q}\) which are not normal crossings (as described in Proposition 3) lie in the image of the map

\[ \tau : \mathbb{C} \times \mathbb{C}^{p-2} \times \mathbb{C}^{q-2} \to R_{p,q}, \]

\[ \tau(t, \alpha, \beta) = ((x - t)^2 \tilde{f}_\alpha, (x - t)^2 \tilde{g}_\beta) \]

with \(\tilde{f}_\alpha, \tilde{g}_\beta\) having a meaning similar to \(f_\alpha, g_\beta\). But \(\dim(\text{im } \tau) \leq p + q - 3 = \dim R_{p,q} - 2\) which proves the first assertion above. Next consider the fibration \(F \to \mathbb{C}^{p+q}\setminus R_{p,q} \to \mathbb{C}^*\) with \(F = F^{-1}(1) = \{(f, g) \in \mathbb{C}^{p+q}; R(f, g) = 1\}\). Using the weighted homogeneity of \(R\) given by Lemma 1, we can identify this fibration with the Milnor fibration of the hypersurface singularity \((R_{p,q}, (x^p, y^q))\). It follows by [6] that \(\Pi_1(F) = 0\) and hence we get an isomorphism

\[ R_# = \prod_1(\mathbb{C}^{p+q}\setminus R_{p,q}) \to \prod_1(\mathbb{C}^*) = \mathbb{Z}. \]

This ends the proof of this corollary as well as giving a more precise version of our Theorem above.

**Remark 5.** There is a natural \(\mathbb{C}\)-action on \(\mathbb{C}^{p+q}\) leaving the resultant hypersurface \(R_{p,q}\) invariant. Namely we define the translation of an element \((f, g)\) by the complex number \(\lambda\) to be the element \((f^\lambda, g^\lambda)\) where

\[ f^\lambda = \prod_{i=1,p} (x - x_i - \lambda), \quad g^\lambda = \prod_{j=1,q} (x - y_j - \lambda) \]
with \( x_i \) (resp. \( y_j \)) being the roots of \( f \) (resp. \( g \)). Since the hyperplane \( a_1 = 0 \) is clearly transversal to all the \( C \)-orbits, it follows that

\[
R_{p,q} = \overline{R}_{p,q} \times C \quad \text{with} \quad \overline{R}_{p,q} = R_{p,q} \cap \{ a_1 = 0 \}.
\]

The first non-trivial case of a resultant hypersurface is for \( p = q = 2 \). Then \( \overline{R}_{2,2} \) is just the Whitney umbrella \( W: b_2^2 - b_2^2 a_2 = s \), with \( b_2 = b_2 - a_2 \), called also a \( D_\infty \)-surface singularity for a pinch point. It follows that \( C^4 \setminus R_{2,2} = (C^3 \setminus W) \times C \) and the homotopy groups of \( C^3 \setminus W \) can be derived from the Milnor fibration \( F_\infty \rightarrow C^3 \setminus W \rightarrow C^* \) associated to the \( D_\infty \)-singularity [8]. It is known that \( F_\infty \) has the homotopy type of the 2-sphere \( S^2 \) and hence

\[
\prod_k (C^4 \setminus R_{2,2}) = \prod_k (S^2) \quad \text{for} \quad k \geq 2.
\]

In particular \( C^4 \setminus R_{2,2} \) is not a \( K(Z, 1) \) space, since \( \Pi_2(C^4 \setminus R_{2,2}) = Z \).

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