ON THE RESULTANT HYPERSURFACE

A. D. Raza Choudary
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The resultant $R(f, g)$ of two polynomials $f$ and $g$ is an irreducible polynomial such that $R(f, g) = 0$ if and only if the equations $f = 0$ and $g = 0$ have one common root.

When $g = f'/p$, then $D(f) = R(f, g)$ is called the discriminant of $f$ and the discriminant hypersurface $D_p = \{ f \in \mathbb{C}^p, D(f) = 0 \}$ can be identified to the discriminant of a versal deformation of the simple hypersurface singularity $A_{p-1}: x^p = 0$. In particular, the fundamental group $\pi = \pi_1(C^p \setminus D_p)$ is the famous braid group and $C^p \setminus D_p$ in fact a $K(\pi, 1)$ space.

Here we prove the following.

THEOREM. $\pi_1(C^{p+q} \setminus R_{p,q}) = \mathbb{Z}$.

As $C^p \setminus D_p$ can be regarded as a linear section of $C^{p+q} \setminus R_{p,q}$, this theorem shows that by a nongeneric linear section the fundamental group may change drastically, in contrast with the case of generic section.

Let $f = x^p + a_1 x^{p-1} + \cdots + a_p$ and $g = x^q + b_1 x^{q-1} + \cdots + b_q$ be two monic polynomials with complex coefficients of degree $p$ and $q$ respectively.

The resultant of them $R(f, g)$ is an irreducible polynomial in the coefficients $a_i, b_j$ such that $R(f, g) = 0$ if and only if the equations $f = 0$ and $g = 0$ have at least one common root. Explicitly, the resultant is given by the next formula (see for instance [5], p. 136):

$$R(f, g) = R(a, b) = \begin{vmatrix}
1 & a_1 & \cdots & a_p & \cdots & 0 & \cdots & 0 \\
1 & a_1 & \cdots & \cdots & a_p & \cdots & 0 \\
1 & \cdots & a_p \\
1 & b_1 & \cdots & b_q & \cdots & 0 & \cdots & 0 \\
1 & b_1 & \cdots & \cdots & b_q & \cdots & 0 \\
1 & \cdots & b_q \\
\end{vmatrix} \quad \text{q lines}
$$

When $g = f'/p$, then $D(f) = (f, g)$ is called the discriminant of the polynomial $f$ and the discriminant hypersurface $D_p = \{ f \in \mathbb{C}^p, D(f) = 0 \}$ has occurred several times in Singularity Theory, since it can be identified to the discriminant of a versal deformation of the simple hypersurface singularity $A_{p-1}: x^p = 0$, see for instance [1], [3], [9].
particular, the fundamental group $\pi = \pi_1(C^p \backslash D^p)$ is the famous braid group [1] (with $p$ strings) and $C^p \backslash D^p$ is in fact a $K(\pi, 1)$ space.

In this note we consider the analogous resultant hypersurface

$$R_{p,q} = \{(f, g) \in C^{p+q}; R(f, g) = 0\}$$

and prove the following.

**Theorem.** $\pi_1(C^{p+q} \backslash R_{p,q}) = \mathbb{Z}$.

Since $C^p \backslash D_p$ can be regarded as a linear section of $C^{p+q} \backslash R_{p,q}$, this theorem shows that by a nongeneric linear section the fundamental group may change drastically, in contrast with the case of generic section [4].

It is also interesting to note that the complements $F_{p,q} = C^{p+q} \backslash R_{p,q}$ have already occurred in an important topological problem [7], going back to certain questions in Control Theory [2]. In short, consider the space of rational real functions of the form

$$\phi = \frac{x^n + \alpha_1 x^{n-1} + \cdots + \alpha_n}{x^n + \beta_1 x^{n-1} + \cdots + \beta_n}$$

with $\alpha_i, \beta_j \in \mathbb{R}$ and the numerator and the denominator having no common root. Then $\phi$ induces a continuous map $P^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\} = P^1(\mathbb{C})$ of degree $n$ and its restriction to the equator $R \cup \{\infty\} = S^1 \subset S^2 = P^1(\mathbb{C})$ gives a map $S^1 \to S^1$ having degree $r$ such that $-n \leq r \leq n$ and $n - r \equiv 0 \mod 2$. Let $E_{n-r}$ denote the space of these mappings with $n$ and $r$ fixed, with the obvious topology. Then Segal has shown in [7] that $E_{n,r}$ is homeomorphic to $F_{p,q}$ with $p+q = n$ and $p - q = r$. He has also proved our Theorem in the special case $p = q$, by a method completely different from ours.

We derive our Theorem from some basic properties of the resultant hypersurface (which are also interesting in themselves) combined with a deep result of Lê-Saito [6] on the connectivity of the Milnor fiber of non-isolated singularity.

**Lemma 1.** $R \in \mathbb{C}[a, b]$, is a weighted homogeneous polynomial of degree $pq$ with respect to the weights $\text{wt}(a_i) = \text{wt}(b_i) = i$.

**Proof.** Note that the polynomial $t \cdot f = x^p + t a_1 x^{p-1} + \cdots + t^p a_p$

has as roots the elements $tx_i$, where $x_i$ are the roots of $f$, for any $t \in \mathbb{C}^\ast$. Then, using [5], p.137, we get $R(t \cdot f, t \cdot g) = \prod_{i,j}(tx_i - ty_j) = t^{pq} \prod_{i,j}(x_i - y_j) = t^{pq} R(f, g)$, where $y_j$ are the roots of $g$. \qed
The key remark in the proof is that the resultant hypersurface has a smooth normalization $\nu$ which can be described explicitly as follows:

$$\nu = \mathbb{C} \times \mathbb{C}^{p-1} \times \mathbb{C}^{q-1} \to R_{p,q} \subset \mathbb{C}^{p+q}$$

$$\nu(t, \alpha, \beta) = ((x-t)f_{\alpha}, (x-t)g_{\beta}),$$

where $f_{\alpha} = x^{p-1}+\alpha_1 x^{p-2}+\cdots+\alpha_{p-1}$, $g_{\beta} = x^{q-1}+\beta_1 x^{q-2}+\beta_1 x^{q-2}+\cdots+\beta_{q-1}$. Then $\nu$ is clearly surjective onto $R_{p,q}$ and the cardinal of a fiber $\nu^{-1}(f, g)$ is equal to the number of common roots of the equations $f = 0, g = 0$, counted without taking their multiplicities into account. Hence $\nu$ is a finite morphism which is generically one-to-one so that $\nu$ is indeed a normalization for $R_{p,q}$.

We use $\nu$ to investigate the singularities of the hypersurface $R_{p,q}$. To do this, we first compute the differential of $\nu$ at a point $(t_0, \alpha_0, \beta_0)$:

$$d\nu(t_0, \alpha_0, \beta_0)(t, \alpha, \beta) = ((x-t_0)(f_{\alpha} - x^{p-1}) - tf_{\alpha_0}, (x-t_0)(g_{\beta} - x^{q-1}) - tg_{\beta_0}).$$

Assume that $t_0$ is not a root for $f_{\alpha_0}$ and $g_{\beta_0}$ simultaneously. Then it follows that $d\nu(t_0, \alpha_0, \beta_0)$ is an injective linear map and its image (which is a hyperplane in the vector space $V$ of all the pairs $(A, B)$, with $A, B \in \mathbb{C}[x], \deg A \leq p-1, \deg B \leq q-1$) is given by the equation $f_{\alpha_0}(t_0)B(t_0) - g_{\beta_0}(t_0)A(t_0) = 0$.

Let $d(f, g)$ be the greatest common divisor of the polynomials $f$ and $g$. The above computation gives us the next

**Corollary 2.** The point $(f, g)$ is nonsingular on the hypersurface $R_{p,q}$ if and only if $\deg d(f, g) = 1$.

**Proof.** Use the fact that a point $(f, g) \in R_{p,q}$ is nonsingular if and only if $\nu^{-1}(f, g)$ consists of one point, say $y$, and the corresponding germ $\nu: (\mathbb{C}^{p+q}, y) \to (R_{p,q}, (f, g))$ is an isomorphism. \[Q.E.D.\]

We have also the more general result.

**Proposition 3.** Assume that $d(f, g) = (x-t_1)\ldots(x-t_s)$ is a product of $s$ linear distinct factors. Then the germ $(R_{p,q}, (f, g))$ consists of $s$ smooth hypersurface germs passing through $(f, g)$ with normal crossings.

**Proof.** In this case the fiber $\nu^{-1}(f, g)$ consists of $s$ points, say $y_k$ with $k = 1, \ldots, s$. Moreover, the germs $\nu_i: (\mathbb{C}^{p+q-1}, y_i) \to (R_{p,q}, (f, g)) \subset (\mathbb{C}^{p+q}, (f, g))$ induced by $\nu$ are all imbeddings and $H_i = \operatorname{im}(\nu_i)$ are pre-
cisely the (smooth) irreducible components of the germ \((R_{p,q}, (f, g))\).
The corresponding tangent spaces are \(T_k = T_{(f,g)}H_k; \bar{f}(t_k)B(t_k) - \bar{g}(t_k)A(t_k) = 0\) for \(K - 1, \ldots, s\) and \(\bar{f} = f/d(f, g), \bar{g} = g/d(f, g).\) The condition of normal crossing in this case means that \(\text{codim}(\bigcap_{k=1,s} T_k) = s.\)

But this intersection corresponds to the kernel of the following linear map. \(T: V \rightarrow \mathbb{C}^{p+q} \rightarrow \mathbb{C}[x]/(d(f, g)) \cong \mathbb{C}^s\) such that the \(k\)th component of \(T(A, B)\) is just the evaluation on \(t_k\) of \((\bar{f} \cdot B - \bar{g} \cdot A),\) for \(k = 1, \ldots, s.\) It is easy to check that \(T\) is a surjective map and hence \(\text{codim}(\bigcap_{k=1,s} T_k) = \text{codim}(\ker T) = s.\)

**Corollary 4.** The hypersurface \(R_{p,q}\) has only normal crossings singularities in codimension 1 and hence \(\pi_1(C^{p+q} \setminus R_{p,q}) = \mathbb{Z}.\)

**Proof.** The singularities of \(R_{p,q}\) which are not normal crossings (as described in Proposition 3) lie in the image of the map
\[
\tau: \mathbb{C} \times \mathbb{C}^{p-2} \times \mathbb{C}^{q-2} \rightarrow R_{p,q},
\]
\[
\tau(t, \alpha, \beta) = ((x - t)^2 \tilde{f}_\alpha, (x - t)^2 \tilde{g}_\beta)
\]
with \(\tilde{f}_\alpha, \tilde{g}_\beta\) having a meaning similar to \(f_\alpha, g_\beta.\) But \(\dim(\text{im } \tau) \leq p+q-3 = \dim R_{p,q} - 2\) which proves the first assertion above. Next consider the fibration \(F \rightarrow \mathbb{C}^{p+q} \setminus R_{p,q} \rightarrow \mathbb{C}^*\) with \(F = F^{-1}(1) = \{(f, g) \in \mathbb{C}^{p+q}; R(f, g) = 1\}.\) Using the weighted homogeneity of \(R\) given by Lemma 1, we can identify this fibration with the Milnor fibration of the hypersurface singularity \((R_{p,q}, (x^p, y^q)).\) It follows by [6] that \(\prod_1(F) = 0\) and hence we get an isomorphism
\[
R_\# = \prod_1(C^{p+q} \setminus R_{p,q}) \rightarrow \prod_1(\mathbb{C}^*) = \mathbb{Z}.
\]
This ends the proof of this corollary as well as giving a more precise version of our Theorem above.

**Remark 5.** There is a natural \(\mathbb{C}\)-action on \(\mathbb{C}^{p+q}\) leaving the resultant hypersurface \(R_{p,q}\) invariant. Namely we define the translation of an element \((f, g)\) by the complex number \(\lambda\) to be the element \((f^\lambda, g^\lambda)\) where
\[
f^\lambda = \prod_{i=1,p} (x - x_i - \lambda), \quad g^\lambda = \prod_{j=1,q} (x - y_j - \lambda)
\]
with \( x_i \) (resp. \( y_j \)) being the roots of \( f \) (resp. \( g \)). Since the hyperplane \( a_1 = 0 \) is clearly transversal to all the \( C \)-orbits, it follows that

\[
R_{p,q} = \overline{R}_{p,q} \times C \quad \text{with } \overline{R}_{p,q} = R_{p,q} \cap \{a_1 = 0\}.
\]

The first non-trivial case of a resultant hypersurface is for \( p = q = 2 \). Then \( \overline{R}_{2,2} \) is just the Whitney umbrella \( W: \overline{b}_2^2 - b_1^2a_2 = s \), with \( \overline{b}_2 = b_2 - a_2 \), called also a \( D_\infty \)-surface singularity for a pinch point. It follows that \( C^4 \setminus R_{2,2} = (C^3 \setminus W) \times C \) and the homotopy groups of \( C^3 \setminus W \) can be derived from the Milnor fibration \( F_\infty \rightarrow C^3 \setminus W \rightarrow C^* \) associated to the \( D_\infty \)-singularity [8]. It is known that \( F_\infty \) has the homotopy type of the 2-sphere \( S^2 \) and hence

\[
\prod_k (C^4 \setminus R_{2,2}) = \prod_k (S^2) \quad \text{for } k \geq 2.
\]

In particular \( C^4 \setminus R_{2,2} \) is not a \( K(Z, 1) \) space, since \( \Pi_2(C^4 \setminus R_{2,2}) = \mathbb{Z} \).

**References**


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**Central Washington University**

**Ellensburg, WA 98926**
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