HYPERHOLOMORPHIC FUNCTIONS AND HIGHER ORDER PARTIAL DIFFERENTIAL EQUATIONS IN THE PLANE

R. Z. Yeh
HYPERHOLOMORPHIC FUNCTIONS AND
HIGHER ORDER PARTIAL DIFFERENTIAL
EQUATIONS IN THE PLANE

R. Z. Yeh

We derive a Taylor formula for matrix-valued functions, in particular for hyperholomorphic functions. The latter functions are matrix-valued functions that satisfy a certain type of first order systems, for which we make no ellipticity assumption. For solutions of higher order linear partial differential equations with constant coefficients in the plane we show the existence of hyperconjugates, an obvious generalization of harmonic conjugates in complex analysis. By way of hyperconjugates we find series expansions for solutions of partial differential equations in terms of polynomial solutions. These polynomials form a basis for real analytic solutions at the origin. An algorithm for obtaining all such polynomials is summarized at the end. This paper continues in the tradition of hypercomplex analysis.

1. Matrix-valued functions. Matrix-valued functions are freely added or multiplied whenever their sizes are compatible. In writing the product \( FG \) we automatically assume the number of columns of \( G \) to be equal to the number of rows of \( G \). We shall not single out any particular class of matrix-valued functions to form an algebra. The underlying scalars for the matrices can be real, complex, or perhaps even elements of a Banach algebra. Most of the basic concepts in the calculus of scalar-valued functions can be readily extended to matrix-valued functions by means of “componentwise applications”. However, certain complications are expected because matrix multiplications are not commutative. Although we need not restrict to two independent variables, we will do so in order to simplify our presentation.

Let \( F \) belong to class \( C^1 \) in some unspecified domain, namely every component function \( f_{ij}(x, y) \) has continuous first order partial derivatives, then the differential of \( F \) or \((df_{ij})\) is conveniently expressed as \( dF = F_x \, dx + F_y \, dy \) where the subscripts represent componentwise partial differentiation. More generally we consider a differential form \( P(x, y) \, dx + Q(x, y) \, dy \), which is said to be exact if there exists an \( F \) such that \( dF = P \, dx + Q \, dy \), or equivalently, \( P = F_x \) and \( Q = F_y \).
A differential form $P\,dx + Q\,dy$ is said to be divisible if there exist $F$ and $G$ such that

$$P\,dx + Q\,dy = F(dG) := FG_x\,dx + FG_y\,dy,$$

or alternatively

$$P\,dx + Q\,dy = (dF)G := F_xG\,dx + F_yG\,dy.$$

If the differential of $H$ is divisible, say if

$$dH = F(dG),$$

that is, $H_x = FG_x$ and $H_y = FG_y$, then we say that $H$ is left-differentiable with respect to $G$ with a left-derivative $F$, and we may write $dH/dG = F$. Likewise, if $H$, $F$, and $G$ are such that

$$dH = (dF)G,$$

that is, $H_x = F_xG$ and $H_y = F_yG$, then we say that $H$ is right-differentiable with respect to $F$ with a right-derivative $G$, and we may perhaps write $dF/dH = G$. However, in most cases it will suffice to write $H'$ to denote either of these derivatives.

A differential form is not necessarily exact; neither is a divisible differential form. Nevertheless, the well-known criterion for exactness of differential forms in general may be applied in particular to divisible differential forms to produce a useful criterion.

**Proposition 1.** If $F$ belongs to $C^1$, and $G$ belongs to $C^2$ in some simply-connected domain $\Omega$, then $F(dG)$ is exact if and only if

$$F_yG_x = F_xG_y \quad \text{in } \Omega.$$

**Proof.** As is well known, in a simply-connected domain, $P\,dx + Q\,dy$ is exact if and only if $P_y = Q_x$. Applying this criterion to $F(dG) = FG_x\,dx + FG_y\,dy$, we obtain

$$(FG_x)_y = (FG_y)_x$$

whence follows

$$F_yG_x = F_xG_y$$

since $G_{xy} = G_{yx}$ because $G$ belongs to $C^2$.

We can likewise show that in a simply-connected domain $(dF)G$ is exact if and only if $F_yG_x = F_xG_y$ assuming that $F$ belongs to $C^2$ and $G$ belongs to $C^1$. Although our exactness criterion is valid only in a simply-connected domain, it does not prevent a divisible differential form from being exact in $\Omega$ regardless of whether $\Omega$ is simply
connected or not. If \( F(dG) \) is exact in \( \Omega \), then \( F(dG) = dH \) for some \( H \) in \( \Omega \). In this case, we say that \( F \) is left-antidifferentiable with respect to \( G \), and \( H \) is a left-antiderivative of \( F \) with respect to \( G \). We may write \( H = \int F(dG) \). Although \( \int F(dG) \) is not unique, no statement shall be made about \( \int F(dG) \) unless it is valid independently of choices of antiderivatives. Likewise, if \( (dG)F = dH \), then \( H \) is a right-antiderivative of \( F \) with respect to \( G \), and we may write \( H = \int (dG)F \). However, in most cases it will suffice simply to write \( F^\# \) for the antiderivative of \( F \).

The line integral of a differential form is defined componentwise:

\[
\int_\gamma P \, dx + Q \, dy = \left( \int_\gamma p_{ij} \, dx + q_{ij} \, dy \right)
\]

where \( \gamma \) is a path of integration (having a continuous tangent vector). We have the following fundamental theorem of line integral, which seems quite obvious.

**Theorem 1.** If a divisible differential form \( F(dG) \) is exact in \( \Omega \), then

\[
\int_\gamma F(dG) = F^\#(x_1, y_1) - F^\#(x_0, y_0)
\]

where \( \gamma \) is a path of integration connecting \((x_0, y_0)\) to \((x_1, y_1)\) in \( \Omega \), and \( F^\# \) is an antiderivative of \( F \) with respect to \( G \).

**Proof.** Since \( F(dG) \) is exact, there exists an \( F^\# \) such that \( F(dG) = dF^\# \). Consequently

\[
\int_\gamma F(dG) = \int_\gamma dF^\# = F^\#(x_1, y_1) - F^\#(x_0, y_0).
\]

In practice it may not be easy to find \( F^\# \).

Every matrix-valued function \( Z \) is differentiable with respect to \( Z \) since \( dZ = I \, dZ \), but powers of \( Z \) need not be differentiable with respect to \( Z \). For example, we can go no further than

\[
dZ^2 = d(ZZ) = (dZ)Z + Z(dZ)
\]

unless \( (dZ)Z = Z(dZ) \), in which case we could go on to

\[
\]

Therefore, following Hile [8], we shall say that \( Z \) is self-commuting in \( \Omega \) if

\[
Z(x_1, y_1)Z(x_2, y_2) = Z(x_2, y_2)Z(x_1, y_1)
\]
for any \((x_1, y_1)\) and \((x_2, y_2)\) in \(\Omega\). For example, \(Z = Ax + By\) is self-commuting if the constant matrices \(A\) and \(B\) commute. Needless to say, \(Z\) has to be a square matrix in order to be self-commuting.

**Proposition 2.** If \(Z\) is self-commuting and \(C^1\) in \(\Omega\), we have

\[
\begin{align*}
(1.1) & \quad Z_x, Z_y, \text{ and } Z \text{ commute pairwise, in particular } \\
& \quad Z(dZ) = (dZ)Z. \\
(1.2) & \quad Z_x, Z_y, \text{ and } Z^{-1} \text{ commute if } Z \text{ is invertible, and thus } \\
& \quad Z^{-1}(dZ) = (dZ)Z^{-1}. \\
(1.3) & \quad dZ^n = (nZ^{n-1})dZ = dZ(nZ^{n-1}) \text{ for all integers } n. \\
(1.4) & \quad d(Z - Z_0)^n = n(Z - Z_0)^{n-1}dZ = dZ[n(Z - Z_0)^{n-1}] \\
& \quad \text{where } Z_0 = Z(x_0, y_0) \text{ and } (x_0, y_0) \in \Omega.
\end{align*}
\]

**Proof.** We show (1.1) by writing out the difference quotients for \(Z_x\) and \(Z_y\) and applying the self-commuting property of \(Z\). (1.1) leads to (1.2), and together they imply (1.3) and (1.4). We omit the details.

One interesting thing about self-commuting \(Z\) is that \(Z\)-differentiability and \(Z\)-antidifferentiability are not unrelated, and this in turn leads to some nice theorems. Three such theorems, 2 to 4, are stated below though they are not used in the rest of the paper except Theorem 4.

**Lemma 1.** If \(F\) is right-differentiable with respect to \(Z\) in a simply-connected domain, and \(Z\) is self-commuting and \(C^2\), then \(F\) is right-antidifferentiable with respect to \(Z\).

**Proof.** Suppose \(dF = (dZ)G\), or

\[
F_x = Z_x G \quad \text{and} \quad F_y = Z_y G.
\]

Multiplying with \(Z_y\) and \(Z_x\) respectively, we have

\[
Z_y F_x = Z_y Z_x G \quad \text{and} \quad Z_x F_y = Z_x Z_y G.
\]

But since \(Z_x Z_y = Z_y Z_x\) by Proposition 2, we obtain

\[
Z_x F_y = Z_y F_x,
\]

which is the exactness criterion in Proposition 1; hence \(dH = (dZ)F\) for some \(H\), and \(F\) is right-antidifferentiable.
THEOREM 2 (Cauchy integral theorem). If \( F \) is right-differentiable with respect to a \( C^2 \) and self-commuting \( Z \) in a simply-connected domain \( \Omega \), then

\[
\int_\gamma (dZ)F = 0
\]

for any closed path of integration \( \gamma \) in \( \Omega \).

**Proof.** By Lemma 1, there exists \( H \) such that \( (dZ)F = dH \). Consequently

\[
\int_\gamma (dZ)F = \int_\gamma dH = 0
\]

for any closed \( \gamma \) in \( \Omega \).

THEOREM 3. If \( F \) is right-antidifferentiable with respect to a \( C^2 \) and self-commuting \( Z \) in a simply-connected domain, it is infinitely many times right-antidifferentiable with respect to \( Z \).

**Proof.** If \( F \) is \( Z \)-antidifferentiable with an antiderivative \( F^# \), then since \( F^# \) is \( Z \)-differentiable, by Lemma 1 \( F^# \) is \( Z \)-antidifferentiable with antiderivative \( F^{##} \). Repeating the same argument on \( F^{##} \), we show the existence of the next antiderivative, etc.

We now consider the following reversal of the preceding. Here we no longer need the simply-connectedness of the domain, but instead we need the invertibility of either \( Z_x \) or \( Z_y \) in the domain.

LEMA 2. If \( F \) is \( C^1 \) and right-antidifferentiable with respect to \( Z \), and \( Z \) is self-commuting, \( C^2 \) and has invertible \( Z_x \) or \( Z_y \), then \( F \) is right-differentiable with respect to \( Z \).

**Proof.** Suppose \( (dZ)F = dH \), or

\[
Z_x F = H_x \quad \text{and} \quad Z_y F = H_y.
\]

Differentiating, we have

\[
Z_{xy} F + Z_x F_y = H_{xy} \quad \text{and} \quad Z_{yx} F + Z_y F_x = H_{yx},
\]

whence follows \( Z_x F_y = Z_y F_x \). Multiplying \( (Z_x)^{-1} \), we obtain

\[
F_y = (Z_x)^{-1} Z_y F_x.
\]
Hence
\[ dF = F_x \, dx + (Z_x)^{-1}Z_y \, F_x \, dy = (I \, dx + Z_x^{-1}Z_y \, dy)F_x \]
\[ = (Z_x)^{-1}(dZ)F_x = (dZ)(Z_x^{-1}F_x). \]

**Theorem 4.** If $F$ is $C^k$, $k \geq 1$, and right-differentiable with respect to $Z$, and $Z$ is self-commuting, $C^k$ and has invertible $Z_x$ or $Z_y$, then $F$ is $k$ times right-differentiable with respect to $Z$.

**Proof.** The case $k = 1$ is trivial. So suppose $k \geq 2$. Let $dF = (dZ)F'$; then $F_x = Z_x F'$ and $F'$ is $C^{k-1}$ with $k - 1 \geq 1$. Hence by Lemma 2 $F'$ is $Z$-differentiable. $dF' = (dZ)F''$ and $F'_x = Z_x F''$, so $F''$ is $C^{k-2}$. Continuing thus, we reach $F^{(k)}$, which is $C^{k-k} = C^0$.

We rely on integration-by-parts to derive our Taylor formula. There are actually two parts to this technique, which we formulate separately as Propositions 3 and 4.

**Proposition 3.** For $F$ and $G$ of class $C^1$ in $\Omega$, and any path of integration $\gamma$ from $(x_0, y_0)$ to $(x_1, y_1)$ in $\Omega$, we have

\[
\int_\gamma (dF)G = FG \bigg|_{(x_0, y_0)}^{(x_1, y_1)} + \int_\gamma (-F) \, dG.
\]

Thus, either both integrals are independent of paths connecting $(x_0, y_0)$ to $(x_1, y_1)$, or neither is.

**Proof.** Since $d(FG) = (dF)G + F(dG)$,

\[
\int_\gamma d(FG) = \int_\gamma (dF)G + \int_\gamma F(dG).
\]

Hence,

\[
\int_\gamma (dF)G = \int_\gamma^{(x_1, y_1)} d(FG) - \int_\gamma F(dG).
\]

Needless to say, we have likewise

\[
\int_\gamma F(dG) = FG \bigg|_{(x_0, y_0)}^{(x_1, y_1)} + \int_\gamma dF(-G).
\]

(1.7)
**Proposition 4.** If $F$ is left-differentiable, and $G$ is right-antidifferentiable, both with respect to some $Z$, then

\begin{equation}
\int_{\gamma} (dF)G = \int_{\gamma} F'(dG^*),
\end{equation}

where $F'$ is the left-derivative, and $G^*$ is a right-antiderivative with respect to $Z$.

**Proof.** The formula follows from

$$(dF)G = (F'dZ)G = F'dG^*.$$

Again needless to say, we have likewise, under suitable assumptions on $F$ and $G,$

\begin{equation}
\int_{\gamma} F(dG) = \int_{\gamma} (dF^*)G'.
\end{equation}

We now attempt to approximate a given $F$ by powers of some self-commuting $Z$. For simplicity of notation we shall use $z$ to represent the point $(x,y)$ without thinking of $z$ as a complex number. The line integral $\int_{\gamma} (dF)G$ may be more fully expressed as $\int_{\gamma} dF(z)G(z)$ or $\int_{\gamma} dF(\bar{z})G(\bar{z})$ with $\bar{z}$ emphasizing its being a variable of integration. The use of dummy variable $\bar{z}$ is especially appropriate when we have to consider, for example, $\int_{z_0}^{z} dF(\bar{z})G(\bar{z}, z)$, in which $G$ depends on $\bar{z}$ as well as a fixed $z$. The upper and lower limits of integration appear only when the line integral is independent of the paths connecting $z_0$ to $z$.

**Theorem 5 (Taylor formula).** Let $F$ be $(k+1)$-times right-differentiable with respect to a self-commuting $Z$ in $\Omega$. Let $z_0$ be a fixed point in $\Omega$; then for any $z$ in $\Omega$, we have

\begin{equation}
F(z) = \sum_{j=0}^{k} \left[ (Z - Z_0)^j / j! \right] F^{(j)}(z_0)
+ \int_{z_0}^{z} \left[ -(Z - \tilde{Z})^{k+1} / (k + 1)! \right] F^{(k+1)}(\tilde{Z}),
\end{equation}

where $\tilde{Z} = Z(\tilde{z})$, $Z_0 = Z(z_0)$, and the line integral is taken along any path of integration connecting $z_0$ to $z$ in $\Omega$.

**Proof.** We give a straightforward derivation, using the integration-by-parts formulas. All the line integrals are independent of the paths
of integration since the first one clearly is
\[ F(z) - F(z_0) = \int_{z_0}^{z} dF(z) = \int_{z_0}^{z} dI^*F'(\tilde{z}) \text{ by (1.9)} \]
\[ = \int_{z_0}^{z} d[(-\tilde{z}Z)]F'(\tilde{z}) \text{ where we picked } \tilde{Z} \text{ for } I^* \]
\[ = [-(Z - \tilde{Z})F'(\tilde{Z})]_{z_0} + \int_{z_0}^{z} (Z - \tilde{Z}) dF'(\tilde{z}) \text{ by (1.6)} \]
\[ = (Z - Z_0)F'(z_0) + \int_{z_0}^{z} d(Z - \tilde{Z})F''(\tilde{z}) \text{ again by (1.9)}. \]

But now \((Z - \tilde{Z})^#\) may be chosen to be \([-(Z - \tilde{Z})^2]/2!\) in view of (1.4) of Proposition 2, and this is where we need the self-commuting property of \(Z\). Thus we have shown the Taylor formula for \(k = 1\). By repeated applications of formulas (1.6) and (1.9) we eventually arrive at the formula (1.10). One can also write out a formal inductive proof.

We state the following Leibniz formula, which will be needed later (in Theorems 7 and 9 below).

**Proposition 5 (Leibniz formula).** If \(G(z, \tilde{z})\) and \(H(z)\) are \(C^1\) in \(\Omega \times \Omega\) and \(\Omega\) respectively, then for any \(z_0\) and \(z\) in \(\Omega\), we have

\(\frac{\partial}{\partial x} \int_{z_0}^{z} G(z, \tilde{z}) dH(\tilde{z}) = \int_{z_0}^{z} G_x(z, \tilde{z}) dH(\tilde{z}) + G(z, z)H_x(z),\)

and ditto for \(\partial/\partial y\), provided that the line integral on the left is independent of the paths of integration from \(z_0\) to \(z\).

**Proof.** We need only add the following equalities:

\[ \frac{\partial}{\partial x} \int_{z_0}^{z} G(z, \tilde{z})H_x(\tilde{z}) d\tilde{x} = \int_{z_0}^{z} G_x(z, \tilde{z})H_x(\tilde{z}) d\tilde{x} + G(z, z)H_x(z), \]

\[ \frac{\partial}{\partial x} \int_{z_0}^{z} G(z, \tilde{z})H_y(\tilde{z}) d\tilde{y} = \int_{z_0}^{z} G_x(z, \tilde{z})H_y(\tilde{z}) d\tilde{y} + 0. \]

Both of these follow from the Leibniz formula for scalar-valued functions.

**2. Hyperholomorphic functions.** Let \(M\) be a constant square matrix. A matrix-valued function \(F\) is said to be \(M\)-holomorphic in a domain if it belongs to \(C^1\) and satisfies the first order system

\(F_y = MF_x\)
in the domain. Among all the $M$-holomorphic functions for a given $M$, the key role is played by

\begin{equation}
Z = xI + yM,
\end{equation}

which is clearly self-commuting and satisfies (2.1). It turns out that $M$-holomorphicity is equivalent to $Z$-differentiability (see Theorem 6 below). This allows us to apply results of the last section to $M$-holomorphic functions.

Following complex analysis, we say that a matrix-valued function $F$ is $M$-analytic (or $Z$-analytic) at the origin if it has a series expansion in powers of $Z$:

\begin{equation}
F = \sum_{j=0}^{\infty} \frac{(Z^j / j!)}{j!} A_j
\end{equation}

in an open disk around the origin, where $A_j$ are constant matrices having the same number of columns as $F$.

Clearly, if $F$ is $M$-analytic, it is $M$-holomorphic as termwise differentiation of (2.3) will verify (2.1). However, the converse is not true (see Theorem 7 below), and here we part company with complex analysis.

**THEOREM 6.** $F$ is $M$-holomorphic if and only if $F$ is differentiable with respect to $Z = xI + yM$. The derivative $F'$ is equal to $F_x$.

**Proof.** If $F$ is $M$-holomorphic,

\[
dF = F_x \, dx + F_y \, dy = F_x \, dx + MF_x \, dy = (I \, dx + M \, dy)F_x = (dZ)F_x.
\]

Thus, $F$ is $Z$-differentiable with the derivative equal to $F_x$, also to be denoted by $D^{(1,0)}F$.

Conversely, if $F$ is $Z$-differentiable,

\[
dF = dZF'
\]

so that $F_x = Z_x F' = IF'$ and $F_y = Z_y F' = MF'$. Consequently $F_y = MF_x$, and $F$ is $M$-holomorphic.

If $Z$ is self-commuting, then $Z^n$ are $Z$-differentiable by Proposition 2, and hence we have

**COROLLARY 6a.** $Z^n$ are $M$-holomorphic for all integers $n$.

**COROLLARY 6b.** $F$ is $M$-holomorphic and $C^k$ if and only if $F$ has a continuous kth order $Z$-derivative $F^{(k)}$. 

Proof. First assume $F$ to be $C^k$ and $M$-holomorphic, hence $Z$-differentiable by Theorem 6. Then in view of Theorem 4, since $Z$ is trivially $C^k$ and $Z_x$ trivially invertible, we have $F^{(k)} = D^{(k,0)} F$, which is continuous because $F$ is $C^k$.

Conversely, if a continuous $F^{(k)}$ exists, $F'$ easily exists, and so $F$ is $M$-holomorphic by Theorem 6. To show $F$ is $C^k$, using the continuity of $F^{(k)} = D^{(k,0)} F$, we see

$$D^{(k-j, j)} F = M^j D^{(0,0)} F$$

are all continuous for $0 \leq j \leq k$. In other words, $F$ is $C^k$.

Corollary 6c (Taylor formula). If $F$ is $M$-holomorphic and $C^k$ in a neighborhood around the origin, then

$$F(z) = \sum_{j=0}^{k-1} (Z^j / j!) F^{(j)}(0) + \int_0^z d[-(Z - \tilde{Z})^k / k!] F^{(k)}(\tilde{z}).$$

The proof follows from Theorem 5 and Corollary 6b.

Theorem 7 (Taylor expansion). $F$ is $M$-holomorphic and real analytic at the origin if and only if $F$ is $M$-analytic at the origin with

$$F(z) = \sum_{j=0}^{\infty} (Z^j / j!) F^{(j)}(0).$$

Proof. First suppose $F$ is $M$-holomorphic and real analytic at the origin; then in an open disk around the origin we have

$$F(z) = \sum_{j=0}^{\infty} F_j(z),$$

where $F_j$ is a matrix consisting of $j$th degree homogeneous polynomials in $x$ and $y$ and possibly also of zero polynomials. On the other hand we also have the Taylor formula (see Corollary 6c above):

$$F(z) = \sum_{j=0}^{k} (Z^j / j!) F^{(j)}(0)$$

$$= \int_0^z d[-(Z - \tilde{Z})^{k+1} / (k+1)!] F^{(k+1)}(\tilde{z}) \quad \text{for } k \geq 0,$$
where we used the fact that $F$ is infinitely many times $Z$-differentiable in view of Corollary 6b. By combining (2.6) and (2.7) we now show inductively

\[(2.8) \quad F_j(z) = (Z^j / j!) F^{(j)}(0) \quad \text{for } j \geq 0.\]

Letting $z = 0$ in (2.6) and (2.7), we see $F_0 = F(0)$ since all homogeneous polynomials of degree one or higher vanish at $x = y = 0$. Next, assume as induction hypothesis

\[(2.9) \quad F_j = (Z^j / j!) F^{(j)}(0) \quad \text{for } 0 \leq j \leq k - 1.\]

Now (2.6), (2.7) and (2.9) imply

\[(2.10) \quad F_k + \sum_{j=k+1}^{\infty} F_j = (Z^k / k!) F^{(k)}(0)\]

\[+ \int_0^z [(Z - \tilde{Z})^k / k!] dF^{(k)}(\tilde{z}),\]

where we wrote the integral term in an alternative form via integration-by-parts formula (1.8) in order to apply the Leibniz formula (1.11). Differentiating both sides of (2.10) by applying $\partial^k / \partial x^i \partial y^{k-i} = D^{(i,k-i)}$, for $0 \leq i \leq k$, we obtain

\[(2.11) \quad D^{(i,k-i)} F_k + \sum_{j=k+1}^{\infty} D^{(i,k-i)} F_j\]

\[= D^{(i,k-i)} (Z^k / k!) F^{(k)}(0)\]

\[+ \int_0^z D^{(i,k-i)} [(Z - \tilde{Z})^k / k!] dF^{(k)}(\tilde{z}).\]

Note that in applying the Leibniz formula (1.11) the term $G(z, z) = (Z - Z)^k / k!$ vanishes. Setting $z = 0$ in (2.11), we obtain

\[(2.12) \quad D^{(i,k-i)} F_k = D^{(i,k-i)} (Z^k / k!) F^{(k)}(0) \quad \text{for } 0 \leq i \leq k\]

since all homogeneous polynomials of degree one or higher vanish at $z = 0$, and so does the line integral. Having all the $k$th order partial derivatives equal by (2.12), $F_k$ and $(Z^k / k!) F^{(k)}(0)$ can differ only by a polynomial of degree at most $k - 1$. But since $F_k$ and $(Z^k / k!) F^{(k)}(0)$ are both homogeneous of degree $k$, they must be equal.

\[(2.13) \quad F_k = (Z^k / k!) F^{(k)}(0).\]

This proves the first half of the theorem.
To prove the converse, assume $F$ is $M$-analytic at the origin. We have then

\[(2.14) \quad F = \sum_{j=0}^{\infty} (Z^j / j!) A_j,\]

where it is easily checked that $A_j = F^{(j)}(0)$, and (2.14) easily implies that $F$ is real analytic since each component of $F$ has a power series expansion. To see $F$ is $M$-holomorphic, we differentiate (2.14) termwise to show $F_y = MF_x$. This completes the proof of the theorem.

We shall call an $M$-holomorphic column-vector-valued function an $M$-conjugation. Note that if $F$ is $M$-holomorphic, then each column of $F$ is an $M$-conjugation since equation (2.1) can be split into as many equations as there are columns in $F$. We shall refer to every $M$-conjugation as an $M$-conjugation of its first component, and all lower components as $M$-conjugates of the first component. The existence of such “hyperconjugates” is important if we are to apply any theory of hyperholomorphic functions to solutions of higher order partial differential equations.

3. Partial differential equations. We consider equations in the $(x, y)$-plane of the form

\[(3.1) \quad a_0 \frac{\partial^m u}{\partial x^m} + a_1 \frac{\partial^m u}{\partial x^{m-1} \partial y} + \cdots + a_{m-1} \frac{\partial^m u}{\partial x \partial y^{m-1}} + \frac{\partial^m u}{\partial y^m} = 0\]

where $m \geq 2$ and the coefficients $a_0, a_1, \ldots, a_{m-1}$ are real or complex constants, with the last coefficient $a_m$ normalized as 1. Letting

$L = a_0 D^{(m, 0)} + a_1 D^{(m-1, 1)} + \cdots + D^{(0, m)},$

we condense (3.1) to $Lu = 0$, and refer to $u$ loosely as a “solution” of $L$. Letting $A = (a_0, a_1, \ldots, a_{m-1}, 1)$, we can also write (3.1) as

\[(3.1a) \quad A \nabla^{(m)} u = 0,\]

where the column vector $\nabla^{(m)} u$ is the $m$th order hypergradient of $u$. The above equation in $\nabla^{(m)} u$ can be further rewritten as a first order equation in $\nabla^{(m-1)} u$, namely

\[(3.1b) \quad (a_0, \ldots, a_{m-1})[\nabla^{(m-1)} u]_x + (0, \ldots, 0, 1)[\nabla^{(m-1)} u]_y = 0.\]

This last equation will be put into a first order system satisfied by $\nabla^{(m-1)} u$ (see Theorem 8 below).
In elementary calculus one learns to recover a function from its gradient via a line integral, namely

\[ u(x, y) = u(x_0, y_0) + \int_{(x_0, y_0)}^{(x, y)} u_x(\tilde{x}, \tilde{y}) \, d\tilde{x} + u_y(\tilde{x}, \tilde{y}) \, d\tilde{y}, \]

which can be written more concisely as

\[ u(z) = u(z_0) + \int_{z_0}^{z} d\tilde{z} \nabla u(\tilde{z}), \]

or somewhat artificially as

\[ u(z) = u(z_0) + \int_{z_0}^{z} d[-(z - \tilde{z})] \nabla u(\tilde{z}). \]

It turns out that this last formula can be generalized so that one can also recover \( u \) from its hypergradient \( \nabla^{(m)} u \). We state the formula for \( m = 2 \), and explain the notations.

\[ u(z) = u(z_0) + (z - z_0) \nabla u(z_0) + \int_{z_0}^{z} d[-(z - \tilde{z})^2 / 2!] \nabla^{(2)} u(\tilde{z}). \]

Here \( z \) is not meant to be a complex number, but rather a “hypernumber” obeying the following conventions: \( z = (x, y) \), \( z^2 = (x^2, 2xy, y^2) \), \( z^3 = (x^3, 3x^2y, 3xy^2, y^3) \), and so on; also

\[ (z - z_0)^2 = (x - x_0, y - y_0)^2 = ((x - x_0)^2, 2(x - x_0)(y - y_0), (y - y_0)^2), \]
\[ d(z - \tilde{z})^2 = (d(x - \tilde{x})^2, 2d(x - \tilde{x})(y - \tilde{y}), d(y - \tilde{y})^2). \]

The general recovery formula, proved in [9], is as follows:

\[ (3.1c) \quad u(z) = \sum_{j=0}^{m-1} [(z - z_0)^j / j!] \nabla^{(j)} u(z_0) + \int_{z_0}^{z} d[-(z - \tilde{z})^m / m!] \nabla^{(m)} u(\tilde{z}). \]

For a later application we will write this formula in yet another form. Let \( X \) be an infinite square matrix whose entries are all zero except the diagonal entries consisting of \( x \)'s and the supradiagonal entries
consisting of \( y \)'s, so that the first few rows of \( X \) look like
\[
E_1 X = (x, y, 0, 0, \ldots),
E_2 X = (0, x, y, 0, \ldots),
\]
where \( E_i \) shall always denote the \( i \)th unit row vector of appropriate dimension determined by the context in which it appears. We note that the power \( X^j \) has rows each of which is essentially a copy of \( z^j \).

For example, for \( j = 2 \) we have
\[
E_1 X^2 = (x^2, 2xy, y^2, 0, 0, 0, \ldots),
E_2 X^2 = (0, x^2, 2xy, y^2, 0, 0, \ldots),
E_3 X^2 = (0, 0, x^2, 2xy, y^2, 0, \ldots).
\]

Using \( E_1 X^2 \) in the place of \( z^2 \), we would like to rewrite \( z^2 \nabla^{(2)} u(z_0) \) as \( E_1 X^2 \nabla^{(2)} u(z_0) \) except that on the right \( \nabla^{(2)} u(z_0) \) as a column vector is too short to match the row vector \( E_1 X^2 \). Therefore, henceforth whenever \( \nabla^{(m)} u \) takes part in a matrix multiplication, we shall automatically assume \( \nabla^{(m)} u \) to have been extended to an appropriate length by addition of as many 0's as necessary. With these notational agreements we can now rewrite our recovery formula (3.1c), where for simplicity we take \( z_0 = 0 \), as follows:

\[
(3.1d) \quad u(z) = E_1 \sum_{j=0}^{m-1} [X^j / j!] \nabla^{(j)} u(0) + E_1 \int_0^z d[-(X - \tilde{X})^m / m!] \nabla^{(m)} u(\tilde{z}).
\]

We now go on to Theorem 8 mentioned after the equation (3.1b).

**Theorem 8.** If \( u \) is \( C^m \) and \( Lu = 0 \), then \( \nabla^{(m-1)} u \) is \( C^1 \) and \( M \)-holomorphic, \([\nabla^{(m-1)} u]_y = M[\nabla^{(m-1)} u]_x\), where \( M \) is the \( m \times m \) associated matrix of \( L \), consisting of 0's everywhere except the supradiagonal consisting of 1's and the bottom row consisting of \( -a_0, -a_1, \ldots, -a_{m-1} \) (see \( M \) below in the proof).

Conversely, if an \( m \times 1 \) column vector \( f \) is \( C^1 \) and \( M \)-holomorphic in a simply-connected domain, i.e.,
\[
(3.2) \quad f_y = M f_x,
\]
then there exists a \( C^m \) solution \( Lu = 0 \) such that \( \nabla^{(m-1)} u = f \). If
furthermore \( f \) is \( C^k \), \( k \geq m \), then every component of \( f \) is a \( C^k \) solution of \( L \).

Proof. To see \( \nabla^{(m-1)}u \) is \( M \)-holomorphic we need to check

\[
\begin{pmatrix}
  u_{xx...x} \\
  u_{xx...y} \\
  \vdots \\
  u_{yy...y}
\end{pmatrix}
= 
\begin{pmatrix}
  0 & 1 & 0 & \ldots & 0 \\
  0 & 0 & 1 & \ldots & 0 \\
  0 & \ddots & \ddots & \ddots & \ddots \\
  -a_0 & -a_1 & -a_2 & \ldots & -a_{m-1}
\end{pmatrix}
\begin{pmatrix}
  u_{xx...x} \\
  u_{xx...y} \\
  \vdots \\
  u_{yy...y}
\end{pmatrix},
\]

But the bottom row of (3.3) is just a restatement of \( Lu = 0 \), and the upper rows merely state the well-known equalities of mixed partial derivatives under the sufficient smoothness condition \( C^m \).

Conversely, if \( f \) satisfies (3.2), the upper rows of (3.2) imply compatibility among the components of \( f \) so that under the assumptions of \( C^1 \) and simply-connectedness of domain \( f \) is guaranteed to be "hyperexaxt", namely \( f = \nabla^{(m-1)}u \) for some \( u \) (see [9]). The substitution of \( f = \nabla^{(m-1)}u \) in (3.2) shows \( Lu = 0 \) from the bottom row of (3.2). If furthermore \( f \) happens to be at least \( C^m \), then

\[
Lf = L(\nabla^{(m-1)}u) = \nabla^{(m-1)}(Lu) = 0,
\]

where \( L \) and \( \nabla^{(m-1)} \) commute because \( u \) is \( C^{2m-1} \). This completes the proof.

According to the last statement of the theorem just proved, if \( f \) is \( C^k \), \( k \geq m \), and \( M \)-holomorphic where \( M \) is the associated matrix of \( L \), then every component of \( f \) is a \( C^k \) solution of \( L \). Can we have all the \( C^k \) solutions of \( L \) by merely looking into the components of all the \( C^k \) \( M \)-holomorphic \( f \)? The following theorem guarantees this. In fact, it turns out that we need only look at just the first component of \( f \). We see, therefore, that every \( C^k \) solution of \( Lu = 0 \) has a \( C^k \) \( M \)-conjugation, \( k \geq m \). The proof is a bit cumbersome.

**Theorem 9.** If \( u \) is \( C^k \), \( k \geq m \), and \( Lu = 0 \) in an open disk around the origin, then there exists an \( f \) also \( C^k \) with \( f_y = Mf_x \) such that \( u = f_1 \) where \( f_1 \) is the first component of \( f \) and \( M \) is the associated matrix of \( L \). For a real analytic \( u \), \( f \) will be real analytic.
Proof. With \( Z = xI + yM \) we claim that \( f \) defined below satisfies the requirement of the theorem.

\[
(3.4) \quad f(z) = \sum_{j=0}^{m-2} (Z^j/j!)\nabla^{(j)}u(0)
+ \int_0^z d[-(Z - \tilde{Z})^{m-1}/(m - 1)!]\nabla^{(m-1)}u(\tilde{z}).
\]

Note that we are regarding \( \nabla^{(j)}u(0) \) as an \( m \times 1 \) column through the a forementioned convention of appending \( m-j \) zeros if \( j \leq m-2 \). We point our however that if constants other than 0's are used, we would end up with other \( M \)-conjugations of \( u \). The fact that \( M \)-conjugations of \( u \) are not unique is not altogether unexpected. We justify our claim by checking the following four points.

**First.** The line integral in (3.4) is independent of the path going from 0 to \( z \). Referring to Proposition 1 we need only show the equality

\[
[-(Z - \tilde{Z})^{m-1}/(m - 1)!]_\tilde{\hat{x}}[\nabla^{(m-1)}\tilde{u}]_{\tilde{\hat{y}}}
= \frac{\{(Z - \tilde{Z})^{m-1}/(m - 1)!\}}{[\nabla^{(m-1)}\tilde{u}]_{\tilde{\hat{x}}},
\]

which in view of (3.3) is equivalent to the identity

\[
[(Z - \tilde{Z})^{m-2}I]M[\nabla^{(m-1)}\tilde{u}]_{\tilde{\hat{x}}} = [(Z - \tilde{Z})^{m-2}M][\nabla^{(m-1)}\tilde{u}]_{\tilde{\hat{x}}}.
\]

**Second.** \( f_1 = u \). To see this, we note from (3.4)

\[
f_1(z) = E_1 \sum_{j=0}^{m-2} [Z^j/j!]\nabla^{(j)}u(0)
+ E_1 \int_0^z d[-(Z - \tilde{Z})^{m-1}/(m - 1)!]\nabla^{(m-1)}u(\tilde{z}).
\]

We also note from (3.1d)

\[
u(z) = E_1 \sum_{j=0}^{m-2} [X^j/j!]\nabla^{(j)}u(0)
+ E_1 \int_0^z d[-(X - \tilde{X})^{m-1}/(m - 1)!]\nabla^{(m-1)}u(\tilde{z}).
\]

If we write out the matrices \( Z, Z^2, \ldots, Z^{m-1} \) (see also Example 1 below), and compare their first rows with those of \( X, X^2, \ldots, X^{m-1} \), then it becomes clear that the two expressions for \( f_1(z) \) and \( u(z) \) above are equal.

**Third.** If \( u \) is \( C^k \) or real analytic, so is \( f \). To see this we need to differentiate the integral term, call it \( R(z) \), in (3.4). But in order to
apply the differentiation formula (1.11) we first rewrite $R(z)$ using the integration-by-parts formula (1.6):

$$R(z) = \frac{Z^m - \iota}{(m-1)!} \nabla^{(m-1)} u(0)$$

$$+ \int_0^Z \frac{(Z - \tilde{Z})^{m-1}}{(m-1)!} d[\nabla^{(m-1)} u(\tilde{z})].$$

Differentiating according to (1.11), we have for $0 < j < m - 1$

$$D^{(m-1-j,i)} \int_0^Z \frac{(Z - \tilde{Z})^{m-1}}{(m-1)!} d[\nabla^{(m-1)} u(\tilde{z})]$$

$$= \int_0^Z M^{j} d[\nabla^{(m-1)} u(\tilde{z})]$$

$$= M^{j}[\nabla^{(m-1)} u(z) - \nabla^{(m-1)} u(0)].$$

Now if $u$ is $C^k$, $\nabla^{(m-1)} u$ is $C^{k-(m-1)}$, which makes $f C^{k-(m-1)+(m-1)} = C^k$. Clearly, if $u$ is real analytic, so is $f$.

Fourth. $f_y = Mf_x$. Since $Z, Z^2, \ldots, Z^{m-1}$ are all $M$-holomorphic by Corollary 6a, we need only concentrate on the integral term, call it $\tilde{R}(z)$, in (3.5). Using the formula (1.11), we see

$$[\tilde{R}(z)]_y = \int_0^Z \frac{(Z - \tilde{Z})^{m-2}}{(m-2)!} M d[\nabla^{(m-1)} u(\tilde{z})]$$

$$= M \int_0^Z \frac{(Z - \tilde{Z})^{m-2}}{(m-2)!} I d[\nabla^{(m-1)} u(\tilde{z})]$$

$$= M[\tilde{R}(z)]_x.$$

This completes the proof of Theorem 9.

**Theorem 10.** If $u$ is real analytic at the origin, and $Lu = 0$, then $u$ has the following series expansion in an open disk around the origin.

$$u(z) = E_1 \sum_{j=0}^{m-1} (Z^j / j!) \nabla^{(j)} u(0)$$

$$+ E_1 \sum_{k=0}^{\infty} (Z^{m+k} / (m+k)! D^{(k+1,0)} \nabla^{(m-1)} u(0)$$

where $E_1 = (1, 0, 0, \ldots, 0), Z = xI + yM$, and $M$ is the associated matrix of $L$.

**Proof.** Since $u$ is a real analytic solution of $L$, by Theorem 9 it has a real analytic $M$-conjugation $f$, which by Theorem 7 has an expansion
in powers of $Z$,
\begin{equation}
(3.6) \quad f(z) = \sum_{j=0}^{\infty} \frac{Z^j}{j!} f^{(j)}(0)
\end{equation}
where $f^{(j)}$ is the $j$th derivative of $f$ with respect to $Z$ and has the expression $D^{(j,0)} f$ (see Theorem 6). Taking only the top row of (3.6), we have
\begin{equation}
(3.7) \quad u(z) = E_1 \sum_{j=0}^{\infty} \frac{Z^j}{j!} f^{(j)}(0).
\end{equation}
Note that all the entries of every $Z^j$ are solutions of $L$ (see the last statement of Theorem 8), and hence (3.7) may be considered as the rearrangement of the ordinary Taylor series of $u$, which is in terms of powers of $x$ and $y$, into one which is in terms of polynomial solutions of $u$. However these two series are distinct only from $m$th degree terms onward. For $j \leq m - 1$ the top of $Z^j$ consists merely of those powers in $x$ and $y$ that appear in the binomial expansion of $(x+y)^j$. In other words we must have
\[ E_1[(Z^j/j!)] f^{(j)}(0) = E_1[(Z^j/j!) \nabla^{(j)} u(0)] \]
for $0 \leq j \leq m - 1$. For higher degree terms we will show, for $0 \leq k < \infty$,
\[ E_1[(Z^{m+k}/(m+k)!)] f^{(m+k)}(0) = E_1[(Z^{m+k}/(m+k)!)] D^{(m+1,0)} \nabla^{(m-1)} u(0)]. \]
Now since $f$ is $M$-holomorphic, we have by Theorem 6
\[ f^{(m+k)} = D^{(m+k,0)} f. \]
Also since $f$ is $M$-holomorphic, there exists $\phi$ by Theorem 8 such that
\[ f = \nabla^{(m-1)} \phi. \]
Consequently,
\[ f^{(m+k)} = D^{(m+k,0)} \nabla^{(m-1)} \phi \]
\[ = D^{(k+1,0)} D^{(m-1,0)} \nabla^{(m-1)} \phi \]
\[ = D^{(k+1,0)} \nabla^{(m-1)} D^{(m-1,0)} \phi \]
\[ = D^{(k+1,0)} \nabla^{(m-1)} E_1 \nabla^{(m-1)} \phi \]
\[ = D^{(k+1,0)} \nabla^{(m-1)} E_1 f = D^{(k+1,0)} \nabla^{(m-1)} u \]
which completes the proof.
**Theorem 11.** The totality of real analytic solutions of $Lu = 0$ at the origin is given by

\[(3.8) \quad u = E_1 \sum_{k=0}^{\infty} Z^k c_k\]

where $c_k$'s are any $m \times 1$ constant column vectors for which the series converges within a certain radius of convergence. $Z = xI + yM$, and $M$ is the associated matrix of $L$.

**Proof.** First it is clear from Theorem 10 that every real analytic solution of $Lu = 0$ at the origin is of the form (3.8). Next clearly (3.8) is a solution of $Lu = 0$ within the radius of convergence, for termwise differentiations give

\[Lu = E_1 \sum_{k=0}^{\infty} L(Z^k c_k) = 0.\]

To check $L(Z^k c_k) = 0$, note that $Z^k$ is $M$-holomorphic by Corollary 6a, hence every column of $Z^k$ is also $M$-holomorphic, and so is their linear combination $Z^k c_k$. Thus $Z^k c_k$ is an $M$-conjugation, and consequently $L(Z^k c_k) = 0$ by the last statement of Theorem 8. This completes the proof.

**Theorem 12.** For $k \geq m$ the $m$ polynomials appearing in the top row of $Z^k$ constitute a basis for the $k$th degree homogeneous polynomial solutions of $Lu = 0$.

**Proof.** In view of Theorem 11 we need only show the linear independence of polynomials in $E_1 Z^k$. We proceed by induction. First note that polynomials in $E_1 Z^{m-1}$ are just the terms in the binomial expansion $(x + y)^{m-1}$, and they are easily shown to be linearly independent. Next, assuming polynomials in $E_1 Z^k$ are independent, we show the polynomials in $E_1 Z^{k+1}$ are independent. So suppose

\[E_1 Z^{k+1} c_{k+1} = 0\]

for some constant column vector $c_{k+1}$. Differentiating with respect to $x$, we have

\[E_1 (k+1) Z^k c_{k+1} = 0,\]

which implies by the induction hypothesis that $c_{k+1} = 0$.

Note that for $k < m$ all $k$th degree homogeneous polynomials are trivially solutions of $Lu = 0$, and the terms in the binomial expansion $(x + y)^k$ constitute a basis.
We exhibit polynomial bases for real analytic solutions of some familiar partial differential equations.

**Example 1.** Consider the biharmonic equation

\[ u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0. \]

Here we have

\[ L = D^{(4,0)} + 0D^{(3,1)} + 2D^{(2,2)} + 0D^{(1,3)} + D^{(0,4)}, \]

and accordingly,

\[ M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \end{pmatrix}, \]

and since \( Z = xI + yM, \)

\[ Z = \begin{pmatrix} x & y & 0 & 0 \\ 0 & x & y & 0 \\ 0 & 0 & x & y \\ -y & 0 & -2y & x \end{pmatrix}. \]

The top rows of \( Z^0, Z^1, Z^2, Z^3 \) produce lower degree polynomials, which trivially satisfy the 4th order biharmonic equation. From \( Z^4 \) on we begin to obtain all the nontrivial biharmonic polynomials. These polynomials form a basis for the real analytic solutions. We list these polynomials up to degree 5.

- **Example 2.** For the wave equation \( u_{xx} - u_{yy} = 0, \) the top rows of \( Z^k \) for \( 0 \leq k \leq 5 \) are

  \[
  \begin{array}{c|c|c|c|c|c}
  k & 0 & 1 & 0 & 0 & 0 \\
  k = 1 & x & y & 0 & 0 \\
  k = 2 & x^2 & 2xy & y^2 & 0 \\
  k = 3 & x^3 & 3x^2y & 3xy^2 & y^3 \\
  k = 4 & x^4 - y^4 & 4x^3y & 6x^2y^2 - 2y^4 & 4xy^3 \\
  k = 5 & x^5 - 5x^2y^4 & 5x^4y - y^5 & 10x^3y^2 - 10xy^4 & 10x^2y^3 - 2y^5 \\
  \end{array}
  \]
EXAMPLE 3. For the Laplace equation $u_{xx} + u_{yy} = 0$, the top rows of $Z^k$ for $0 \leq k \leq 5$ are

\begin{align*}
  k = 0 & \quad 1 & \quad 0 \\
  k = 1 & \quad x & \quad y \\
  k = 2 & \quad x^2 - y^2 & \quad 2xy \\
  k = 3 & \quad x^3 - 3xy^2 & \quad 3x^2y - y^3 \\
  k = 4 & \quad x^4 - 6x^2y^2 + y^4 & \quad 4x^3y - 4xy^3 \\
  k = 5 & \quad x^5 - 10x^3y^2 + 5xy^4 & \quad 5x^4y - 10x^2y^3 + y^5
\end{align*}

The examples above, especially Example 1, demonstrate our simple but quite universal algorithm for obtaining all the polynomial solutions of all the equations of the type (3.1): From the coefficients in (3.1) we construct the square matrix $M$ as shown in (3.3), thence the “generating” matrix $Z = xI + yM$; we then find all the basic $k$th degree homogeneous polynomial solutions of (3.1) in the top row of the matrix $Z^k$.

The author is grateful to G. N. Hile for his encouragement in writing this paper.

REFERENCES


Received April 18, 1988.

UNIVERSITY OF HAWAII
HONOLULU, HI 96822
PACIFIC JOURNAL OF MATHEMATICS

EDITORS

V. S. VARADARAJAN
(Managing Editor)
University of California
Los Angeles, CA 90024-1555-05

HERBERT CLEMENS
University of Utah
Salt Lake City, UT 84112

THOMAS ENRIGHT
University of California, San Diego
La Jolla, CA 92093

FRANK FINN
Stanford University
Stanford, CA 94305

HERMANN FLASCHKA
University of Arizona
Tucson, AZ 85721

VAUGHAN F. R. JONES
University of California
Berkeley, CA 94720

ROBION KIRBY
University of California
Berkeley, CA 94720

R. FINN
Stanford University
Stanford, CA 94305

C. C. MOORE
University of Utah
Salt Lake City, UT 84112

STEVEN KERCKHOFF
Stanford University
Stanford, CA 94305

HERMANN FLASCHKA
University of Arizona
Tucson, AZ 85721

ROBION KIRBY
University of California
Berkeley, CA 94720

C. C. MOORE
University of California
Los Angeles, CA 90024-1555-05

HAROLD STARK
University of California, San Diego
La Jolla, CA 92093

ASSOCIATE EDITORS

R. ARENS
E. F. BECKENBACH
B. H. NEUMANN
F. WOLF
K. YOSHIDA

(1906–1982)
(1904–1989)

SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA
UNIVERSITY OF CALIFORNIA
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
OREGON STATE UNIVERSITY

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF SOUTHERN CALIFORNIA
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California 90024-1555-05.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 50 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is issued monthly as of January 1966. Regular subscription rate: $190.00 a year (6 Vols., 12 issues). Special rate: $95.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924 (ISSN 0030-8730) publishes 6 volumes per year. Second-class postage paid at Carmel Valley, California 93924, and additional mailing offices. Postmaster: send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Copyright © 1990 by Pacific Journal of Mathematics
Christopher J. Bishop, Bounded functions in the little Bloch space ........ 209
Lutz Bungart, Piecewise smooth approximations to q-plurisubharmonic functions .................................................. 227
Donald John Charles Bures and Hong Sheng Yin, Outer conjugacy of shifts on the hyperfinite II$_1$-factor ..................................... 245
A. D. Raza Choudary, On the resultant hypersurface ......................... 259
Luis A. Cordero and Robert Wolak, Examples of foliations with foliated geometric structures ........................................... 265
Peter J. Holden, Extension theorems for functions of vanishing mean oscillation .......................................................... 277
Detlef Müller, A geometric bound for maximal functions associated to convex bodies ...................................................... 297
John R. Schulenberger, Time-harmonic solutions of some dissipative problems for Maxwell’s equations in a three-dimensional half space .... 313
Mark Andrew Smith and Barry Turett, Normal structure in Bochner $L^p$-spaces ............................................................. 347
Jun-ichi Tanaka, Blaschke cocycles and generators ............................. 357
R. Z. Yeh, Hyperholomorphic functions and higher order partial differential equations in the plane .................................. 379