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**EXTENSION THEOREMS FOR FUNCTIONS OF VANISHING  
MEAN OSCILLATION**

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## EXTENSION THEOREMS FOR FUNCTIONS OF VANISHING MEAN OSCILLATION

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A locally integrable function is said to be of vanishing mean oscillation (VMO) if its mean oscillation over cubes in  $\mathbf{R}^d$  converges to zero with the volume of the cubes. We establish necessary and sufficient conditions for a locally integrable function defined on a bounded measurable set of positive measure to be the restriction to that set of a VMO function.

**1. Introduction.** Let  $F$  be a locally integrable function on  $\mathbf{R}^d$  and let  $Q$  be a cube in  $\mathbf{R}^d$  with sides parallel to the axes. (We denote the set of all such cubes in  $\mathbf{R}^d$  by  $\mathfrak{F}'$ .) We denote the Lebesgue measure of  $Q$  by  $|Q|$  and the length of  $Q$  by  $l(Q)$ . We denote the average of  $F$  on  $Q$  by  $F_Q$ ; that is  $F_Q = \frac{1}{|Q|} \int_Q F dt$ . We say  $F$  is of bounded mean oscillation (abbreviated  $\text{BMO}(\mathbf{R}^d)$  or simply  $\text{BMO}$ ) if

$$(1.1) \quad \sup_{Q \in \mathfrak{F}'} \frac{1}{|Q|} \int_Q |F - F_Q| < \infty.$$

We denote this supremum by  $\|F\|_*$ .  $\|\cdot\|_*$  defines a norm on  $\text{BMO}$  and  $\text{BMO}$  is a Banach space with respect to this norm. (We identify functions which differ by a constant.) If in (1.1) we restrict the cubes to be dyadic we obtain the space dyadic-BMO and we denote the corresponding norm by  $\|\cdot\|_{*,d}$ . (By a dyadic cube we mean a cube of the form  $Q = \{k_j < x_j < (k_j + 1)2^{-n}; 1 \leq j \leq d\}$  where  $n$  and  $k_j$ ,  $1 \leq j \leq d$ , are integers.) We will denote the set of dyadic cubes of length  $2^{-n}$  by  $D_n$  and  $Q_0$  will denote the dyadic unit cube. The function space  $\text{BMO}$  was introduced in 1961 by John and Nirenberg [7] who proved the following fundamental theorem:

**THEOREM 1.1.** *Let  $F$  be a locally integrable function on  $\mathbf{R}^d$ , and for each  $n \in \mathbf{Z}$  define:*

$$\bar{\mu}_n(F) = \inf \left\{ \frac{1}{\lambda} : \sup_{l(Q) \leq 2^{-n}} \inf_{a \in \mathbf{R}} \frac{1}{|Q|} \int_Q e^{\lambda|F-a|} < 2 \right\}.$$

Then,

- (1)  $F \in \text{BMO}$  if and only if
- (2)  $\sup_{n \in \mathbf{Z}} \bar{\mu}_n(F) < \infty$ .

The implication (2)  $\Rightarrow$  (1) is straightforward while (1)  $\Rightarrow$  (2) is obtained by means of a Calderon-Zygmund stopping time argument. (This result and other basic results on BMO can be found in [4] and [12].)

A closed subspace of BMO that we will be mainly concerned with, is the space of functions of vanishing mean oscillation (VMO) which was introduced by Sarason in [11] and is defined as:

$$\text{VMO} = \left\{ F \in \text{BMO} : \lim_{\delta \rightarrow 0} \left( \sup_{\substack{Q \in \mathfrak{S}' \\ l(Q) < \delta}} \frac{1}{|Q|} \int_Q |F - F_Q| \right) = 0 \right\}$$

Equivalently, by the theorem of John and Nirenberg,  $F \in \text{VMO}$  if and only if  $F \in \text{BMO}$  and  $\lim_{n \rightarrow \infty} \bar{\mu}_n(F) = 0$ .

If  $E$  is a Lebesgue measurable subset of  $\mathbf{R}^d$  of positive measure (throughout we will always assume  $E$  has positive measure), we can ask for necessary and sufficient conditions for a locally integrable function defined on  $E$  to be the restriction to  $E$  of a function in  $\text{BMO}(\mathbf{R}^d)$ . This characterization was given by Wolff [15] and is based upon a technique due to Rubio de Francia [10] which generalizes Jones' factorization theorem for  $A_p$ -weights [8]. The main result of this paper is to obtain a similar characterization for VMO functions and this is the content of the following theorem:

**THEOREM I.** *Let  $E$  be a bounded measurable subset of  $\mathbf{R}^d$  and let  $f$  be a locally integrable function defined on  $E$ . For each  $n \in \mathbf{Z}$  define:*

$$\mu_n(f) = \inf \left\{ \frac{1}{\lambda} : \sup_{l(Q) \leq 2^{-n}} \inf_{a \in \mathbf{R}} \frac{1}{|Q|} \int_{Q \cap E} e^{\lambda|f-a|} < 2 \right\}.$$

*Then the following are equivalent:*

- (1)  $f$  is the restriction of a VMO function on  $\mathbf{R}^d$  to  $E$
- (2)  $\sup_{n \in \mathbf{Z}} \mu_n(f) < \infty$  and  $\lim_{n \rightarrow \infty} \mu_n(f) = 0$ .

The proof of this theorem consists of two parts. In the first part we obtain a dyadic-VMO extension of  $f$ . We then obtain a dyadic-VMO extension for each translation of  $f$  and  $E$  and the second part of the proof consists of averaging these extensions to obtain a VMO extension of  $f$ .

Throughout  $C$  will denote a positive constant which will be independent of the variables in the equation in which it occurs but which may be different at each occurrence.

**2. Preliminary Results.** Let  $E$  be a measurable subset of  $\mathbf{R}^d$  and let  $\mathfrak{F}$  be a collection of cubes in  $\mathbf{R}^d$  with  $E \subseteq \bigcup\{Q: Q \in \mathfrak{F}\}$ .

**DEFINITION.** (1) if  $F$  is a locally integrable function on  $\mathbf{R}^d$ , we define the maximal function of  $F$  relative to  $\mathfrak{F}$  by

$$(M_{\mathfrak{F}}F)(x) = \sup_{\substack{x \in Q \\ Q \in \mathfrak{F}}} \frac{1}{|Q|} \int_Q F dt \quad \text{for all } x \in \bigcup\{Q: Q \in \mathfrak{F}\}.$$

If  $\mathfrak{F} = \mathfrak{F}'$ , this is the usual Hardy-Littlewood maximal function.

(2) If  $f$  is a locally integrable function on  $E$ , we define the maximal function of  $f$  relative to  $\mathfrak{F}$  by

$$(m_{\mathfrak{F}}f)(x) = \sup_{\substack{x \in Q \\ Q \in \mathfrak{F}}} \frac{1}{|Q|} \int_{Q \cap E} f dt \quad \text{for all } x \in E.$$

**DEFINITION.** (1) Let  $w$  be a positive locally integrable function on  $E$  and let  $1 < p < \infty$ . We say  $w$  is an  $A_p(E)$ -weight relative to  $\mathfrak{F}$  if

$$(2.1) \quad \sup_{Q \in \mathfrak{F}} \left( \frac{1}{|Q|} \int_{Q \cap E} w dt \right) \left( \frac{1}{|Q|} \int_{Q \cap E} \left( \frac{1}{w} \right)^{1/(p-1)} dt \right)^{p-1} < \infty$$

and we denote the collection of all such weights by  $A_p(E, \mathfrak{F})$ . If  $E = \mathbf{R}^d$  and  $\mathfrak{F} = \mathfrak{F}'$  we abbreviate  $A_p(E, \mathfrak{F})$  by  $A_p$  and say  $w$  is an  $A_p$ -weight.

(2) We say a positive locally integrable function  $w$  is an  $A_1(E)$ -weight relative to  $\mathfrak{F}$  if

$$\sup_{Q \in \mathfrak{F}} \left\{ \left( \frac{1}{|Q|} \int_{Q \cap E} w dt \right) \operatorname{ess\,sup}_{x \in Q} \frac{1}{w(x)} \right\} < \infty.$$

We denote the collection of all such weights by  $A_1(E, \mathfrak{F})$ .

We record some properties of  $A_p(E, \mathfrak{F})$ -weights in the following proposition.

**PROPOSITION 2.1.**

(i) If  $w \in A_p(E, \mathfrak{F})$  then  $w \in A_r(E, \mathfrak{F})$  for all  $r > p$  and  $(\frac{1}{w})^{1/(p-1)} \in A_{p/(p-1)}$ .

(ii) If  $w_1, w_2 \in A_1(E, \mathfrak{F})$  then  $w_1 w_2^{1-p} \in A_p(E, \mathfrak{F})$  for all  $1 < p < \infty$ .

(iii) If  $w \in A_p$  then  $F = \log w \in \text{BMO}$ . By the theorem of John and Nirenberg (Theorem 1.1), if  $F \in \text{BMO}$  there exists  $\delta > 0$  such that  $e^{\delta F} \in A_p$ .

(iv) We mention here the following result of Coifman and Rochberg [3]: If  $F \in L_1^{\log}(\mathbf{R}^d)$  and  $M_{\mathfrak{F}} f(x) < \infty$  a.e., then for each  $0 < \delta < 1$ ,  $(M_{\mathfrak{F}} f)^{\delta} \in A_1$ .

**DEFINITION.** Let  $1 < p < \infty$  and let  $w \in A_p(E, \mathfrak{F})$ . We say  $w$  satisfies a reverse Hölder inequality if there exists  $\varepsilon > 0$  such that  $w^{1+\varepsilon} \in A_p(E, \mathfrak{F})$ .

**REMARK.** If  $w \in A_p$  then  $w$  satisfies a reverse Hölder inequality with  $\varepsilon$  depending on  $p$  and supremum in (2.1). This fact may be deduced from (2.1) by a repeated application of a Calderon-Zygmund stopping time argument. See [1], [9].

The next theorem is a variation of a theorem of Muckenhoupt [9]. The proof is the same and so will be omitted.

**THEOREM 2.1.** Let  $1 < p < \infty$  and let  $w \in A_p(E, \mathfrak{F})$  where  $\mathfrak{F}$  is a collection of dyadic cubes or  $\mathfrak{F} = \mathfrak{F}'$ . If  $w$  satisfies a reverse Hölder inequality then there exists a constant  $C > 0$  such that

$$(2.2) \quad \int_E \{m_{\mathfrak{F}}(f)\}^p w \, dx \leq C \int_E |f|^p w \, dx$$

and

$$(2.3) \quad \int_E \{m_{\mathfrak{F}}(f)\}^q \left(\frac{1}{w}\right)^{q/p} dx \leq C \int_E |f|^q \left(\frac{1}{w}\right)^{q/p} w \, dx$$

where  $q = p/(p - 1)$ .

By a theorem of Rubio de Francia [10], (2.2) and (2.3) imply that there exist  $w_1, w_2 \in A_1(E, \mathfrak{F})$  such that  $w = w_1 w_2^{1-p}$ . We summarize what we need from the above in the following corollary.

**COROLLARY 2.1.** If  $w \in A_2(E, \mathfrak{F})$  and  $w$  satisfies a reverse Hölder inequality then there exist  $w_1, w_2 \in A_2(E, \mathfrak{F})$  such that  $w = w_1/w_2$ .

We are now in a position to give the BMO extension theorem of Wolff [15].

**THEOREM 2.2.** *If  $f$  is measurable on  $E$ , then the following are equivalent:*

- (1)  $f$  is the restriction of a BMO function on  $\mathbf{R}^d$  to  $E$ ;
- (2)  $\exists \lambda > 0$  such that

$$\sup_{Q \in \mathfrak{F}'} \frac{1}{|Q|} \int_{Q \cap E} e^{\lambda|f - (1/|Q \cap E|) \int_{Q \cap E} f|} < \infty;$$

- (3)  $\exists \lambda > 0$  such that

$$\sup_{Q \in \mathfrak{F}'} \inf_{a \in \mathbf{R}} \frac{1}{|Q|} \int_{Q \cap E} e^{\lambda|f - a|} < \infty.$$

We give the proof as it provides one of the basic steps needed in proving Theorem I.

*Proof.* The equivalence of (2) and (3) are straightforward and the implication (1)  $\Rightarrow$  (3) is similar to the proof of (1)  $\Rightarrow$  (2) in Theorem I which we give in §3.

(2)  $\Rightarrow$  (1): (2) implies that  $w = e^{\lambda f/2} \in A_2(E, \mathfrak{F}')$  and satisfies a reverse Hölder inequality. Hence by Corollary 2.1, there exist  $w_1, w_2 \in A_1(E, \mathfrak{F}')$  such that  $w = e^{\lambda f/2} = w_1/w_2$ . Define  $W_i = M_{\mathfrak{F}'}(\chi_E w_i)^{1/2}$ ,  $i = 1, 2$ . By Proposition 2.1 (iv),  $W_1, W_2 \in A_1$ , i.e.  $\exists C > 0$  such that  $W_i \leq M_{\mathfrak{F}'}(W_i) \leq C W_i$ ,  $i = 1, 2$ . Since  $M_{\mathfrak{F}'}(\chi_E w_i) = m_{\mathfrak{F}'}(w_i)$ , a.e. on  $E$ , ( $i = 1, 2$ ), it follows that  $\exists g \in L_\infty(\mathbf{R}^d)$ ,  $g > 0$ , such that

$$g \left( \frac{W_1}{W_2} \right)^2 = \frac{w_1}{w_2} = e^{\lambda f/2} \quad \text{a.e. on } E.$$

Define  $F = \frac{2}{\lambda} \{\log g + 2 \log(W_1/W_2)\}$ . Then  $F = f$  a.e. on  $E$  and by Proposition 2.1 (ii) and (iii),  $F \in \text{BMO}(\mathbf{R}^d)$ . □

Finally we prove 2 lemmas which are needed in the next section. The first is a variation of the theorem of Coifman and Rochberg mentioned above while the second is based on Lemma 2.2 in [5].

**LEMMA 2.1.** *Let  $n \in \mathbf{N}$  and let  $\mathfrak{F} = \{Q: Q \text{ dyadic, } Q \subseteq Q_0, l(Q) \geq 2^{-n}\}$ . Let  $g \in L_1^{\text{loc}}(Q_0)$  and suppose  $(M_{\mathfrak{F}}g)(x) < \infty$  a.e. Then for all  $0 < \delta < 1$ , there exists  $C_\delta > 0$ , depending only on  $\delta$  and the dimension, such that*

$$M_{\mathfrak{F}} \left( (M_{\mathfrak{F}}g)^\delta \right) (x) \leq C_\delta (M_{\mathfrak{F}}g)^\delta(x).$$

*Proof.* It suffices to show  $\exists C_\delta > 0$  such that  $\forall Q \in \mathfrak{F}$ ,

$$(2.4) \quad \frac{1}{|Q|} \int_Q (M_{\mathfrak{F}}g)^\delta dt \leq C_\delta \inf_{x \in Q} (M_{\mathfrak{F}}g)^\delta(x).$$

This result is established by Coifman and Rochberg in [3] for the usual Hardy-Littlewood maximal function. The proof of (2.4) for  $Q \in \mathfrak{F}$  with  $l(Q) < \frac{1}{2}$  is essentially the same; the relevant property of such cubes is the following:

$\exists Q_1 \in \mathfrak{F}, l(Q_1) = 2l(Q)$  such that  $Q \subseteq Q_1$  and whenever  $Q_2 \in \mathfrak{F}$  satisfies  $|Q_2 \cap Q| > 0$  and  $|Q_2 \cap Q_1^c| > 0$  we have  $Q \subseteq Q_2$ .

For  $Q \in \mathfrak{F}$  with  $l(Q) \geq \frac{1}{2}$ , (2.4) follows by integrating the weak-type estimate for the Hardy-Littlewood maximal function.

LEMMA 2.2. *Let  $E$  be a measurable subset of the unit cube  $Q_0$  with  $0 < |E| < 1$ . Then if  $0 < \beta < \log 1/|E|$ ,  $\exists H \in \text{dyadic-VMO}(Q_0)$ ,  $\|H\|_{*,d} \leq C$  such that:*

- (1)  $0 \leq H \leq \beta, \text{supp}(H) \subseteq Q_0, H = \beta$  on  $E$ ,
- (2)  $\sup_{Q: l(Q) \geq 1} \frac{1}{|Q|} \int_Q H dt \leq C$ .

*Proof.* This is a version of Lemma 2.2 in [5] where  $H$  is constructed in  $\text{BMO}(Q_0)$  satisfying (1) and (2). The argument below is the dyadic version of this construction. (See also the proof of Lemma 1.2 in [14] where a similar result is obtained.) Without loss of generality we may assume  $|E| \leq 2^{-4d}$  (otherwise we may take  $H$  to be constant). Let  $\{Q_i\}_{i \geq 1}$  be the maximal subcubes of  $Q$  for which  $|Q_i \cap E| > \frac{1}{2}|Q_i|$ . Let  $n_0 = 1$  and for each  $j \geq 1$  choose  $n_j$  so that

$$\left| \log \sum_{i \geq n_j} |Q_i| \right| > 2^j \left| \log \left| \bigcup Q_i \right| \right|$$

and define  $G^{(j)} = \{Q_i: n_j \leq i < n_{j+1}\}$  so that  $\sum_{Q \in G^{(j)}} |Q| \leq 4^{-\beta_j d} |Q_0|$  where  $\beta_j = 2^j \beta_0, \beta_0 = \lceil \frac{1}{2d} \lceil \log \left| \bigcup Q_i \right| \rceil \rceil$  and  $\lceil \cdot \rceil$  denotes the greatest integer function.

For each  $j$  we now construct a sequence of generation  $\{G_i^{(j)}\}_{i=1}^{\beta_j}$  as follows:

- (1) Set  $G_1^{(j)} = G^{(j)}$ .
- (2) Suppose  $G_i^{(j)}$  has been defined. For each  $Q \in G_i^{(j)}$  let  $G^{(k)}$  denote that dyadic cube of length  $2^k l(Q)$  containing  $Q$ . Choose  $k$  minimal so that

$$\sum \left\{ |Q_i|: Q_i \in G_i^{(j)}, Q_i \subseteq Q^{(k)} \right\} < 2^{-d} |Q^{(k)}|.$$

We define  $G_{i+1}^{(j)}$  to be the maximal cubes in  $\{Q_r^{(k)} : Q_r \in G_i^{(j)}\}$ . We note that

$$\sum \{ |Q| : Q \in G_{\beta_{j-i}}^{(j)} \} \leq 2^{-d} \sum \{ |Q| : Q \in G_{\beta_{j-i-1}}^{(j)} \}$$

and hence if  $Q$  is any dyadic cube then

$$\sum_{i=1}^{\beta_j} \sum_{\substack{Q_k \in G_i^{(j)} \\ Q_k \subseteq Q}} |Q_k| \leq C|Q|.$$

Hence by an argument similar to the proof of Lemma 2.1 in [5] we see that the function

$$a_j = \sum_{i=1}^{\beta_j} \sum_{Q_k \in G_i^{(j)}} \chi_Q$$

belongs to dyadic-BMO( $Q_0$ ) with  $\|a_j\|_{*,d} \leq C$ . Furthermore  $a_j = \beta_j$  on  $\cup\{Q_k : Q_k \in G_1^{(j)}\}$  and since  $a_j$  is constant on small dyadic cubes, we have in fact that  $a_j \in$  dyadic-VMO( $Q_0$ ). We now define

$$H = \frac{\beta}{\beta_0} \min \left( \sum_{j \geq 0} \frac{a_j}{2^j}, \beta_0 \right). \quad \square$$

REMARK. We note that Lemma 2.2 may be established without using the above construction. Indeed, it is not hard to see that the function

$$f = \begin{cases} \beta_j & \text{on } \cup\{Q : Q \in G_1^{(j)}\}, \\ 0 & \text{on } Q_0^c \end{cases}$$

satisfies the hypothesis (3) of Theorem 2.2 with

$$E = \cup\{Q : Q \in G_1^{(j)}\} \cup Q_0^c$$

and with  $\mathfrak{F}'$  replaced by those dyadic cubes of length  $\geq \min\{l(Q) : Q \in G_1^{(j)}\}$  and so can be extended to a dyadic-BMO function  $F$  with  $\|F\|_{*,d} \leq C$  and which is constant on small dyadic cubes. This will become clear after reading §3(i) below.

**3. Proof of Theorem I.** If  $Q$  is a cube and  $r > 0$  we will denote by  $rQ$  the cube with the same center as  $Q$  and of length  $rl(Q)$ .

Without loss of generality we will assume  $E$  is contained in  $\frac{1}{3}Q_0$  where  $Q_0$  is the unit cube in  $\mathbf{R}^d$ .

(1)  $\Rightarrow$  (2): Let  $F \in \text{VMO}$  with  $F\chi_E = f$  and for each  $n \in \mathbf{Z}$  define

$$\begin{aligned} \bar{\mu}_n(F) &= \inf \left\{ \frac{1}{\lambda} : \sup_{l(Q) \leq 2^{-n}} \inf_{a \in \mathbf{R}} \frac{1}{|Q|} \int_Q e^{\lambda|F-a|} < 2 \right\}, \\ \bar{\mu}_n^*(F) &= \inf \left\{ \frac{1}{\lambda} : \sup_{l(Q) \leq 2^{-n}} \frac{1}{|Q|} \int_Q e^{\lambda|F-F_Q|} < 2 \right\}, \\ \|F\|_{*,n} &= \sup_{l(Q) \leq 2^{-n}} \frac{1}{|Q|} \int_Q |F - F_Q|. \end{aligned}$$

Since  $F \in \text{BMO}$ ,  $\exists C > 0$  such that  $\forall n \geq 0$ ,

$$\|F\|_{*,n} \leq C \quad \text{and} \quad \lim_{n \rightarrow \infty} \|F\|_{*,n} = 0.$$

By Theorem 1.1,  $\exists C > 0$  such that whenever  $0 < \lambda < C/\|F\|_{*,n}$  we have

$$\sup_{l(Q) \leq 2^{-n}} \frac{1}{|Q|} \int_Q e^{\lambda|F-F_Q|} < 2.$$

Hence  $\bar{\mu}_n^*(F) \leq \|F\|_{*,n}/C$ . Since  $\bar{\mu}_n(F) \leq \bar{\mu}_n^*(F)$  and  $\mu_n(f) \leq \bar{\mu}_n(F)$ , it follows that  $\mu_n(f) \leq C$  for  $n = 0, 1, 2, \dots$  and  $\lim_{n \rightarrow \infty} \mu_n(f) = 0$  and this proves (1)  $\Rightarrow$  (2).

*Proof of (2)  $\Rightarrow$  (1):*

Part (i): Extension to dyadic-VMO.

Condition (2) in Theorem I implies there exists a sequence  $\{\lambda_n\}_{n \geq 0}$ ,  $0 < \lambda_n \uparrow \infty$  such that

$$\sup_{l(Q) \leq 2^{-n}} \frac{1}{|Q|} \int_{Q \cap E} e^{\lambda_n|f-f_{Q \cap E}|} < 2.$$

Let  $n_0 = 0$  and define a sequence  $\{n_k\}_{k \geq 1} \subseteq \mathbf{N}$  by the condition  $\lambda_n > 2\lambda_{n_k}$  if and only if  $n \geq n_{k+1}$ . To simplify the notation we will write  $\lambda_k$  for  $\lambda_{n_k}$ . Now define for each  $k \geq 0$

$$\mathfrak{F}_k = \{Q : Q \text{ dyadic, } Q \subseteq Q_0, l(Q) \geq 2^{-n_k}\}.$$

For each  $n = 0, 1, 2, \dots$  we define  $f_n = \sum_{Q \in \mathfrak{D}_n} f_{Q \cap E} \chi_{Q \cap E}$ .

The idea of the proof is as follows: We write

$$f = f_0 + \sum_{k \geq 0} (f_{n_{k+1}} - f_{n_k})$$

and note that in order to extend  $f$  it suffices to extend each of the functions  $f_0, f_{n_{k+1}} - f_{n_k}, k \geq 0$ . These functions are constant at every point of  $E$  which belong to the same small dyadic cube and so

we would like to extend each of these functions to dyadic-BMO functions which are constant on small dyadic cubes and in particular will therefore be in dyadic-VMO. To do this we proceed as in the proof of Theorem 2.2; that is, we first show (in Lemma 3.1) that

$$e^{(\lambda_0/4)f_0} \in A_2(E, \mathfrak{F}_0) \quad \text{and} \quad e^{\lambda_k/4(f_{n_{k+1}}-f_{n_k})} \in A_2(E, \mathfrak{F}_{k+1})$$

and satisfy a reverse Hölder inequality. We then factor and extend these weights using a truncated dyadic maximal function instead of the full dyadic maximal function. We do this to ensure that the  $A_1(Q_0, \mathfrak{F}_k)$ -weights we obtain using Lemma 2.1 (that is, the weights  $W_1, W_2$  in the notation of Theorem 2.2) will be constant on small dyadic cubes. The difficulty that now arises is that we cannot conclude as in Theorem 2.2 that the quotient of each factored  $A_1(E, \mathfrak{F}_k)$ -weight with the corresponding truncated maximal function is uniformly bounded on  $E$ . To accommodate this possibly large quotient into an appropriate VMO function we will appeal to Lemma 2.2. We proceed with the dyadic extension:

**LEMMA 3.1.** *There exists  $C > 0$ , depending only on the dimension, such that for all  $k \geq 0$ ,*

(1)

$$\sup_{Q \in \mathfrak{F}_{k+1}} \left( \frac{1}{|Q|} \int_{Q \cap E} e^{(\lambda_k/2)(f_{n_{k+1}}-f_{n_k})} \right) \left( \frac{1}{|Q|} \int_{Q \cap E} e^{-(\lambda_k/2)(f_{n_{k+1}}-f_{n_k})} \right) \leq C,$$

(2)

$$\sup_{Q \in \mathfrak{F}_0} \left( \frac{1}{|Q|} \int_{Q \cap E} e^{(\lambda_0/2)f_0} \right) \left( \frac{1}{|Q|} \int_{Q \cap E} e^{-(\lambda_0/2)f_0} \right) \leq C.$$

*Proof.* Let  $Q$  be a dyadic cube with  $l(Q) \geq 2^{-n_j}$  and let  $\{Q_i\}$  be those dyadic subcubes of  $Q$  of length  $2^{-n_j}$ . Then for all  $\lambda \leq \lambda_j$ ,

$$\begin{aligned} \frac{1}{|Q|} \int_{Q \cap E} e^{\lambda|f-f_{n_j}|} &= \sum \frac{|Q_i|}{|Q|} \left( \frac{1}{|Q_i|} \int_{Q_i \cap E} e^{\lambda|f-f_{n_j}|} \right) \\ &\leq \sum \frac{|Q_i|}{|Q|} \left( \frac{1}{|Q_i|} \int_{Q_i \cap E} e^{\lambda|f-f_{Q_i \cap E}|} \right) \leq C \end{aligned}$$

and hence

$$(3.1) \quad \frac{1}{|Q|} \int_{Q \cap E} e^{\lambda(f-f_{n_j})} \frac{1}{|Q|} \int_{Q \cap E} e^{-\lambda(f-f_{n_j})} \leq C.$$

Now if  $Q$  is a dyadic cube with  $l(Q) \leq 2^{-n_j}$  then for all  $\lambda \leq \lambda_j$  we are given

$$\frac{1}{|Q|} \int_{Q \cap E} e^{\lambda(f-f_{Q \cap E})} \frac{1}{|Q|} \int_{Q \cap E} e^{-\lambda(f-f_{Q \cap E})} \leq C$$

and hence

$$(3.2) \quad \left( \frac{1}{|Q|} \int_{Q \cap E} e^{\lambda f} \right) \left( \frac{1}{|Q|} \int_{Q \cap E} e^{-\lambda f} \right) \leq C.$$

We note that if  $j = 0$ , (3.2) holds for all  $Q \in \mathfrak{F}_0$  and for all  $\lambda \leq \lambda_0$ . Now (3.2) implies that for all  $Q \in \mathfrak{F}_{j+1}$ ,  $l(Q) \leq 2^{-n_j}$  and for all  $\lambda \leq \lambda_j$

$$(3.3) \quad \frac{1}{|Q|} \int_{Q \cap E} e^{\lambda(f-f_{n_j})} \frac{1}{|Q|} \int_{Q \cap E} e^{-\lambda(f-f_{n_j})} \leq C.$$

Since

$$\begin{aligned} & \frac{1}{|Q|} \int_{Q \cap E} e^{(\lambda/2)(f_{n_{k+1}}-f_{n_k})} \\ & \leq \left( \frac{1}{|Q|} \int_{Q \cap E} e^{\lambda(f-f_{n_k})} \right)^{1/2} \left( \frac{1}{|Q|} \int_{Q \cap E} e^{-\lambda(f-f_{n_{k+1}})} \right)^{1/2} \end{aligned}$$

(and similarly for  $\frac{1}{|Q|} \int_{Q \cap E} e^{-(\lambda/2)(f_{n_{k+1}}-f_{n_k})}$ ) we see that (1) follows from (3.1) and (3.3).

Now (3.1), (3.2) and Hölder's inequality imply that for all  $\lambda \leq \lambda_0/2$  and for all  $Q \in \mathfrak{F}_0$

$$\left( \frac{1}{|Q|} \int_{Q \cap E} e^{\lambda f_0} \right) \left( \frac{1}{|Q|} \int_{Q \cap E} e^{-\lambda f_0} \right) \leq C$$

and this gives (2) which completes the proof of the lemma. □

To simplify the notation we set  $M_k(g) = M_{\mathfrak{F}_k}(g)$  and  $m_k(g) = m_{\mathfrak{F}_k}(g)$ . Corollary 2.1 implies the following:

For each  $k = 0, 1, 2, \dots$  there exist  $u_k, v_k \in A_1(E, \mathfrak{F}_k)$  such that

$$\frac{u_0}{v_0} = \exp \left( \frac{\lambda_0}{4} f_0 \right)$$

and

$$\frac{u_k}{v_k} = \exp \left( \frac{\lambda_{k-1}}{4} (f_{n_k} - f_{n_{k-1}}) \right) \quad \forall k = 1, 2, \dots$$

Furthermore, since the  $A_2(E, \mathfrak{F}_k)$ -weights and the maximal function  $m_k(\cdot)$  are constant at every point of  $E$  which belong to the same dyadic cube of length  $2^{-n_k}$ , the proof of Corollary 2.1 given in [2] shows that the same is true for the  $A_1(E, \mathfrak{F}_k)$ -weights  $u_k$  and  $v_k$ .

Now for each  $k = 0, 1, 2, \dots$  and for each  $x \in \mathbf{R}^d$  we define

$$\begin{aligned} U_k(x) &= M_k(\chi_E u_k)(x), \\ V_k(x) &= M_k(\chi_E v_k)(x). \end{aligned}$$

Then, for all  $x \in E$ ,

$$\exp\left(\frac{\lambda_{k-1}}{4}(f_{n_k} - f_{n_{k-1}})(x)\right) = \frac{U_k(x)}{V_k(x)} w_k(x) \quad \forall k \geq 1$$

and

$$\exp\left(\frac{\lambda_0}{4} f_{k_0}(x)\right) = \frac{U_0(x)}{V_0(x)} w_0(x)$$

where

$$w_k = \frac{u_k}{v_k} \frac{m_k(v_k)}{m_k(u_k)} \quad \forall k \geq 0.$$

Hence,

$$f_{n_k} - f_{n_{k-1}} = \frac{8}{\lambda_{k-1}} \log\left(\frac{U_k}{V_k}\right)^{1/2} + \frac{4}{\lambda_{k-1}} \log w_k$$

and

$$f_0 = \frac{8}{\lambda_0} \log\left(\frac{U_0}{V_0}\right)^{1/2} + \frac{4}{\lambda_0} \log w_0.$$

Now Lemma 2.1 implies  $\exists C > 0$  such that

$$M_k(U_k^{1/2}) \leq C U_k^{1/2} \quad \text{and} \quad M_k(V_k^{1/2}) \leq C V_k^{1/2}$$

and so by Proposition 2.1 (ii),  $(U_k/V_k)^{1/2} \in A_2(Q_0, \mathfrak{F}_k)$ .

Then, by Proposition 2.1 (iii), we conclude that

$$\sup_{Q \in \mathfrak{F}_k} \frac{1}{|Q|} \int_Q \left| \log\left(\frac{U_k}{V_k}\right) - \left(\log\left(\frac{U_k}{V_k}\right)\right)_Q \right| \leq C.$$

In particular since  $U_k, V_k$  are constant on dyadic cubes of length  $2^{-n_k}$ , we have  $\log(U_k/V_k) \in \text{dyadic-VMO}(Q_0)$ .

Finally we need to extend the functions  $\log w_k$  to dyadic-VMO  $(Q_0)$ . Let  $Q_k$  be a dyadic cube of length  $2^{-n_k}$  with  $|Q_k \cap E| > 0$ . Then since  $u_k$  and  $v_k$  are constant on  $Q_k \cap E$  we have for all  $x \in Q_k \cap E$

$$u_k(x) \leq \frac{1}{|Q_k \cap E|} \int_{Q_k \cap E} u_k(t) dt \leq \frac{|Q_k|}{|Q_k \cap E|} m_k(u_k)(x)$$

and similarly

$$v_k(x) \leq \frac{|Q_k|}{|Q_k \cap E|} m_k(v_k)(x)$$

and hence

$$|\log w_k(x)| \leq \log C + \log \frac{|Q_k|}{|Q_k \cap E|}.$$

Hence by Lemma 2.2, there exists  $h_k \in \text{dyadic-VMO}(Q_0)$  with  $\text{supp}(h_k) \subseteq Q_k$  satisfying

$$|\log w_k(x) - h_k(x)| \leq C, \quad \forall x \in Q_k \cap E \quad \text{and}$$

$$\sup_{l(Q) \geq l(Q_k)} \frac{1}{|Q|} \int_Q h_k(t) dt \leq C.$$

We now define

$$H_k(x) = \begin{cases} h_k(x) & \text{if } x \in Q_k, Q_k \in D_{n_k}, |Q_k \cap E| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check  $H_k \in \text{dyadic-VMO}(Q_0)$  with  $\|H_k\|_{*,d} \leq C$  and  $|\log w_k(x) - H_k(x)| \leq C, \forall x \in E$ . This implies  $\exists R_k(x) \in L_\infty(\mathbf{R}^d)$  with  $\|R_k\|_\infty \leq C$  and which is constant on dyadic cubes of length  $2^{-n_k}$  and satisfies  $R_k(x) = \log w_k(x) - H_k(x), \forall x \in E$ . In particular,  $R_k \in \text{dyadic-VMO}(Q_0)$  with  $\|R_k\|_{*,d} \leq C, \forall k$ . Hence the function  $W_k = \exp(R_k + H_k)$  satisfies  $\log W_k \in \text{dyadic-VMO}(Q_0), \|\log W_k\|_{*,d} \leq C$  and  $W_k \chi_E = w_k$ .

Now define

$$(3.4) \quad F(x) = \begin{cases} \sum_{k \geq 0} \frac{2}{\lambda_{k-1}} (G_k + R_k + H_k) & \text{if } x \in Q_0, \\ 0 & \text{otherwise,} \end{cases}$$

where  $G_k = 4 \log(U_k/V_k)^{1/2}$  and by  $\lambda_{-1}$  we mean  $\lambda_0$ . Since  $G_k + R_k + H_k \in \text{dyadic-VMO}(Q_0)$  with  $\|G_k + R_k + H_k\|_{*,d} \leq C$  and since  $\sum_{k \geq 0} 1/(\lambda_{k-1}) < \infty$ , it follows that  $F \in \text{dyadic-VMO}$  and  $\|F\|_{*,d} \leq C$ .

Furthermore

$$\sum_{k=0}^{\infty} \frac{2}{\lambda_{k-1}} (G_k + R_k + H_k) \chi_E = \sum_{k \geq 0} (f_{n_{k+1}} - f_{n_k}) + f_0 = f \quad \text{a.e. on } E.$$

Hence  $F$  is a dyadic-VMO extension of  $f$ .

Part (ii): Extension to non-dyadic VMO.

If  $S$  is a set in  $\mathbf{R}^d$  we will denote by  $S - \alpha$  the set  $\{x - \alpha : x \in S\}$ .

For each  $\alpha \in \frac{1}{3}Q_0$  the function  $f^{(\alpha)}(x) = f(x - \alpha)$  satisfies the hypothesis of Theorem I with the set  $E$  replaced by  $E - \alpha$ . Hence by the proof in part (i) above, there exists  $F^{(\alpha)} \in \text{dyadic-VMO}$  with

$\|F^{(\alpha)}\|_* \leq C$ ,  $F^{(\alpha)} = f^{(\alpha)}$  a.e. on  $E - \alpha$  and  $\text{supp}(F^{(\alpha)}) \subseteq Q_0$ . We claim that the function

$$F(x) = 3^d \int_{\alpha \in Q_0/3} F^{(\alpha)}(x + \alpha) d\alpha$$

is a VMO extension of  $f$ . It is clear that  $F = f$  a.e. on  $E$ . It will follow from Lemma 3.2 below that  $F \in \text{VMO}$ .

**LEMMA 3.2.** *Let  $Q$  be a cube and let  $\{f^{(\alpha)} : \alpha \in Q\}$  be a collection of dyadic-VMO functions satisfying*

- (1)  $\sup_{\alpha \in Q} \|f^{(\alpha)}\|_{*,d} \leq 1$ ,
- (2) *there exists  $r > 0$  such that for all  $\alpha \in Q$ ,  $\text{supp}(f^{(\alpha)}) \subseteq 2^r Q$ .*

*Then the function*

$$f(x) = \frac{1}{|Q|} \int_{\alpha \in Q} f^{(\alpha)}(x + \alpha) d\alpha$$

*belongs to VMO with  $\|f\|_* \leq C_r$  where  $C_r$  depends only on  $r$  and the dimension.*

**REMARKS.** (1) The idea of averaging dyadic-BMO comes from [6] where the BMO version of Lemma 3.2 is stated (and proved implicitly). Our argument is different to that used in [6] and the simpler BMO version of our argument can be used to provide an easy proof of the theorem in [6].

(2) Let  $\text{BMO}(\rho)$  denote the space of those VMO functions whose mean oscillation over any cube is  $O(\rho(l(Q)))$  where  $\rho$  is a positive, non-decreasing function with  $\rho(0+) = 0$ . In the case when each  $f^{(\alpha)}$  belongs to  $\text{BMO}(\rho)$ , Lemma 3.2 may be deduced from the results in §3 of [13]; the conclusion in this case being that  $f \in \text{BMO}(\tilde{\rho})$  where  $\tilde{\rho}(t) = t \int_t^\infty \rho(\theta)/\theta^2 d\theta$ . The argument in [13] is a version of the averaging argument in [6]. The proof below can be easily modified to obtain this result.

*Proof of Lemma 3.2. Without loss of generality we may assume  $Q = Q_0$ , the unit cube in  $\mathbb{R}^d$ .*

We first show that for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $Q$  is any cube with  $l(Q) < \delta$  then

$$\frac{1}{|Q|} \int_Q |f - f_Q| < \varepsilon.$$

Fix  $0 < \varepsilon < \frac{1}{2}$ . For each  $\alpha \in Q_0$ ,  $f^{(\alpha)} \in \text{dyadic-VMO}$  and hence there exists  $n_\alpha \in \mathbb{N}$  so that

$$\frac{1}{|Q|} \int_Q |f^{(\alpha)} - (f^{(\alpha)})_Q| < \varepsilon^{d+1}$$

whenever  $Q$  is a dyadic cube of length  $\leq 2^{-n_\alpha}$ . Hence by Egoroff's theorem there exists  $N_1 \in \mathbb{N}$  so that the set

$$S_0 = \left\{ \alpha \in Q_0 : \frac{1}{|Q|} \int_Q |f^{(\alpha)} - (f^{(\alpha)})_Q| < \varepsilon^{d+1} \right. \\ \left. \text{whenever } Q \text{ is a dyadic cube of length } \leq 2^{-N_1} \right\}$$

has measure  $\geq 1 - \varepsilon$ . Choose  $N_2$  so that  $N_2 2^{-N_2} < \varepsilon 2^{-N_1}$  and let  $Q$  be any cube with  $l(Q) \leq 2^{-N_2}$ . Let

$$S_1 = \{ \alpha \in Q_0 : Q - \alpha \text{ is contained in a dyadic cube of length } \leq \frac{1}{\varepsilon} l(Q) \}$$

and note that  $|Q_0 \cap S_1^c| \leq C\varepsilon$ .

If  $\alpha \in S_0 \cap S_1$  we claim that

$$(3.5) \quad \frac{1}{|Q|} \int_Q |f^{(\alpha)}(x + \alpha) - f^{(\alpha)}(\cdot + \alpha)_Q| dx \leq 2\varepsilon.$$

To see this let  $Q_1$  be that dyadic cube of length  $\leq \frac{1}{\varepsilon} l(Q)$  that contains  $Q - \alpha$ . Then,

$$\begin{aligned} & \frac{1}{|Q|} \int_Q |f^{(\alpha)}(x + \alpha) - (f^{(\alpha)}(\cdot + \alpha))_Q| dx \\ & \leq 2 \inf_{a \in \mathbb{R}} \frac{1}{|Q|} \int_Q |f^{(\alpha)}(x + \alpha) - a| dx \\ & \leq 2 \frac{1}{|Q|} \int_Q |f^{(\alpha)}(x + \alpha) - (f^{(\alpha)})_{Q_1}| dx \\ & \leq 2 \frac{|Q_1|}{|Q|} \frac{1}{|Q_1|} \int_{Q_1} |f^{(\alpha)}(x) - (f^{(\alpha)})_{Q_1}| dx \leq 2\varepsilon^{-d} \varepsilon^{d+1} = 2\varepsilon. \end{aligned}$$

For each  $k \geq 0$  let  $f_k^{(\alpha)} = \sum_{Q \in D_k} (f^{(\alpha)})_Q \chi_Q$  and choose  $m$  so that  $2^{-(m+1)} \leq l(Q) < 2^{-m}$ . Let

$$g_1^{(\alpha)} = f^{(\alpha)} - f_m^{(\alpha)} \quad \text{and} \quad g_2^{(\alpha)} = f_m^{(\alpha)}.$$

Now for all  $\alpha \in Q_0$ ,

$$\begin{aligned} \frac{1}{|Q|} \int_Q |g_1^{(\alpha)}(x + \alpha)| dx & \leq C 2^{md} \int_{Q - \alpha} |f^{(\alpha)}(x) - f_m^{(\alpha)}(x)| dx \\ & \leq C \sum \frac{1}{|Q_i|} \int_{Q_i} |f^{(\alpha)} - (f^{(\alpha)})_{Q_i}| dx \end{aligned}$$

where the sum is taken over those dyadic cubes  $Q_i$  of length  $2^{-m}$  which intersect  $Q - \alpha$ . If  $\alpha \in S_0$  then for each such  $Q_i$ ,

$$\frac{1}{|Q_i|} \int_{Q_i} |f^{(\alpha)} - (f^{(\alpha)})_{Q_i}| < \varepsilon^{d+1}$$

while if  $\alpha \in Q_0 \cap S_0^c$

$$\frac{1}{|Q_i|} \int_{Q_i} |f^{(\alpha)} - (f^{(\alpha)})_{Q_i}| \leq 1.$$

Since there are at most a fixed number of such cubes we have

$$(3.6) \quad \frac{1}{|Q|} \int_Q |g_1^{(\alpha)}(x + \alpha)| dx \leq \begin{cases} C\varepsilon & \text{if } \alpha \in S_0, \\ C & \text{if } \alpha \in Q_0 \cap S_0^c. \end{cases}$$

Now for each  $0 \leq j \leq m$  let

$$A_j = \{\alpha \in Q_0 : Q - \alpha \text{ is contained in a dyadic cube of length } 2^{-j}\}$$

and note that  $|A_j^c| \leq C2^{j-m}$ .

Since  $\text{supp}(f^{(\alpha)}) \subseteq 2^r Q_0$ , and  $\|f^{(\alpha)}\|_* \leq 1$ , it follows that there exists  $C > 0$  depending only on  $r$  and the dimension such that if  $\alpha \in (A_j \cap A_{j+1}^c) \cap S_0$  then

$$\sup_{x, y \in Q - \alpha} |g_2^{(\alpha)}(x + \alpha) - g_2^{(\alpha)}(y + \alpha)| \leq \begin{cases} C\varepsilon(m - j) & \text{if } j \geq N_1, \\ C(m - j) & \text{if } 0 \leq j < N_1, \end{cases}$$

and if  $\alpha \in A_0^c \cap S_0$  then

$$\sup_{x, y \in Q - \alpha} |g_2^{(\alpha)}(x + \alpha) - g_2^{(\alpha)}(y + \alpha)| \leq C(m + r)$$

while if  $\alpha \in (A_j \cap A_{j+1}^c) \cap S_0^c$

$$\sup_{x, y \in Q - \alpha} |g_2^{(\alpha)}(x + \alpha) - g_2^{(\alpha)}(y + \alpha)| \leq C(m - j)$$

and if  $\alpha \in A_0^c \cap S_0^c$  then

$$\sup_{x, y \in Q - \alpha} |g_2^{(\alpha)}(x + \alpha) - g_2^{(\alpha)}(y + \alpha)| \leq C(m + r).$$

Hence if  $j \geq N_1$  then

$$\begin{aligned} & \int_{A_j \cap A_{j+1}^c \cap S_0} \sup_{x, y \in Q - \alpha} |g_2^{(\alpha)}(x + \alpha) - g_2^{(\alpha)}(y + \alpha)| d\alpha \\ & \leq C\varepsilon(m - j)2^{j-m} \end{aligned}$$

and if  $0 \leq j < N_1$  then

$$\int_{A_j \cap A_{j+1}^c \cap S_0} \sup_{x, y \in Q^{-\alpha}} |g_2^{(\alpha)}(x + \alpha) - g_2^{(\alpha)}(y + \alpha)| d\alpha \leq C(m - j)2^{j-m}$$

Also

$$\int_{A_0^c \cap S_0} \sup_{x, y \in Q^{-\alpha}} |g_2^{(\alpha)}(x + \alpha) - g_2^{(\alpha)}(y + \alpha)| d\alpha \leq C(m + r)2^{-m}$$

and

$$\begin{aligned} & \int_{A_j \cap A_{j+1}^c \cap S_0^c} \sup_{x, y \in Q^{-\alpha}} |g_2^{(\alpha)}(x + \alpha) - g_2^{(\alpha)}(y + \alpha)| d\alpha \\ & \leq C(m - j) \min(2^{j-m}, \varepsilon), \\ & \int_{A_0^c \cap S_0^c} \sup_{x, y \in Q^{-\alpha}} |g_2^{(\alpha)}(x + \alpha) - g_2^{(\alpha)}(y + \alpha)| d\alpha \\ & \leq C(m + r) \min(2^{-m}, \varepsilon). \end{aligned}$$

Hence

$$\begin{aligned} (3.7) \quad & \int_{Q_0 \setminus (S_0 \cap S_1)} \sup_{x, y \in Q} |g_2^{(\alpha)}(x + \alpha) - g_2^{(\alpha)}(y + \alpha)| d\alpha \\ & \leq \sum_{j=0}^m C(m - j) \min(2^{j-m}, \varepsilon) + C(m + r) \min(2^{-m}, \varepsilon) \\ & \quad + \sum_{j=N_1}^m C\varepsilon(m - j)2^{j-m} \\ & \quad + \sum_{j=0}^{N_1} C(m - j)2^{j-m} + C(m + r)2^{-m} \\ & \leq C\varepsilon \log \frac{1}{\varepsilon} + Cm2^{-m} + C\varepsilon + (m - N_1)2^{N_1-m} + Cm2^{-m} \\ & \leq C\varepsilon \log \frac{1}{\varepsilon} + C\varepsilon + C(m2^{-m})2^{N_1} \\ & \leq C\varepsilon \log \frac{1}{\varepsilon} + C\varepsilon + C\varepsilon \leq C\varepsilon \log \frac{1}{\varepsilon}. \end{aligned}$$

Combining (3.5), (3.6) and (3.7) we get

$$\begin{aligned}
& \frac{1}{|Q|} \int_Q |f - f_Q| dx \\
& \leq \int_{Q_0} \left( \frac{1}{|Q|} \int_Q |f^{(\alpha)}(x + \alpha) - (f^{(\alpha)}(\cdot + \alpha))_Q| dx \right) d\alpha \\
& \leq \int_{S_0 \cap S_1} \left( \frac{1}{|Q|} \int_Q |f^{(\alpha)}(x + \alpha) - (f^{(\alpha)}(\cdot + \alpha))_Q| dx \right) d\alpha \\
& \quad + \int_{Q_0 \setminus (S_0 \cap S_1)} \sup_{x, y \in Q} |g_2^{(\alpha)}(x + \alpha) - g_2^{(\alpha)}(y + \alpha)| d\alpha \\
& \quad + 2 \int_{S_0} \left( \frac{1}{|Q|} \int_Q |g_1^{(\alpha)}(x + \alpha)| dx \right) d\alpha \\
& \quad + 2 \int_{Q_0 \setminus S_0} \left( \frac{1}{|Q|} \int_Q |g_1^{(\alpha)}(x + \alpha)| dx \right) d\alpha \\
& \leq C\varepsilon + C\varepsilon \log \frac{1}{\varepsilon} + C\varepsilon + C\varepsilon \leq C\varepsilon \log \frac{1}{\varepsilon}.
\end{aligned}$$

It remains to show that

$$(3.8) \quad \sup_Q \frac{1}{|Q|} \int_Q |f - f_Q| < \infty.$$

It is clear from what we established above that it suffices to prove (3.8) when the supremum is taken over all cubes with  $l(Q) \geq 1$ . Fix  $Q$  with  $l(Q) \geq 1$ . Then for all  $\alpha \in Q_0$ ,

$$\begin{aligned}
& \frac{1}{|Q|} \int_Q |f^{(\alpha)}(x + \alpha)| dx \\
& \leq \frac{1}{|Q|} \int_Q |f_0^{(\alpha)}(x + \alpha)| dx + \frac{1}{|Q|} \int_Q |f^{(\alpha)} - f_0^{(\alpha)}| dx.
\end{aligned}$$

Since each  $f^{(\alpha)}$  is supported on  $2^r Q_0$  there exists  $C > 0$ , depending only on  $r$  and the dimension, such that  $|f_0^{(\alpha)}(x)| \leq C$  for all  $x \in \mathbf{R}^d$ . Hence

$$\frac{1}{|Q|} \int_Q |f_0^{(\alpha)}(x + \alpha)| dx \leq C.$$

Let  $\{Q_i\}$  be those dyadic cubes of length 1 with  $Q_i \cap (Q - \alpha) \neq \emptyset$ . Then for all  $\alpha \in Q_0$ ,

$$\begin{aligned}
& \frac{1}{|Q|} \int_Q |f^{(\alpha)}(x + \alpha) - f_0^{(\alpha)}(x + \alpha)| dx \\
& \leq C \sum_{Q_i} \frac{1}{|Q_i|} \int_{Q_i} |f^{(\alpha)}(x) - (f^{(\alpha)})_{Q_i}| dx \leq C.
\end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{|Q|} \int_Q |f - f_Q| &\leq C \frac{1}{|Q|} \int_Q |f| \\ &\leq C \int_{Q_0} \left( \frac{1}{|Q|} \int_Q |f^{(\alpha)}(x + \alpha)| dx \right) d\alpha \leq C. \end{aligned}$$

and (3.8) now follows. This completes the proof of Lemma 3.2 and Theorem I.

A consequence of Theorem I which has useful applications is the following corollary:

**COROLLARY 3.1.** *Let  $E_1, E_2$  be measurable subsets of the unit cube in  $\mathbb{R}^d$  and suppose there exists an increasing sequence of positive numbers  $\{\lambda_n\}_{n=0}^\infty$  with  $\lambda_n \rightarrow \infty$  such that for each  $n \in \mathbb{N}$  and for each cube  $Q$  with  $l(Q) \leq 2^{-n}$  we have*

$$\min \left( \frac{|Q \cap E_1|}{|Q|}, \frac{|Q \cap E_2|}{|Q|} \right) < e^{-\lambda_n}.$$

*Then there exists  $F \in \text{VMO}$ ,  $\|F\|_* \leq C_{\lambda_0}$  with  $F = 0$  on  $E_1$  and  $F = 1$  on  $E_2$ .*

*Proof.* Set  $E = E_1 \cup E_2$  in Theorem I and define

$$f(x) = \begin{cases} 0 & \text{if } x \in E_1, \\ 1 & \text{if } x \in E_2, \end{cases}$$

and

$$a_Q = \begin{cases} 1 & \text{if } |Q \cap E_1|/|Q| < e^{-\lambda_n}, \\ 0 & \text{otherwise.} \end{cases}$$

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