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## **DIFFERENTIAL GEOMETRY OF SYSTEMS OF PROJECTIONS IN BANACH ALGEBRAS**

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Let  $A$  be a Banach algebra,  $n$  a positive integer and  $Q_n = \{(q_1, \dots, q_n) \in A^n : q_i q_k = \delta_{ik} q_i, q_1 + \dots + q_n = 1\}$ . The differential geometry of  $Q_n$ , as a discrete union of homogeneous spaces of the group  $G$  of units of  $A$  is studied, a connection on the principal bundle  $G \rightarrow Q_n$  is defined and invariants of the associated connection on the tangent bundle  $TQ_n$  are determined.

**Introduction.** The structure of the set  $Q$  of all idempotent elements of a Banach algebra  $A$  plays a fundamental role in several aspects of spectral theory. This work deals with the differential structure of the space

$$Q_n = \left\{ (q_1, \dots, q_n) \in A^n : q_i q_k = \delta_{ik} q_i, \sum_{i=1}^n q_i = 1 \right\}$$

of systems of  $n$  “orthogonal” projections in  $A$ .

The manifold  $Q_n$  appears as a universal model when certain polynomial equations are considered. More precisely, if  $\alpha_1, \dots, \alpha_n$  are *different* complex numbers and  $\alpha(X)$  denotes the polynomial  $(X - \alpha_1) \cdots (X - \alpha_n)$ , then the set  $A_\alpha = \{a \in A : \alpha(a) = 0\}$  is a closed submanifold which is diffeomorphic to  $Q_n$ . Thus  $Q_n$  is the model for all simple algebraic elements of  $A$  of degree  $n$ . Moreover,  $Q_n$  plays a role in the study of arbitrary algebraic (in particular, nilpotent) elements (see [AS]).

Section 1 contains the description of the differential structure of  $Q_n$  and  $A_\alpha$  as closed analytic submanifolds of  $A^n$  and  $A$ , respectively; it contains also the proof that  $Q_n$  and  $A_\alpha$  are diffeomorphic.

Using Kaplansky’s notion of SBI-rings, we recover a result of Barnes [Ba] concerning the surjectivity of  $A_\alpha \rightarrow B_\alpha$  when  $B$  is the quotient of  $A$  by its Jacobson radical. In §2 we show that  $Q_n$  is a discrete union of homogeneous spaces of  $G$ , the group of units of  $A$ ; this fact, together with a classical result of Michael [Mi], shows that an epimorphism  $f: A \rightarrow B$  of Banach algebras induces Serre fibrations  $Q_n(A) \rightarrow Q_n(B)$  and  $A_\alpha \rightarrow B_\alpha$ . In §3 we obtain an explicit way of

lifting differentiable curves in  $Q_n$  to  $G$  by solving a linear differential equation which we call the *transport equation*; this fact is due to Daleckii and S. G. Krein [DK] and T. Kato [Ka1] but its geometrical meaning is new. In fact, in §4 we define a connection in the principal bundle  $G \rightarrow Q_n$  and show that the horizontal liftings of differentiable curves in  $Q_n$  are precisely the solutions of the transport equation.

Several invariants of the tangent bundle of  $Q_n$  are calculated in §5 (covariant derivative, curvature, geodesics, etc.). As observed by Kato [Ka1], [Ka2, II.4] the lifting theorem has important applications in quantum mechanics (see [Ga], [GS]). A remark about  $C^*$ -algebras is in order: our results extend to the case of some involution algebras, in particular to all  $C^*$ -algebras. For instance, the transport equation has a unitary solution if the curve has selfadjoint values; in a forthcoming paper the immersion of

$$P_n = \{p \in Q_n : p_i^* = p_i, i = 1, \dots, n\}$$

into  $Q_n$  will be studied, together with associated fibrations  $Q_n \rightarrow P_n$ .

Concerning the references, the reader may consult Rickart's book [Ri] for the literature up to 1960; the topology of the space of idempotents  $Q = Q_2$  has been considered in [PR1], [Ra], [Ko], [Ze], [Au], [Gr] and with special emphasis on the differential structure of  $Q$  in [Ra], [Gr], [Ki], [HK]; for the transport equation the reader may consult [Ka1] and [DK2]; in [PR2] the differential geometry of  $P = P_2$  is needed for the study of minimality of geodesics; see also [CPR2] for a related problem; finally, the case of algebraic operators on Hilbert space, the reader may consult the books [He] and [AFVH]. In particular, some problems concerning the set  $P_n$  in this context are discussed in [CH]. The set  $Q_n$  appears, implicitly or explicitly, in various works; we only mention [Ja, p. 54], [Ka2, II.5] and [DK2, Chapter IV].

**1. Differential structure of systems of projections.** Let  $A$  be a real or complex algebra with identity 1. Denote by  $G = G(A)$  the group of units of  $A$  and by  $Q = Q(A)$  the set of all idempotents of  $A$ .

Suppose that the polynomial  $\alpha(X) = \prod_{i=1}^n (X - \alpha_i)$  has different roots  $\alpha_1, \dots, \alpha_n$  in the field. Let  $g_j(X) = \prod_{i \neq j} (X - \alpha_i)$  and  $q_j(X) = g_j(X)/g_j(\alpha_j)$ . Then  $q_j(X)$  has degree  $n - 1$ ,  $q_j(\alpha_i) = \delta_{ji}$ , for  $i \neq j$   $q_i(X)q_j(X) = h(X)\alpha(X)$  for some polynomial  $h(X)$  and  $\sum_{i=1}^n q_i(X) = 1$  (because  $1 - \sum_{i=1}^n q_i(X)$  has degree  $\leq n - 1$  and it vanishes at  $n$  values, the  $\alpha_j$ ).

Let  $A_\alpha$  denote the solution set of  $\alpha$ , i.e., the set of all  $a \in A$  with  $\alpha(a) = 0$ .

1.1. PROPOSITION. *Let  $a \in A(\alpha)$ . Then*

- (i)  $\sum_{i=1}^n q_i(a) = 1$ ;
- (ii)  $q_i(a)q_j(a) = 0$  if  $i \neq j$ ;
- (iii)  $q_i(a) \in Q$ ,  $i = 1, \dots, n$ ;
- (iv)  $q_i(a)a = aq_i(a) = \alpha_i q_i(a)$ ,  $i = 1, \dots, n$ .

*Proof.* (i) follows from  $\sum_{i=1}^n q_i(X) = 1$  and (ii) follows from the equality  $q_i(X)q_j(X) = h(X)\alpha(X)$ . From (i) and (ii),

$$q_i(a) = q_i(a) \sum_{k=1}^n q_k(a) = \sum_{k=1}^n q_i(a)q_k(a) = q_i(a)^2,$$

which gives (iii). Finally from  $\alpha(X) = c(X - \alpha_i)q_i(X)$  (with  $c = g_i(\alpha_i) \neq 0$ ) it follows that  $0 = \alpha(a) = c(aq_i(a) - \alpha_i q_i(a))$  and this completes the proof because  $q_i(a)$  commutes with  $a$ .

Let  $Q_n = Q_n(A)$  denote the set of all  $n$ -tuples of idempotents  $q_i$  of  $A$  which satisfy  $q_i q_j = 0$  if  $i \neq j$  and  $\sum_{i=1}^n q_i = 1$ .

1.2. PROPOSITION. *The mapping  $a \rightarrow (q_1(a), \dots, q_n(a))$  is a bijection from  $A_\alpha$  onto  $Q_n$  whose inverse is  $(q_1, \dots, q_n) \rightarrow \sum_{i=1}^n \alpha_i q_i$ .*

The proof is a straightforward application of Proposition 1.1. Thus, from a set-theoretical view point,  $Q_n$  is a universal model for the sets  $A_\alpha$ . We shall extend this result to the differential geometry setting.

1.3. REMARK. I. Kaplansky introduced the notion of SBI-rings (SBI = suitable for building idempotents) as those rings  $A$  such that the natural mapping  $Q(A) \rightarrow Q(A/R)$  is onto, where  $R$  is the Jacobson radical of  $A$ .

It is known that for a SBI-ring  $A$ , the map  $Q_n(A) \rightarrow Q_n(A/R)$  is also onto for each  $n = 1, 2, \dots$  (see [Ja, p. 54]).

It is also known that all Banach algebras are SBI [Ri, p. 58]. These facts and 1.2 imply that, for every  $\alpha = (\alpha_1, \dots, \alpha_n)$  (with  $\alpha_i \neq \alpha_k$ ),  $A_\alpha \rightarrow (A/R)_\alpha$  is onto, a result due to Barnes [Ba, Theorem 7].

From now on, we will assume that  $A$  is a real or complex Banach algebra with identity. For  $n$ -tuples  $Z = (Z_1, \dots, Z_n)$  in  $A^n$  we use the norm  $\|Z\| = \max_{1 \leq i \leq n} \|Z_i\|$ . The general facts on Banach algebras and Banach manifolds needed below can be found in [Ri] and [La], respectively.

1.4. THEOREM. Let  $a \in A_\alpha$  be a fixed element,  $q = q(a) = (q_1(a), \dots, q_n(a)) \in Q_n$  the corresponding system of idempotents. Set

$$T = \{X \in A; q_i X q_i = 0 \text{ for all } i = 1, \dots, n\},$$

$$S = \{Y \in A; q_k Y q_l = 0 \text{ for all } k \neq l\}.$$

1.4.(i)  $A$  is the Banach space direct sum  $A = T \oplus S$ .

1.4.(ii) For each  $Z = X + Y$ ,  $X \in T$ ,  $Y \in S$ , set

$$X' = \sum_{i \neq k} q_i X q_k / (\alpha_k - \alpha_i)$$

and define

$$\phi(Z) = \exp(X')(a + Y) \exp(-X').$$

Then  $\phi$  is a diffeomorphism from a neighborhood  $U$  of  $O \in A$  onto a neighborhood  $V$  of  $a$ . Moreover,  $\phi|_{U \cap T}$  is a homeomorphism onto  $V \cap A_\alpha$ .

*Proof.* It is clear that every  $Z \in A$  decomposes as  $X + Y$ , where

$$X = \sum_{j \neq k} q_j Z q_k \in T \quad \text{and}$$

$$Y = \sum_l q_l Z q_l \in S, \quad \text{for } \sum_{l=1}^n q_l = 1 \quad \text{and}$$

$$Z = \left( \sum q_l \right) Z \left( \sum q_l \right) = \sum_{j \neq k} q_j Z q_k + \sum_l q_l Z q_l.$$

It is also clear that the decomposition is topological, for  $T$  and  $S$  are respectively defined as the images of the projections

$$Z \rightarrow \sum_{j \neq k} q_j Z q_k \quad \text{and} \quad Z \rightarrow \sum_l q_l Z q_l.$$

An easy computation shows that the derivative of  $\phi$  at  $O$  is the identity: in fact, for  $Y \in S$   $D\phi(O)Y = Y$  obviously; for  $X \in T$   $D\phi(O)X = [X', a] = X'a - aX' = X$ ; the assertion follows from the decomposition  $A = T \oplus S$ .

Then, by the inverse function theorem, there exist open neighborhoods  $U'$  of  $O$  and  $V'$  of  $a$  such that  $\phi$  maps  $U'$  diffeomorphically onto  $V'$ . Consider next  $Z = X + Y$  with  $\phi(Z) \in A_\alpha$ . Since

$\phi(Z) = M(a + Y)M^{-1}$ , then  $a + Y$  is also a root of  $\alpha$ . Then  $O = \prod_i(a + Y - \alpha_i)$  and using Prop. 1.1.(iv):

$$\begin{aligned} O &= q_j \prod_i(a + Y - \alpha_i) = q_j \prod_i(\alpha_j + Y - \alpha_i) \\ &= q_j YL \end{aligned}$$

where  $L = \prod_{j \neq i}(Y - (\alpha_i - \alpha_j))$ . If  $Y$  has small norm ( $\|Y\| < \min\{|\alpha_i - \alpha_j|, i \neq j\}$  suffices) then  $L$  is invertible and therefore  $q_j Y = 0$  for each  $j$ . Hence  $\phi(Z) \in A_\alpha$  with  $Y$  small implies  $Z \in T$ . This means that (perhaps for smaller neighborhoods)  $\phi$  is a homeomorphism from  $U' \cap T$  onto  $V' \cap V_\alpha$ .

Considering the maps  $\phi$  as analytic local coordinates in  $A$ , we obtain:

**1.5. COROLLARY.**  $A_\alpha$  is a closed analytic submanifold of  $A$  whose tangent space at  $a \in A_\alpha$  can be identified to the Banach space  $T$ .

**1.6. REMARKS.** (i) The choice of the chart  $\phi$  may seem rather artificial; for instance, the derivative at  $O$  of  $\phi_1(X + Y) = \exp(X)(a + Y)\exp(-X)$  is  $X + Y \rightarrow Xa - aX + Y = [X, a] + Y$  and the equalities  $q_i[X, a]q_j = (\alpha_j - \alpha_i)q_i X q_j$  ( $i \neq j$ ) show that  $D\phi_1(O)$  maps  $T$  onto  $T$  and  $S$  onto  $S$ . Thus,  $\phi_1$  also provides charts for the analytic structure of  $A_\alpha$ . However, we have chosen the map  $\phi$  because it is the exponential map of the natural connection to be studied later (see §4). This remarks applies also to the charts chosen below for  $Q_n$ .

(ii) An obvious consequence of 1.3 is that  $A_\alpha$  is locally arcwise connected for all  $\alpha$  as above. For the simpler case of  $\alpha(X) = X(X - 1)$  this is a result of Zemanek [Ze, 3.2] for complex Banach algebras, which was generalized for real algebras by Aupetit [Au, p. 413]. However both results have been also proved in [PR1, 4.3] (see also 2.2(iii) below).

**1.7. THEOREM.**  $Q_n$  is a closed submanifold of  $A^n$ .

*Proof.* Fix  $q \in Q_n$  and define  $T' = \{X = (X_1, \dots, X_n) \in A^n: q_r X_i q_s = 0 \text{ for } r \neq i \text{ and } s \neq i \text{ or } r = s = i, \text{ and } q_i X_i q_k + q_i X_k q_k = 0 \text{ for } i \neq k\}$ .

The map  $\theta: A^n \rightarrow A^n$ ,  $\theta(Z_1, \dots, Z_n) = (X_1, \dots, X_n)$  defined by

$$\begin{aligned} X_1 &= \sum_{i>1} q_1 Z_1 q_i + q_i Z_1 q_1, \\ X_2 &= \left( \sum_{i>2} q_2 Z_2 q_i + q_i Z_2 q_i \right) - (q_1 Z_1 q_2 + q_2 Z_1 q_1), \\ &\vdots \\ X_k &= \sum_{i>k} (q_k Z_k q_i + q_i Z_k q_k) - \sum_{i<k} (q_i Z_i q_k + q_k Z_i q_i) \quad (k \leq n-1), \\ X_n &= - \sum_{k=1}^{n-1} X_k \end{aligned}$$

is a projection onto  $T'$  whose kernel is the set  $S'$  of all  $Y = (Y_1, \dots, Y_n) \in A^n$  with  $q_r Y_i q_s = 0$  for  $r = i$  and  $s > i$  or  $s = i$  and  $r > i$ .

Thus  $T' \oplus S' = A^n$ . For  $X \in T'$  put

$$\tilde{X} = \sum_{i \neq j} \tilde{X}_{ij} \quad \text{where } \tilde{X}_{ij} = \begin{cases} q_i X_j q_j & \text{if } j < i, \\ -q_i X_i q_j & \text{if } i < j. \end{cases}$$

Observe that  $q_i \tilde{X} q_i = 0$  for  $i = 1, \dots, n$ .

Consider now the map  $\psi: A^n \rightarrow A^n$  defined by

$$\psi(Z)_i = \psi(X + Y)_i = \exp(\tilde{X})(q_i Y_i) \exp(-\tilde{X})$$

for  $X \in T'$ ,  $Y \in S'$ . Then  $D\psi(O)Y = Y$  for  $Y \in S'$  and, calculating,

$$(D\psi(O)X)_i = [\tilde{X}, q_i] = X_i \quad \text{for } X \in T', \quad i = 1, \dots, n.$$

This means that  $D\psi(O) = \text{identity}$  and  $\psi$  is a diffeomorphism from a neighborhood of  $O$  onto a neighborhood of  $q$ . For  $Y \in S'$  such that  $\|Y\| < 1$  it is easily shown that  $q + Y \in Q_n$  if and only if  $Y = O$ . This completes the proof.

**REMARK.** According to Proposition 1.2, the bijections connecting  $A_\alpha$  and  $Q_n$  are given by algebraic expressions.

The next result, whose proof follows easily from the theorems above, shows that  $Q_n$  is a universal model for the sets  $A_\alpha$  of simple algebraic elements of degree  $n$ .

**1.8. THEOREM.** *The map  $a \rightarrow (q_1(a), \dots, q_n(a))$  is a diffeomorphism from  $A_\alpha$  onto  $Q_n$  whose inverse is given by  $(q_1, \dots, q_n) \rightarrow$*

$\sum_{i=1}^n \alpha_i q_i$ . Consequently, for any other  $\beta = (\beta_1, \dots, \beta_n)$  with  $\beta_i \neq \beta_j$  the map  $a \rightarrow \sum_{i=1}^n \beta_i q_i(a)$  is a diffeomorphism from  $A_\alpha$  onto  $A_\beta$ .

**2. Fibrations.** The group  $G$  of invertible elements of  $A$  acts on  $Q_n$  by inner automorphisms on each coordinate: if  $g \in G$  and  $q = (q_1, \dots, q_n) \in P_n$  then  $gqg^{-1} = (gq_1g^{-1}, \dots, gq_ng^{-1}) \in Q_n$ .

**2.1. THEOREM.** Let  $q$  be a fixed element of  $Q_n$  and define  $\pi: G \rightarrow Q_n$  by  $\pi(g) = gqg^{-1}$ . Then

(i) there exist an open neighborhood  $U$  of  $q$  in  $Q_n$  and a local section  $\sigma: U \rightarrow G$  of  $\pi$ ;

(ii) the orbit  $V_q = \{gqg^{-1}: g \in G\}$  is open (and closed) in  $Q_n$ ;

(iii)  $\pi: G \rightarrow V_q$  is a principal fiber bundle with structure group  $G_0 = \{g \in G: gq_i = q_i g, i = 1, \dots, n\}$ .

Therefore  $Q_n$  is a discrete union of homogeneous spaces of  $G$ .

*Proof.* Given  $q' \in Q_n$  define

$$\sigma(q') = \langle q, q' \rangle = q'_1 q_1 + \dots + q'_n q_n.$$

It is clear that  $\sigma(q) = 1$  and  $\sigma(q)q_i = q'_i \sigma(q')$ . Thus, for every  $q'$  in a neighborhood  $U$  of  $q$ , we have  $\sigma(q') \in G$  and  $\sigma(q')q\sigma(q')^{-1} = q'$ . This proves (i) and (ii) and the rest of the statement follows from standard arguments (see [St, §7]).

**2.2. REMARKS.** (i) An invertible element  $g$  belongs to  $G_0$  if and only if  $q_k g q_l = 0$  for all  $k \neq l$ . Thus, the Lie algebra of  $G_0$  can be identified to  $\{X \in A: q_k X q_l = 0 \text{ for all } k \neq l\}$ .

(ii) With the notations of 2.1 and 1.6 it is easy to describe trivializations of the tangent bundle  $TQ_n$  and of a supplement  $NQ_n$  of  $TQ_n$  in the trivial bundle  $\varepsilon: Q_n \times A^n \rightarrow Q_n$ . We call  $NQ_n$  the “normal bundle” of  $Q_n$ . Given  $q \in Q_n$ , let  $U_q = \{q' \in Q_n: \sigma(q') \in G\}$ . Then  $h: U_q \times A^n \rightarrow U_q \times A^n$ , defined by

$$h(q', Z) = (q', \sigma(q')Z\sigma(q')^{-1}),$$

is a diffeomorphism which trivializes simultaneously  $\tau: TQ_n \rightarrow Q_n$  and a bundle  $\nu: NQ_n \rightarrow Q_n$  where  $(NQ_n)_q = S'$  (as in 1.6).

(iii) Given  $q \in Q_n$ , its connected component (in  $Q_n$ ) can be described as the set  $\{gqg^{-1}: g \in G^0\}$ , where  $G^0$  is the connected component of 1 in  $G$ : in fact, it suffices to replace  $G$  by  $G^0$  in the proof of 2.1. Of course, similar statements hold for  $A_\alpha$ . This generalizes [Ze, Theorem 3.3] and [Au].

**2.3. COROLLARY.** *Consider a fixed  $q \in Q_n$  and a continuous curve  $\gamma: [0, 1] \rightarrow Q_n$  such that  $\gamma(0) = q$ . Then, there exists a continuous curve  $\Gamma: [0, 1] \rightarrow G$  such that  $\Gamma(0) = 1$  and  $\pi \circ \gamma = \gamma$ , where  $\pi(g) = gqg^{-1}$ .*

We consider now the behaviour of the functor  $Q_n$  under epimorphisms.

Let  $f: A \rightarrow B$  be a continuous homomorphism of Banach algebras which preserves the identity

Clearly  $f$  induces maps  $G(f): G(A) \rightarrow G(B)$ , and  $f_n: Q_n(A) \rightarrow Q_n(B)$ . We shall prove that  $f_n$  is a Serre fibration when  $f$  is an epimorphism [Sp].

**2.4. THEOREM.** *Let  $f: A \rightarrow B$  be a (continuous) epimorphism of Banach algebras. Then  $f_n: Q_n(A) \rightarrow Q_n(B)$  is a Serre fibration. In particular,  $f_n$  is onto if and only if its image intersects every connected component of  $Q_n(B)$ .*

*Proof.* Replacing  $A$  and  $B$  by  $C(I^m, A)$  (= algebra of all maps  $I^m \rightarrow A$ ) and  $C(I^m, B)$  respectively (where  $I = [0, 1]$ ), it suffices to show that if  $\gamma: I \rightarrow Q_n(B)$  is such that  $\gamma(0) = q' = f_n(q)$  for some  $q \in Q_n(A)$  there exists a curve  $\tilde{\gamma}: I \rightarrow Q_n(A)$  such that  $f_n \circ \tilde{\gamma} = \gamma$ .

For this, we consider the commutative diagram

$$\begin{array}{ccc} G(A) & \xrightarrow{f} & G(B) \\ \pi_q \downarrow & & \downarrow \pi_{q'} \\ Q_n(A) & \xrightarrow{f_n} & Q_n(B) \end{array}$$

where  $\pi_q(g) = gqg^{-1}$ ,  $\pi_{q'}(h) = hq'h^{-1}$  ( $g \in G(A)$ ,  $h \in G(B)$ ). By the local triviality of  $\pi_{q'}$  proved in 2.1, there is a curve  $\delta: I \rightarrow G(B)$  with  $\delta(0) = 1$  and  $\pi_{q'}\delta = \gamma$ . Michael [Mi] proved that  $f: G(A) \rightarrow G(B)$  is a Serre fibration; therefore, there is a curve  $\varepsilon: I \rightarrow G(A)$  such that  $\varepsilon(0) = 1$  and  $f \circ \varepsilon = \delta$ . To finish the proof it suffices to define  $\tilde{\gamma} = \pi_q \circ \varepsilon$ , which satisfies  $f_n \circ \tilde{\gamma} = \gamma$ .

The next theorem extends results of Raeburn [Ra] concerning the set  $\pi_0(P(A \hat{\otimes} B))$  of all connected components of the idempotents of  $A \hat{\otimes} B$ , where  $A$  is supposed to be commutative.

We omit its proof and that of the proposition below because they are simple combination of Raeburn's techniques without previous results.

**2.5. PROPOSITION** (cf. [Ra, p. 383]). *Let  $A$  be a Banach algebra and  $B_1, \dots, B_n$  be open balls in  $\mathbf{C}$  with pairwise disjoint closures, centered at  $\alpha_1, \dots, \alpha_n$ , respectively. Let  $U = B_1 \cup \dots \cup B_n$  and  $A_U = \{a \in A : \text{the spectrum of } a \text{ is contained in } U\}$ . Then  $A_U$  is open in  $A$  and  $f = (f_1, \dots, f_n): A_U \rightarrow A^n$  is an analytic retraction onto  $Q_n$ , where  $f_i: U \rightarrow \mathbf{C}$  is defined by  $f_i(z) = \delta_{ik}$  for  $z \in B_k$  and  $f_n(a)$  is obtained by means of the holomorphic functional calculus.*

**2.6. THEOREM** (cf. [Ra, 4.5, 4.7]). *Let  $A$  and  $B$  be complex Banach algebras. Suppose that  $A$  is commutative with spectrum  $X$ . Then the Gelfand map  $A \rightarrow C(X)$  induces bijections*

$$\begin{aligned} \pi_0(Q_n(A \hat{\otimes} B)) &\rightarrow [X, Q_n(B)], \\ \{Q_n(A \hat{\otimes} B)\} &\rightarrow \{Q_n(C(X, B))\} \end{aligned}$$

where  $[ , ]$  denotes homotopy classes of maps and  $\{Q_n(C)\}$  is the set of orbits of the action of  $G(C)$  on  $Q_n(C)$ .

**2.7. REMARK.** If  $A$  is the algebra of complex continuous functions on the 3-sphere,  $B$  is the algebra of all  $2 \times 2$ -matrices over  $\mathbf{C}$  and  $n = 2$ , we reobtain the example of [PR1, 7.13].

**3. Lifting  $C^1$ -curves. The transport equation.** In this section we describe a method which leads to a lifting  $\Gamma$  of a curve  $\gamma: [a, b] \rightarrow Q_n$ , as in Corollary 2.3, valid when  $\gamma$  is rectifiable and continuous. For the sake of simplicity we only consider  $n = 2$ , the general case being similar and somewhat more involved. The reader can find the details (for  $n = 2$ ) in [PR1]. Our present interest in this construction lies in that it leads to the transport equation.

Consider a continuous rectifiable curve  $\gamma: [a, t] \rightarrow Q$  and a partition  $\Pi: t_0 = a < t_1 < \dots < t_n = t$  such that  $\|\gamma_k - \gamma_{k+1}\| < 1$  ( $k = 0, \dots, n - 1$ ), where  $\gamma_k = \gamma(t_k)$ ; then

$$\sigma_k = \gamma_k \gamma_{k-1}^{-1} + (1 - \gamma_k)(1 - \gamma_{k-1}) \in G \quad (k = 0, \dots, n - 1) \quad \text{and}$$

$$\begin{aligned} \sigma_k \gamma_0 \sigma_1^{-1} &= \gamma_1, \\ \sigma_2 \sigma_1 \gamma_0 \sigma_1^{-1} \sigma_2^{-1} &= \sigma_2 \gamma_1 \sigma_2^{-1} = \gamma_2, \dots, \sigma_n \dots \sigma_1 \gamma_0 \sigma_1^{-1} \dots \sigma_n^{-1} = \gamma_n. \end{aligned}$$

Thus,  $\sigma$  can be thought of as a “discrete” curve of units which conjugates  $\gamma_0$  with  $\gamma_n$ . Putting  $u(\Pi) = \sigma_n \dots \sigma_1$ , it can be shown [PR1, §5] that the limit  $\Gamma(t) = \lim u(\Pi)$ , when the length of the partition  $\Pi$  tends to zero, exists and defines a unit of the algebra. Moreover

$\Gamma: [a, b] \rightarrow G$  is continuous and rectifiable. If the original curve  $\gamma$  has a continuous derivative, then the value

$$\begin{aligned} (1/h)(\Gamma(t+h) - \Gamma(t)) & \text{ is, approximately,} \\ (1/h)(\sigma_{t+h}\Gamma(t) - \Gamma(t)), & \text{ where} \\ \sigma_{t+h} & = \gamma(t+h)\gamma(t) + (1 - \gamma(t+h))(1 - \gamma(t)). \end{aligned}$$

Then,

$$\begin{aligned} (1/h)(\Gamma(t+h) - \gamma(t)) & \cong (1/h)(\sigma_{t+h} - 1)\Gamma(t) \\ & = (1/h)(2\gamma(t+h)\gamma(t) - \gamma(t+h) - \gamma(t))\Gamma(t) \\ & = (1/h)\{\gamma(t+h)(\gamma(t) - \gamma(t+h)) + (\gamma(t+h) - \gamma(t))\gamma(t)\}\Gamma(t) \end{aligned}$$

and

$$\begin{aligned} \dot{\Gamma}(t) & = \lim_{h \rightarrow 0} (1/h)(\Gamma(t+h) - \Gamma(t)) \\ & = \{-\gamma(t)\dot{\gamma}(t) + \dot{\gamma}(t)\gamma(t)\}\Gamma(t). \end{aligned}$$

Thus, the lifting  $\Gamma$  of  $\gamma$  constructed by the limiting process described above satisfies the initial values problem

$$\begin{aligned} \dot{\Gamma} & = (\dot{\gamma}\gamma - \gamma\dot{\gamma}), \\ \Gamma(0) & = 1. \end{aligned}$$

In the general case  $n > 2$  the initial value problem is

$$\begin{aligned} \dot{\Gamma} & = \left( \sum_1^n \dot{\gamma}_k \gamma_k \right) \Gamma, \\ \Gamma(0) & = 1, \end{aligned}$$

where  $\gamma = (\gamma_1, \dots, \gamma_n): [a, b] \rightarrow Q_n$  is of class  $C^1$ . Observe that  $\sum_1^2 \dot{\gamma}_k \gamma_k = \dot{\gamma}_1 \gamma_1 - \dot{\gamma}_1 (1 - \gamma_1) = \dot{\gamma}_1 \gamma_1 - \gamma_1 \dot{\gamma}_1$  because  $\gamma_2 = 1 - \gamma_1$  and  $\dot{\gamma}_1 = \dot{\gamma}_1 \gamma_1 + \gamma_1 \dot{\gamma}_1$  (differentiate  $\gamma_1^2 = \gamma_1$ ).

As we said before, we shall not justify all the assertions about  $\Gamma$ . Instead we include the proof of the following result due to Daleckii, Krein and Kato, for the sake of completeness (see [DK2, IV, Theorem 1.1]).

**3.1. THEOREM.** *Let  $\gamma: [a, b] \rightarrow Q_n$  be a  $C^1$  curve. Then, the unique solution in  $A$  of the initial conditions problem*

$$\begin{aligned} \dot{\Gamma} & = \hat{\gamma}\Gamma, \\ \Gamma(a) & = 1, \end{aligned}$$

where  $\hat{\gamma} = \sum_{k=1}^n \dot{\gamma}_k \gamma_k$ , satisfies

- (i)  $\Gamma(t) \in G \quad (t \in [a, b])$ ,
- (ii)  $\Gamma(t)\gamma(a)\Gamma(t)^{-1} = \gamma(t) \quad (t \in [a, b])$ .

*Proof.* Existence and uniqueness of  $\Gamma$  follow from general facts [La, p. 71]. To prove (i) consider the companion problem

$$\begin{cases} \dot{\Delta} = -\Delta\hat{\gamma}, \\ \Delta(a) = 1, \end{cases}$$

and observe that  $(\Delta\Gamma)' = \dot{\Delta}\Gamma + \Delta\dot{\Gamma} = 0$ . Then  $\Delta\Gamma$  is constant on  $[a, b]$  and, since  $\Delta(a) = \Gamma(a) = 1$ , it is  $\Delta\Gamma \equiv 1$ . Thus  $\Gamma(t)$  is left invertible in  $A$ ; moreover,  $\Gamma(t)$  belongs to the connected component of the identity in the set of left invertible elements. It is easy to see that this component is completely contained in  $G$ . This proves (i).

To see (ii) we compute  $(\Gamma^{-1}\gamma_k\Gamma)'$  ( $k = 1, \dots, n$ ):

$$\begin{aligned} (\Gamma^{-1}\gamma_k\Gamma)' &= -\Gamma^{-1}\dot{\Gamma}\Gamma^{-1}\gamma_k\Gamma + \Gamma^{-1}\dot{\gamma}_k\Gamma + \Gamma^{-1}\gamma_k\dot{\Gamma} \\ &= -\Gamma^{-1}\{\hat{\gamma}\gamma_k - \dot{\gamma}_k - \gamma_k\hat{\gamma}\}\Gamma; \end{aligned}$$

observe that  $\hat{\gamma}\gamma_k = (\sum \dot{\gamma}_i \gamma_i)\gamma_k = \dot{\gamma}_k \gamma_k$ , because  $\gamma_i \gamma_k = 0$  for  $i \neq k$ , and that  $\gamma_k \hat{\gamma} = \gamma_k(\sum \dot{\gamma}_i \gamma_i) = -\gamma_k(\sum \gamma_i \dot{\gamma}_i) = -\gamma_k \dot{\gamma}_k$ , because  $\dot{\gamma}_k = \dot{\gamma}_k \gamma_k + \gamma_k \dot{\gamma}_k$  and  $\sum \dot{\gamma}_k = (\sum \dot{\gamma}_k)' = 1' = 0$ . Thus

$$(\Gamma^{-1}\gamma_k\Gamma)' = -\Gamma^{-1}\{\dot{\gamma}_k \gamma_k - \dot{\gamma}_k + \gamma_k \dot{\gamma}_k\}\Gamma = 0$$

and  $\Gamma^{-1}\gamma_k\Gamma$  is constantly  $\gamma_k(a)$ . This completes the proof of (ii).

**3.2. REMARK.** The proof of part (i) could have been omitted because it is a general fact that the solution of  $\dot{\Gamma} = \varphi\Gamma$ ,  $\Gamma(a) = 1$ , where  $\varphi: [a, b] \rightarrow A$  is a continuous curve, is a curve of invertible element of  $A$ .

If  $A$  is an involutive Banach algebra, i.e. there exists a continuous antilinear mapping  $x \rightarrow x^*$  such that  $(xy)^* = y^*x^*$ ,  $1^* = 1$  and  $x^{**} = x$  ( $x, y \in A$ ), we consider the unitary group of  $A$

$$U = \{u \in G: u^{-1} = u^*\}$$

and the selfadjoint part of  $Q_n$

$$P_n = \{p = (p_1, \dots, p_n) \in Q_n: p_k^* = p_k \quad (k = 1, \dots, n)\}.$$

For these algebras more specific results hold. We omit the details about the differential structure of  $P_n$ .

**3.3. COROLLARY.** *If  $\gamma: [a, b] \rightarrow P_n$  is a  $C^1$  curve then the solution of  $\dot{\Gamma} = \hat{\gamma}\Gamma$ ,  $\Gamma(a) = 1$ , defines a curve  $\Gamma: [a, b] \rightarrow U$  which conjugates the curve  $\gamma$ .*

*Proof.* It suffices to show that  $\Gamma(t) \in U$  for every  $t \in [a, b]$ . Observe first that

$$\begin{aligned} \dot{\Gamma}^* &= \left\{ \left( \sum \dot{\gamma}_k \gamma_k \right) \Gamma^* = \Gamma^* \left( \sum \dot{\gamma}_k \gamma_k \right)^* \right. \\ &= \Gamma^* \left( \sum \gamma_k \dot{\gamma}_k \right) = -\Gamma^* \left( \sum \dot{\gamma}_k \gamma_k \right), \end{aligned}$$

because

$$\sum \dot{\gamma}_k \gamma_k + \gamma_k \dot{\gamma}_k = \sum \dot{\gamma}_k = \left( \sum \gamma_k \right)^{\cdot} = 1^{\cdot} = 0.$$

Thus  $(\Gamma^*\Gamma)^{\cdot} = \dot{\Gamma}^*\Gamma + \Gamma^*\dot{\Gamma} = 0$  and  $\Gamma^*\Gamma$  is constant. But  $\Gamma(0) = \Gamma^*(0) = 1$ , so  $\Gamma^*\Gamma = 1$ . Now,  $\Gamma(t)$  is invertible for all  $t$ , by Theorem 3.1, so  $\Gamma(t)^* = \Gamma(t)^{-1}$ .

**3.4. REMARK.** Of course many liftings of  $\gamma$  may exist. But  $\Gamma$  is the unique horizontal lifting of  $\gamma$  with respect to the connection we shall define in the next section. This fact completes Kato's remark [Ka, II.4.2, Remark 4.4]. Moreover, if our  $\sigma$ 's, used to obtain the transport equation, are multiplied (at left or at right) by  $(1 - (\gamma_k - \gamma_{k-1})^2)^{-1/2}$ , where  $(1 - r)^{-1/2} = \sum_{m=0}^{\infty} \binom{-1/2}{m} (-r)^m$  for  $\|r\| < 1$ , we get a different "discrete" lifting of  $\gamma$  but in the limit it becomes the same continuous curve  $\Gamma$ . In this sense, the local solution [Ka, p. 102, (4.18)]

$$\Gamma_1(t) = (1 - (\gamma(t) - \gamma(0))^2)^{-1/2} (\gamma(t)\gamma(0) + (1 - \gamma(t))(1 - \gamma(0)))$$

is related to the global solution  $\Gamma$ .

**4. The connection.** Let  $q \in Q_n$  be fixed and  $\pi: G \rightarrow Q_n$  defined by  $\pi(g) = gqg^{-1} = (gq_1g^{-1}, \dots, gq_n g^{-1})$ . It is very easy to show that the derivative of  $\pi$  at  $g \in G$  is  $(T\pi)_g: (TG)_g \rightarrow (TQ_n)_{\pi(g)}$  is given by

$$(T\pi)_g(X) = g[g^{-1}X, q]g^{-1} \quad (X \in (TG)_g)$$

where  $[Z, q] = ([Z, q_1], \dots, [Z, q_n])$  for all  $Z \in A$ .

We say that  $X \in (TG)_g$  is *vertical* if  $(T\pi)_g(X) = 0$  or, what is the same, if  $[g^{-1}X, q] = 0$ . Then, if  $V_g = \{X \in (TG)_g: [g^{-1}X, q] = 0\}$ ,

it is clear that  $V_g = g \cdot V_1$  and that

$$\begin{aligned} V_1 &= \{X \in A = (TG)_g : [X, q] = 0\} \\ &= \{X \in A : q_k X q_i = 0 \text{ for all } i \neq k\} \\ &= \left\{ \sum_{i=1}^n q_i X q_i : X \in A \right\}. \end{aligned}$$

This shows that

$$\begin{aligned} H_1 &= \{X \in A : q_i X q_i = 0 \ (i = 1, \dots, n)\} \\ &= \left\{ \sum_{k \neq i} q_k X q_i : X \in A \right\} \end{aligned}$$

is a supplement of  $V_1$  in  $A (= (TG)_1)$  and, in general  $H_g = gH_1$  is a supplement of  $V_g$  in  $A (= (TG)_g)$ . Moreover,  $H_g \cdot h = H_{gh}$  ( $g \in G, h \in H$ ). Finally, the projections  $h_g : (TG)_g \rightarrow H_g, v_g : (TG)_g \rightarrow V_g$  given by

$$\begin{aligned} h_g(X) &= g \sum_{i \neq k} q_k g^{-1} X q_i, \\ v_g(X) &= g \sum_{i=1}^n q_i g^{-1} X q_i, \end{aligned}$$

verify

$$\begin{aligned} h_g(X) &= gh_1(g^{-1}X), \\ v_g(X) &= gv_1(g^{-1}X). \end{aligned}$$

Clearly the mappings  $g \rightarrow h_g$  and  $g \rightarrow v_g$  from  $G$  into the bounded linear operators on  $A$  are differentiable. All these facts show that  $g \rightarrow H_g$  defines a connection in the principal bundle  $\pi : G \rightarrow Q'_n$ .

For the theory of connections we refer the reader to [KN]. However, we are dealing with Banach manifolds and bundles, which requires a few notational changes.

From now on by "curve" we mean a  $C^\infty$  curve.

Given a curve  $\gamma : [\alpha, \beta] \rightarrow Q_n$ , a *horizontal lifting* of  $\gamma$  is a curve  $\Gamma : [\alpha, \beta] \rightarrow G$  such that  $\pi\Gamma = \gamma$  and  $\dot{\Gamma}(t) \in H_{\Gamma(t)}$  ( $t \in [\alpha, \beta]$ ).

It is a general fact that, for each  $g_0 \in G$  such that  $\gamma(\alpha) = g_0 p g_0^{-1}$ , there is a unique horizontal lifting  $\Gamma$  such that  $\Gamma(\alpha) = g_0$ . In particular, if  $\gamma(\alpha) = q$  there is a unique horizontal lifting  $\Gamma$  such that  $\Gamma(\alpha) = 1$ .

4.1. THEOREM. Given a curve  $\gamma: [\alpha, \beta] \rightarrow \mathcal{Q}_n$  the horizontal lifting  $\Gamma$  such that  $\Gamma(\alpha) = 1$  is the solution of the transport equation

$$(4.2) \quad \dot{\Gamma} = \hat{\gamma}\Gamma, \quad \text{where } \hat{\gamma} = \sum_{i=1}^n \dot{\gamma}_i \gamma_i,$$

with initial condition  $\Gamma(\alpha) = 1$ .

*Proof.* We have seen that the solution  $\Gamma$  of (4.2) is a lifting of  $\pi$ , i.e.  $\pi \circ \Gamma = \gamma$  (see 3.1). By the uniqueness of both objects it suffices to show that the horizontal lifting  $\Gamma$  with  $\Gamma(\alpha) = 1$  satisfies (4.2). We recall that  $\Gamma$  satisfies

$$(4.3) \quad \Gamma(t)q\Gamma(t)^{-1} = \gamma(t) \quad (t \in [\alpha, \beta]),$$

$$(4.4) \quad \dot{\Gamma} \in H_\Gamma = \Gamma H_1, \quad \text{i.e. } \dot{\Gamma}(t) \in \Gamma(t)H_1 \quad (t \in [\alpha, \beta])$$

or, what is the same

$$(4.5) \quad \Gamma^{-1}\gamma\Gamma = q$$

and

$$(4.6) \quad \Gamma^{-1}\dot{\Gamma} \in H_1.$$

Differentiating (4.5) we get  $0 = \Gamma^{-1}(-\dot{\Gamma}\Gamma^{-1}\gamma + \dot{\gamma} + \gamma\dot{\Gamma}\Gamma^{-1})\Gamma$  and cancelling  $\Gamma^{-1}$  and  $\Gamma$ , we get

$$(4.7) \quad \dot{\gamma} = [\dot{\Gamma}\Gamma^{-1}, \gamma].$$

Now, (4.6) means that  $q_i\Gamma^{-1}\dot{\Gamma}q_1 = 0$ , ( $i = 1, \dots, n$ ), which can also be written as

$$(4.8) \quad q\Gamma^{-1}\dot{\Gamma} = \Gamma^{-1}\dot{\Gamma}(1 - q).$$

Replacing (4.5) in (4.8) we get  $\Gamma^{-1}\gamma\dot{\Gamma} = \Gamma^{-1}\dot{\Gamma} - \Gamma^{-1}\dot{\Gamma}\Gamma^{-1}\gamma\Gamma$  which, after cancellation, gives

$$(4.9) \quad \gamma\dot{\Gamma}\Gamma^{-1} = \dot{\Gamma}\Gamma^{-1}(1 - \gamma)$$

and

$$(4.10) \quad \dot{\Gamma}\Gamma^{-1}\gamma = (1 - \gamma)\dot{\Gamma}\Gamma^{-1}.$$

Finally,

$$\begin{aligned} \hat{\gamma}\Gamma &= \left( \sum_i^n \dot{\gamma}_i \gamma_i \right) \Gamma \\ &= \sum_1^n [\dot{\Gamma}\Gamma^{-1}, \gamma_i] \gamma_i \Gamma \quad (\text{by 4.7}) \\ &= \sum_1^n \{ \dot{\Gamma}\Gamma^{-1} \gamma_i - \gamma_i \dot{\Gamma}\Gamma^{-1} \gamma_i \} \Gamma. \end{aligned}$$

This last expression coincides with  $\dot{\Gamma}$  because  $\gamma_i \dot{\Gamma} \Gamma^{-1} = \dot{\Gamma} \Gamma^{-1} (1 - \gamma_i)$  by (4.9) and therefore  $\gamma_i \dot{\Gamma} \Gamma^{-1} \gamma_i = \dot{\Gamma} \Gamma^{-1} (1 - \gamma_i) \gamma_i = 0$ . This proves the theorem.

4.11. REMARK. In general, if  $\gamma: [\alpha, \beta] \rightarrow Q_n$  is a curve with origin  $q' = g_0 q g_0^{-1}$  then  $\Gamma$  is the horizontal lifting with origin  $g_0$  if and only if it is the solution of the problem  $\dot{\Gamma} = \hat{\gamma} \Gamma$ ,  $\Gamma(\alpha) = g_0$ .

We compute next the 1-form, the 2-form and the curvature form of the connection.

We recall that the 1-form  $\theta$  assigns to each  $X \in (TG)_g$  the horizontal component of  $g^{-1} X \in (TG)_1 = \mathcal{L}$ , the Lie algebra of  $H$ . More explicitly,

$$\theta_g X = v_1(g^{-1} X) = g^{-1} v_g(X) = \sum_{i=1}^n q_i g^{-1} X q_i.$$

The 2-form  $d\theta$  of the connection is defined by

$$d\theta(X, Y) = \frac{1}{2} \{ X \cdot \theta Y - Y \cdot \theta X - \theta([X, Y]) \},$$

where  $X, Y \in (TG)_g$ ,  $[ , ]$  denotes the Lie bracket and  $Z \cdot W$  denotes the derivative of  $W$  in the direction of  $Z$ , i.e.  $W$  is extended to a vector field on a neighborhood of  $g$  and given a curve  $\delta: (-\varepsilon, \varepsilon) \rightarrow G$  such that  $\delta(0) = g$  and  $\dot{\delta}(0) = Z$ ,

$$Z \cdot W = \frac{d}{dt_{t=0}} W(\delta(t)).$$

Although the notation is the same, the Lie bracket should not be confused with the commutator bracket of the algebra.

From the computations

$$\begin{aligned} X \cdot \theta Y &= X \cdot \left( \sum_{i=1}^n q_i g^{-1} Y q_i \right) \\ &= - \sum_{i=1}^n q_i g^{-1} X g^{-1} Y q_i + \sum_{i=1}^n q_i g^{-1} X \cdot Y q_i, \\ Y \cdot \theta X &= - \sum_{i=1}^n q_i g^{-1} Y g^{-1} X q_i + \sum_{i=1}^n q_i g^{-1} Y \cdot X q_i, \end{aligned}$$

and

$$\theta([X, Y]) = \sum_{i=1}^n q_i g^{-1} [X, Y] q_i,$$

we get

$$\begin{aligned} d\theta(X, Y) &= \frac{1}{2} \sum_{i=1}^n q_i [g^{-1}Y, g^{-1}X]q_i \\ &= \left(-\frac{1}{2}\right) \sum_{i=1}^n q_i [g^{-1}X, g^{-1}Y]q_i. \end{aligned}$$

The *horizontal differential of  $\theta$* , also called the *curvature form of the connection* is  $\Omega(X, Y) = d\theta(h_g X, h_g Y)$  for  $[X, Y] \in (TG)_g$ . Explicitly

$$\begin{aligned} \Omega(X, Y) &= \left(-\frac{1}{2}\right) \sum_{i=1}^n q_i [g^{-1}h_g X, g^{-1}h_g Y]q_i \\ &= \left(-\frac{1}{2}\right) \sum_{i=1}^n q_i \left[ \sum_{k \neq l} q_k g^{-1}Xq_l, \sum_{r \neq s} q_r g^{-1}Yq_s \right] q_i \\ &= \left(-\frac{1}{2}\right) \sum_{i=1}^n q_i g^{-1} \{X(1 - q_i)g^{-1}Y - Y(1 - q_i)g^{-1}X\}q_i \\ &= \left(-\frac{1}{2}\right) \sum_{i=1}^n q_i g^{-1} \{X\bar{q}_i g^{-1}Y - Y\bar{q}_i g^{-1}X\}q_i, \\ &\quad \left( \text{where } \bar{q}_k = 1 - q_k = \sum_{i \neq k} q_i \right) \\ &= \left(-\frac{1}{2}\right) \sum_{i=1}^n q_i g^{-1} (Xg^{-1}Y - Yg^{-1}X - Xq_i g^{-1}Y + Yq_i g^{-1}X)q_i. \end{aligned}$$

The *structure equation*  $\Omega(X, Y) = d\theta(X, Y) + (\frac{1}{2})[\theta X, \theta Y]$  is thus trivially satisfied.

**5. Calculations on the tangent bundle, geodesics.** Consider  $q \in Q_n$  fixed and let  $A_1 = \{X \in A: q_i X q_i = 0, i = 1, \dots, n\}$  (in §4 we called it  $H_1$ ). It is clear that  $H = \{g \in G: gq_i = q_i g, i = 1, \dots, n\}$  operates at left on  $A_1$  by  $h \cdot X := hXh^{-1}$ .

Thus we define the associated bundle of  $\pi: G \rightarrow Q_n$  with standard fibre  $A_1$ , denoted by  $G \otimes A_1 \rightarrow Q_n$ , where  $G \otimes A_1 := G \times A_1 / \sim$ ,  $(g, X) \sim (gh, h^{-1}X)$  for  $h \in H$  and the map  $G \otimes A_1 \rightarrow Q_n$  is determined by  $(g, X) \rightarrow \pi(g)$ . It is a general fact that this vector bundle is isomorphic to the tangent bundle  $TQ_n$ , by means of  $(g, X) \rightarrow (\pi(g), gXg^{-1}) \in (TQ_n)_{\pi(g)}$ . Given a curve  $\gamma: [\alpha, \beta] \rightarrow$

$Q_n$  the parallel displacement of the fibre  $(TQ_n)_{\gamma(\alpha)}$  along  $\gamma$  from  $\alpha$  to  $t \in [\alpha, \beta]$  is defined by  $\tau_\alpha^t: (TQ_n)_{\gamma(\alpha)} \rightarrow (TQ_n)_{\gamma(t)}$ ,  $\tau_\alpha^t(Z) = \Gamma(t)Z\Gamma(t)^{-1}$ , where  $\Gamma$  is the horizontal lifting of  $\gamma$  with origin  $\Gamma(\alpha) = 1$ .

Given  $X \in (TQ_n)_q$  and a vector field  $Z$  defined near  $q$  the covariant derivative  $D_X Z$  is  $D_X Z := X \cdot Z + [Z, \tilde{X}]$ , where

$$\tilde{X} = \sum_{i=1}^n X_i q_i \quad \text{and} \quad X \cdot Z = \frac{d}{dt_{t=0}} Z(\delta(t))$$

for a curve  $\delta: (-\varepsilon, \varepsilon) \rightarrow Q_n$  such that  $\delta(0) = q$  and  $\delta'(0) = X$ .

**5.1. PROPOSITION.** *For every curve  $a: [\alpha, \beta] \rightarrow A^n$  the element  $Da/dt = \dot{a} + [a, \hat{\gamma}]$  is well defined and has the following properties:*

(a) *if  $\gamma_i a \gamma_i = 0$  for all  $i = 1, \dots, n$  then  $\gamma_i (Da/dt) \gamma_i = 0$  for all  $i = 1, \dots, n$  (in other words,  $Da/dt$  is tangent if  $a$  is tangent).*

(b) *if  $\gamma_i a \gamma_k = 0$  for all  $i \neq k$  then  $\gamma_i (Da/dt) \gamma_k = 0$  for all  $i \neq k$  (i.e.  $Da/dt$  is normal if  $a$  is normal).*

*Proof.* (a) Differentiating  $\gamma_i a \gamma_i = 0$  we get

$$0 = \dot{\gamma}_i a \gamma_i + \gamma_i \dot{a} \gamma_i + \gamma_i a \dot{\gamma}_i.$$

Multiplying by  $\gamma_i$  at right and left we have

$$(5.2) \quad \gamma_i \dot{\gamma}_i a \gamma_i + \gamma_i \dot{a} \gamma_i + \gamma_i a \dot{\gamma}_i \gamma_i = 0.$$

On the other hand

$$\begin{aligned} \gamma_i \frac{Da}{dt} \gamma_i &= \gamma_i \dot{a} \gamma_i + \gamma_i [a, \hat{\gamma}] \gamma_i \\ &= \gamma_i \dot{a} \gamma_i + \gamma_i \left( a \sum \dot{\gamma}_k \gamma_k - \sum \dot{\gamma}_k \gamma_k a \right) \gamma_i \\ &= \gamma_i \dot{a} \gamma_i + \gamma_i a \dot{\gamma}_i \gamma_i - \gamma_i \sum \dot{\gamma}_k \gamma_k a \gamma_i \end{aligned}$$

and  $\gamma_i \sum_k \dot{\gamma}_k \gamma_k = \gamma_i \sum_k (1 - \gamma_k) \dot{\gamma}_k$  because  $\dot{\gamma}_k = \dot{\gamma}_k \gamma_k + \gamma_k \dot{\gamma}_k$  (differentiate  $\gamma_k^2 = \gamma_k$ ); thus

$$\gamma_i \sum_k \dot{\gamma}_k \gamma_k = \gamma_i \sum_k \dot{\gamma}_k - \gamma_i \sum_k \gamma_k \dot{\gamma}_k = -\gamma_i \dot{\gamma}_i,$$

because  $\sum_k \dot{\gamma}_k = 0$  and  $\gamma_i \gamma_k = 0$  if  $i \neq k$ .

This shows that

$$\gamma_i \frac{Da}{dt} \gamma_i = \gamma_i \dot{a} \gamma_i + \gamma_i a \dot{\gamma}_i \gamma_i + \gamma_i \dot{\gamma}_i a \gamma_i = 0, \quad \text{by (4.2).}$$

The proof of (b) is similar.

This shows that for every vector field  $Y$  of  $Q_n$  along  $\gamma$ , the formula  $Da/dt = \dot{Y} + [Y, \hat{\gamma}]$  defines another vector field of  $Q_n$ , the *covariant derivative* of  $Y$ .

The *torsion* of the connection, defined by  $T(X, Y) = D_X Y - D_Y X - [X, Y]$  in general, turns out to be in our case

$$(5.3) \quad T(X, Y) = [Y, \tilde{X}] - [X, \tilde{Y}],$$

where  $X, Y \in (TQ_n)_g$  and  $\tilde{X} = \sum_{i=1}^n X_i q_i, \tilde{Y} = \sum_{i=1}^n Y_i q_i$ .

**5.4. REMARK.** For  $n = 2$  the connection is symmetric, in the sense that its torsion is zero everywhere: in fact, for  $n = 2$  we have  $X_1 + X_2 = 0, Y_1 + Y_2 = 0, q_1 + q_2 = 1, q_i X_i = X_i(1 - q_i), q_i X_j = -X_i q_j$ .

These equalities, when replaced in (4.3), prove the assertion. However, for  $n > 3$  this is no longer true.

The *curvature* of the connection, expressed by  $R(X, Y)Z = D_X(D_Y Z) - D_Y(D_X Z) - D_{[X, Y]}Z$  for  $X, Y, Z \in (TQ_n)_q$ , is given, in our case, by

$$(5.5) \quad R(X, Y)Z = \left[ \sum_{i=1}^n [X_i, Y_i] q_i, Z \right]$$

or, abbreviating

$$(5.6) \quad R(X, Y)Z = [[X, Y]^\sim, Z].$$

We study now the geodesic curves of the connection, that is, the curves  $\gamma: [\alpha, \beta] \rightarrow Q_n$  such that  $D\dot{\gamma}/dt = 0$ . It is a well-known fact that this condition is equivalent to  $\tau_\alpha^t(\dot{\gamma}(\alpha)) = \dot{\gamma}(t), (t \in [\alpha, \beta])$ . The equation defining the geodesic curves can be written as

$$(5.7) \quad \ddot{\gamma}_k + [\dot{\gamma}_k, \hat{\gamma}] = 0, \quad k = 1, \dots, n.$$

Using the commutation rules obtained from  $\sum \gamma_i = 1, \gamma_i^2 = \gamma_i$  and  $\gamma_i \gamma_k = 0$  for  $i \neq k$ , we get

- (i)  $\dot{\gamma}_i \gamma_i = (1 - \gamma_i) \dot{\gamma}_i \quad (i = 1, \dots, n);$
- (ii)  $\dot{\gamma}_i \gamma_k + \gamma_i \dot{\gamma}_k = 0 \quad (i \neq k);$
- (iii)  $\sum_i^n \dot{\gamma}_k = 0;$
- (iv)  $\gamma_i \dot{\gamma}_i^2 = \dot{\gamma}_i^2 \gamma_i \quad (i = 1, \dots, n);$
- (v)  $\gamma_i \dot{\gamma}_i \gamma_i = 0 \quad (i = 1, \dots, n).$

These equalities imply that (5.7) is equivalent to

$$(5.8) \quad \ddot{\gamma}_k + \gamma_k \left( \sum_1^n \dot{\gamma}_i^2 \right) + \left( \sum_1^n \dot{\gamma}_i^2 \right) \gamma_k - 2\dot{\gamma}_k^2 = 0, \quad (k = 1, \dots, n).$$

It is easy to exhibit all the solutions of (5.8) which satisfy  $\gamma(t) \in Q_n$  for all  $t$ . In fact, for  $q \in Q_n$ ,  $X \in (TQ_n)_q$ ,  $\gamma(t) = e^{t\tilde{X}} q e^{-t\tilde{X}}$  ( $t \in R$ ), satisfies (5.8) and all the solutions of (5.8) with the additional condition  $\gamma(t) \in Q_n$ , have this form. The connection is also complete, in the sense that its geodesics are defined for all  $t \in R$ , and the exponential map of the connection is given by

$$\text{Exp}_q: (TQ_n)_q \rightarrow Q_n, \quad \text{Exp}_q(X) = e^{\tilde{X}} q e^{-\tilde{X}}.$$

Properties of minimality of length of geodesics are studied in a forthcoming paper ([CPR2]).

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