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**NOTE ON THE INEQUALITY OF THE ARITHMETIC AND
GEOMETRIC MEANS**

HAO ZHI CHUAN

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We show how to insert a continuum of additional terms (defined by an integral and depending on an arbitrary positive parameter) between the two sides of the generalized arithmetic-geometric mean inequality with weights. Applications give an inequality involving positive definite matrices and also a refinement of the inequality connecting the inscribed and circumscribed radii of a triangle.

We suppose throughout that

$$(1) \quad n \in \mathbb{N} \quad \text{and} \quad a_j > 0, \quad q_j > 0 \quad (j = 1, \dots, n), \quad q_1 + \dots + q_n = 1.$$

Then we have the well-known inequality of the means (e.g. [2, #9])

$$(2) \quad \prod_{j=1}^n a_j^{q_j} \leq \sum_{j=1}^n q_j a_j,$$

with equality if and only if $a_j = a_1$ ($j = 1, \dots, n$).

THEOREM 1. *If (1) holds and if $p > 0$, then*

$$(3) \quad \prod_{j=1}^n a_j^{q_j} \leq \left\{ p \int_0^\infty \left[\prod_{j=1}^n (x + a_j)^{q_j} \right]^{-p-1} dx \right\}^{-1/p} \leq \sum_{j=1}^n q_j a_j.$$

Proof. For $x \geq 0$, we replace a_j by $x + a_j$ in (2); then

$$0 < \prod_{j=1}^n (x + a_j)^{q_j} \leq \sum_{j=1}^n q_j (x + a_j) = x + \sum_{j=1}^n q_j a_j.$$

Hence (for $p > 0$)

$$(4) \quad \int_0^\infty \left[\prod_{j=1}^n (x + a_j)^{q_j} \right]^{-p-1} dx \\
 \geq \int_0^\infty \left[x + \sum_{j=1}^n q_j a_j \right]^{-p-1} dx = \frac{1}{p} \left(\sum_{j=1}^n q_j a_j \right)^{-p}.$$

In addition, by Hölder's integral inequality for n functions [2, #188],

$$\begin{aligned}
 (5) \quad \int_0^\infty \left[\prod_{j=1}^n (x + a_j)^{q_j} \right]^{-p-1} dx &= \int_0^\infty \prod_{j=1}^n [(x + a_j)^{-p-1}]^{q_j} dx \\
 &\leq \prod_{j=1}^n \left[\int_0^\infty (x + a_j)^{-p-1} dx \right]^{q_j} \\
 &= \prod_{j=1}^n \left[\int_0^\infty (x + a_j)^{-p-1} dx \right]^{q_j} = \prod_{j=1}^n \frac{1}{p} (a_j)^{-p q_j}.
 \end{aligned}$$

Multiplying (4) and (5) through by p and raising both sides to the power $-1/p$, we obtain respectively the right and left sides of (3). \square

REMARKS. (a) As examination of the proof of Theorem 1 shows, there is strict inequality in each part of (3) unless $a_j = a_1$ ($j = 1, \dots, n$).

(b) If $q_1 = \dots = q_n = 1/n$ in (3) then, for any $p > 0$,

$$\begin{aligned}
 (6) \quad (a_1 a_2 \cdots a_n)^{1/n} &\leq \left\{ p \int_0^\infty [(x + a_1) \cdots (x + a_n)]^{-(p+1)/n} dx \right\}^{-1/p} \\
 &\leq \frac{1}{n} (a_1 + \cdots + a_n).
 \end{aligned}$$

Suppose $a_{ij} > 0$ ($i = 1, \dots, m; j = 1, \dots, n$); then ([2, #11])

$$(7) \quad \sum_{i=1}^m \prod_{j=1}^n a_{ij}^{q_j} \leq \prod_{j=1}^n \left(\sum_{i=1}^m a_{ij} \right)^{q_j}.$$

If we then use $a_j := \sum_{i=1}^m a_{ij}$ in (3) and combine (6) with the left side of (3), we obtain the apparently more general result (reducing to (3) for $m = 1$):

COROLLARY 1.1. Let $a_{ij} > 0$ ($i = 1, \dots, m; j = 1, \dots, n$), $p > 0$. Then

$$\begin{aligned}
 (8) \quad \sum_{i=1}^m \prod_{j=1}^n a_{ij}^{q_j} &\leq \left\{ p \int_0^\infty \left[\prod_{j=1}^n \left(x + \sum_{i=1}^m a_{ij} \right)^{q_j} \right]^{-p-1} dx \right\}^{-1/p} \\
 &\leq \sum_{i=1}^m \sum_{j=1}^n q_j a_{ij}.
 \end{aligned}$$

If A is a real positive definite $n \times n$ matrix (namely the quadratic form vAv is positive for all non-trivial n -vectors v) then it is well known that the eigenvalues a_j ($j = 1, \dots, n$) are all positive. Indeed, since the a_j are the solutions of the polynomial equation $|\lambda I - A| := \det(\lambda I - A) = 0$, we clearly also have

$$|xI + A| = \prod_{j=1}^n (x + a_j) \quad \text{for any } x \in \mathbb{R}.$$

From this equation ($x = 0$) and the definition of the trace, we therefore have

$$|A| = \prod_{j=1}^n a_j > 0, \quad \text{tr } A = \sum_{j=1}^n a_j.$$

Using these a_j in (6), we then obtain

COROLLARY 1.2. *Let A be a real positive definite $n \times n$ matrix and $p > 0$. Then*

$$(9) \quad |A|^{1/n} \leq \left\{ p \int_0^\infty |xI + A|^{-(p+1)/n} dx \right\}^{-1/p} \leq \frac{1}{n} \text{tr } A.$$

There is an analogue of this result similar to Corollary 1.1; replace A by $\sum_{i=1}^m A_i$ in Corollary 1.2 and use Minkowski's inequality for positive definite $n \times n$ matrices (e.g. [1, p. 70, Theorem 15]), that

$$\sum_{i=1}^m |A_i|^{1/n} \leq \left| \sum_{i=1}^m A_i \right|^{1/n}$$

(this is really (7) in disguise, with $q_1 = \dots = q_n = 1/n$). We obtain immediately

COROLLARY 1.3. *Let A_i ($i = 1, \dots, m$) be real positive definite $n \times n$ matrices, and $p > 0$. Then*

$$(10) \quad \sum_{i=1}^m |A_i|^{1/n} \leq \left\{ p \int_0^\infty \left| xI + \sum_{i=1}^m A_i \right|^{-(p+1)/n} dx \right\}^{-1/p} \\ \leq \frac{1}{n} \text{tr} \left(\sum_{i=1}^m A_i \right).$$

As a further application of Theorem 1, we show how to insert additional terms between the two sides of Euler's inequality $2r \leq R$ (e.g.

see [3, p. 79] or [4, v.2: §17.3, p. 161]) connecting the circumscribed radius R and the inscribed radius r of a triangle. If the triangle has angles A, B, C with sides a, b, c opposite these angles, and area Δ , then, using the sine rule,

$$(11) \quad R = \frac{a + b + c}{2(\sin A + \sin B + \sin C)} \geq \frac{a + b + c}{3\sqrt{3}}.$$

Also

$$2r(a + b + c) = 4\Delta \leq \sqrt{3}(abc)^{2/3}$$

(see [4, v.2: §17.3, pp. 161, 372]) and so, by the arithmetic-geometric mean inequality,

$$(12) \quad 2r \leq \frac{\sqrt{3}(abc)^{2/3}}{a + b + c} \leq \frac{\sqrt{3}(abc)^{2/3}}{3(abc)^{1/3}} = \frac{(abc)^{1/3}}{\sqrt{3}}.$$

Now, by (11), (12) and (6), we have:

COROLLARY 1.4. *If a, b, c are the sides of a triangle, with inscribed radius r and circumscribed radius R , then, for any $p > 0$,*

$$(13) \quad 2r\sqrt{3} \leq (abc)^{1/3} \leq J(a, b, c; p) \leq \frac{1}{3}(a + b + c) \leq R\sqrt{3},$$

where

$$J(a, b, c; p) := \left\{ p \int_0^\infty [(x + a)(x + b)(x + c)]^{-(p+1)/3} dx \right\}^{-1/p}.$$

There is strict inequality throughout (13) unless $a = b = c$.

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