AMENABILITY OF DISCRETE CONVOLUTION ALGEBRAS, THE COMMUTATIVE CASE

NIELS GRONBAEK
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Niels Grønbæk

A Banach algebra $A$ is called amenable if all bounded derivations into dual Banach $A$-modules are inner. Let $S$ be a semigroup and let $l^1(S)$ be the corresponding discrete convolution algebra. This paper is on the theme: “On the hypothesis that $l^1(S)$ is amenable, what conclusions can be drawn about the (algebraic) structure of $S$?” We give a complete characterization of commutative semigroups carrying amenable semigroup algebras. If $S$ is commutative, then $l^1(S)$ is amenable if and only if $S$ is a finite semilattice of groups, that is, there is a finite semilattice $Y$ and disjoint commutative groups $G_\alpha$ ($\alpha \in Y$) such that $S = \bigcup_{\alpha \in Y} G_\alpha$ and $G_\alpha G_\beta \subseteq G_{\alpha \beta}$ ($\alpha, \beta \in Y$).

The theme above has previously been studied in [3] and [4]. In both papers it is apparent that the condition of amenability imposes strong algebraic constraints on the semigroup. In [3] a rather complete description of inverse semigroups carrying amenable semigroup algebras is given. Of particular interest for this paper is that a semilattice carries an amenable semigroup algebra if and only if it is finite [3, Theorem 10]. In [4] it is proved that, if a one-sided cancellative semigroup carries an amenable semigroup algebra, then it is a group. The result of this paper, that for a commutative semigroup $S$, the semigroup algebra $l^1(S)$ is amenable if and only if $S$ is a finite lattice of groups, is proved by looking at the gross structure of $S$ by means of the “principle of maximal homomorphic image of a given type”. Using the fact that homomorphic images of $S$ carry amenable semigroup algebras when $S$ does, we establish the necessity of the characterization by showing that each archimedean component of $S$ is a group. This is obtained by applying the results from [3] and [4], mentioned above, to the maximal semilattice, the maximal cancellative, and the maximal separative homomorphic images of $S$. The sufficiency of the characterization is easily verified. Alternatively, it follows from [3, Theorem 8].

1. Preliminaries. We shall need some elementary semigroup theory. We prefer to keep our exposition self-contained, so although most of what follows can be found in standard texts on the subject,
we shall, with a few exceptions, give proofs in some detail. For a further discussion the reader is referred to [1]. Throughout $S$ will denote a commutative semigroup, with the binary operation written multiplicatively.

1.1. **Definitions.** Consider the following conditions on $S$:

(A) Each element of $S$ is an idempotent.

(B) For all $s, t \in S$ there is $n \in \mathbb{N}$ such that

$$s^n \in tS \quad \text{and} \quad t^n \in sS.$$  

(C) $s^2 = t^2 = st \Rightarrow s = t \ (s, t \in S)$. If $S$ satisfies (A) we call $S$ a semilattice.

If $S$ satisfies (B) we call $S$ archimedean.

If $S$ satisfies (C) we call $S$ separative.

An ideal in $S$ is a subset $I$ such that $SI \subseteq I$. A prime ideal in $S$ is an ideal, whose complement is a subsemigroup of $S$.

A congruence on $S$ is an equivalence relation which is compatible with the semigroup operation.

A congruence $\sim$ on $S$ will be called separative (cancellative, archimedean, etc.) if the semigroup $S/\sim$ is separative (cancellative, archimedean, etc.).

1.2. **Definition.** (Principle of maximal homomorphic image of a given type). Let $\mathcal{C}$ be a class of congruences on $S$, closed under intersections. Put $\rho_0 = \bigcap \{\rho | \rho \in \mathcal{C}\}$. Then $S/\rho_0$ is the maximal “type class $\mathcal{C}$” homomorphic image of $S$.

See also [1, p. 18] and [7, §1].

**Example.** Let $\rho_0 = \bigcap \{\rho | s^2 \rho s \ (s \in S)\}$. Then $S/\rho_0$ is the maximal semilattice homomorphic image of $S$.

1.3. **Definition.** Let $s \in S$ and choose $m \in \mathbb{N}$ smallest possible so that $s^m = s^{m+r}$ for some $r \in \mathbb{N}$. Then $\text{order}(s) = m$ and the smallest possible $r$ is called period $(s)$. If no such $m \in \mathbb{N}$ can be found we put $\text{order}(s) = \infty$.

1.4. **Definition.** Let $S$ be a semigroup and suppose that there is a semilattice $Y$ and disjoint subsemigroups $S_\alpha \ (\alpha \in Y)$ of $S$ such that $S = \bigcup_{\alpha \in Y} S_\alpha$ and $S_\alpha S_\beta \subseteq S_{\alpha \beta} \ (\alpha, \beta \in Y)$. Then $S$ is called a semilattice of the subsemigroups $S_\alpha \ (\alpha \in Y)$.

The following lemma is the main structure theorem for commutative semigroups.
1.5. **Lemma.** Let \( S \) be a commutative semigroup and let \( Y \) be the maximal semilattice homomorphic image of \( S \). Then there are disjoint archimedean subsemigroups \( S_\alpha (\alpha \in Y) \) of \( S \) such that \( S \) is a semilattice of the semigroups \( S_\alpha (\alpha \in Y) \). This decomposition of \( S \) into archimedean subsemigroups is unique up to isomorphism of \( Y \), and \( S \) is separative if and only if each archimedean component \( S_\alpha \) is cancellative.

**Proof.** See [1, §4.3].

1.6. **Lemma.** On \( S \) define the relations:

\[
sc \equiv \exists u \in S \ su = tu
\]

and

\[
s\sigma t \equiv \exists n_0 \in \mathbb{N} \forall n \geq n_0 \ s^n = t^n.
\]

Then \( c \) and \( \sigma \) are congruences and \( S/c \) is the maximal cancellative homomorphic image of \( S \) and \( S/\sigma \) is the maximal separative homomorphic image of \( S \).

**Proof.** It is clear that both relations are congruences. Now suppose \( \rho \) is a cancellative congruence; that is, \( su \rho tu \Rightarrow s \rho t \ (s, t, u \in S) \). Then clearly \( sc \Rightarrow s \rho t \ (s, t \in S) \) so that \( c \subseteq \rho \). Since \( c \) is cancellative we are done with the statements about \( c \).

Now suppose that \( s^2 \sigma t^2 \sigma st \); that is, there is \( n_0 \in \mathbb{N} \) so that \( s^{2n} = t^{2n} = s^nt^n \) for \( n \geq n_0 \). Then \( s^{4n_0+1}t = ss^{2n_0} . t^{2n_0} . t = s^{2n_0+1}t^{2n_0+1} = s^{4n_0+2} \) so that for \( n \geq 8n_0 + 2 \) we have \( s^n = t^n \). Hence \( s\sigma t \), proving that \( \sigma \) is separative. Let \( \rho \) be a separative congruence. If \( s\sigma t \), then there is \( k \in \mathbb{N} \) so that \( st^k = t^{k+1} \). In particular \( st^k \rho t^{k+1} \). This gives

\[
(st^{k-1})^2 = st^{k-2}st^k \rho st^{k-2}t^{k+1} = st^{k-1}t^k \rho t^{k+1}t^{k-1} = (t^k)^2.
\]

With \( x = st^{k-1} \) and \( y = t^k \) we have \( x^2 \rho y^2 \rho xy \) so that \( x \rho y \), that is, \( st^{k-1} \rho t^k \). Repeating as necessary, we get \( st \rho t^2 \rho s^2 \), where the second relation follows from symmetry. Thus \( s \rho t \), proving that \( \sigma \subseteq \rho \). \( \Box \)

1.7. **Lemma.** \( s^2 \sigma s \Leftrightarrow \text{order}(s) < \infty \) and period\( (s) = 1 \). If \( e, f \) are idempotents in \( S \), then \( e\sigma f \Leftrightarrow e = f \).

**Proof.** Suppose \( s^2 \sigma s \). Then there is \( n_0 \in \mathbb{N} \) so that \( s^{2n} = s^n \) for \( n \geq n_0 \). If \( r \) is the period of \( s \) we have \( 2n \equiv n \) \( \pmod{r} \) for \( n \geq n_0 \) so that \( r = 1 \). The rest is obvious. \( \Box \)
1.8. Lemma. $S/\sigma$ is a group if and only if $S$ is archimedean with unique idempotent.

Proof. First suppose that $S/\sigma$ is a group. From Lemma 1.7 it follows that $S$ has a unique idempotent. Let $s, t \in S$. Since $S/\sigma$ is a group there are $u, v \in S$ so that $su \sigma t$ and $tv \sigma s$. By definition of $\sigma$, $s$ divides a power of $t$ and $t$ divides a power of $s$, that is, $S$ is archimedean. Conversely, let $s \in S$ and let $e$ denote the unique idempotent in $S$. Since $S$ is archimedean there are $t, u \in S$ so that $st = e$ and $ue = s^{n_0}$ for some $n_0$. We have $(es)^{n_0+p} = e^{n_0+p}s^{n_0}s^p = e^{n_0+p}ues^p = ues^p = s^{n_0+p}$ ($p \in \mathbb{N}$) so that $es \sigma s$. Clearly $st \sigma e$, so $S/\sigma$ is a group. \(\square\)

2. The main theorem. For the remainder of this paper we shall assume that $S$ is a commutative semigroup such that $l^1(S)$ is amenable. We shall make frequent use of the fact that, if $T$ is a homomorphic image of $S$, then $l^1(T)$ is amenable, and if $I$ is an ideal in $S$ which is generated by an idempotent, then $l^1(I)$, being a closed $l^1(S)$-ideal which is unital as a Banach algebra, is amenable [6, Proposition 5.1]. Thus, if $S = \bigcup_{\alpha \in \gamma} S_\alpha$ is the decomposition of $S$ into its archimedean components, then the semilattice $Y$ is finite, since $l^1(Y)$ is amenable ([3, Theorem 10]). We give $Y$ the usual semilattice ordering $\alpha \leq \beta \iff \alpha \beta = \alpha \ (\alpha, \beta \in Y)$. Since $Y$ is finite, $Y$ has a minimal element, namely the product of all elements in $Y$.

It is convenient to start with the case where $S$ is separative; that is, we are assuming that each archimedean component is cancellative.

2.1. Lemma. Let $S$ and $Y$ be as above and let $\alpha_0$ be the minimal element of $Y$. Then $S_{\alpha_0}$ is a group.

Proof. By [4, Theorem 2.3] $S/c$ is a group. Let $s \in S_{\alpha_0}$ Then there is $t \in S$ so that for all $u \in S$ stucu, that is, for all $u \in S$ there is $v \in S$ so that stuv = uv. Since $\alpha_0$ is minimal, $st \in S_{\alpha_0}$ and $uv \in S_{\alpha_0}$, so, using the cancellation law in $S_{\alpha_0}$, we see that $st$ is a neutral element in $S_{\alpha_0}$. Consequently $l^1(S_{\alpha_0})$ can be identified canonically with an ideal generated by an idempotent in $l^1(S)$. It follows that $l^1(S_{\alpha_0})$ is amenable and therefore $S_{\alpha_0}$ is a group, again by [4, Theorem 2.3]. \(\square\)

2.2. Lemma. Let $l^1(S)$ be amenable and suppose that $S$ is separative. Then $S$ is a finite semilattice of groups.
Proof. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be the decomposition of $S$ into its archimedean components. Let $\beta \in Y$, and define $T = \bigcup_{\alpha \geq \beta} S_\alpha$. Then $T$ is a subsemigroup of $S$ and $S \setminus T$ is a (prime) ideal in $S$. Hence the canonical Banach space direct sum $l^1(S) = l^1(T) \oplus l^1(S \setminus T)$ is a semidirect product, so that $l^1(T)$ is amenable. Since $\beta$ is minimal in $\{\alpha \in Y | \alpha \geq \beta\}$, Lemma 2.1 implies that $S_\beta$ is a group. But $\beta$ was arbitrary in $Y$. □

We now turn to the general case.

2.3. Lemma. Suppose $l^1(S)$ is amenable. Then $S$ is a finite semilattice of its archimedean components, $S = \bigcup_{\alpha \in Y} S_\alpha$. Each $S_\alpha$ has a unique idempotent $e_\alpha$, and $e_\alpha S_\alpha$ is a group, isomorphic to the maximal separative homomorphic image of $S_\alpha$.

Proof. By Lemma 2.2 $S/\sigma$ is a finite semilattice of groups, $S/\sigma = \bigcup_{\alpha \in Y} G_\alpha$. Let $S_\alpha$ be the preimage of $G_\alpha$ by the canonical map $S \rightarrow S/\sigma$. With slight abuse of notation we have $S_\alpha/\sigma = G_\alpha$, so that $S_\alpha$ is archimedean with unique idempotent, $e_\alpha$ say, by Lemma 1.8. It follows that $S = \bigcup_{\alpha \in Y} S_\alpha$ is the decomposition of $S$ into its archimedean components. Now let $s \in S_\alpha$. Since $G_\alpha$ is a group, there is $t \in S_\alpha$ so that $st \sigma e_\alpha$, i.e. $(st)^n = e_\alpha$ for some $n \in \mathbb{N}$. Hence $e_\alpha s^{n-1} t^n$ is an inverse to $e_\alpha s$. Clearly the canonical map from $e_\alpha S_\alpha$ to $G_\alpha$ is surjective. Assume that $e_\alpha s \sigma e_\alpha$ for some $s \in S_\alpha$. Since $e_\alpha S_\alpha$ is a group it follows from Lemma 1.7 that $e_\alpha s = e_\alpha$, proving injectivity of the canonical map. □

We shall finish the proof of the main theorem by proving that $e_\alpha S_\alpha = S_\alpha$ for each $\alpha \in Y$. This is done by exploiting that $l^1(S)$, being amenable, has a bounded approximate identity. First we need a definition.

2.4. Definition. Let $s \in S$. Then we define

$$[ss^{-1}] = \{u \in S | us = s\}.$$  

Since $l^1(S)$ has a bounded approximate identity $[ss^{-1}] \neq \emptyset$ for all $s \in S$ [4, Theorem 1.1].

2.5. Lemma. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be the decomposition of $S$ into its archimedean components, as in Lemma 2.3, and let $s \in S_\alpha$. If $[ss^{-1}] \cap S_\alpha \neq \emptyset$, then $s \in e_\alpha S_\alpha$. If $\alpha$ is maximal in $Y$, then $S_\alpha$ is a group.
Proof. Let \( u \in [ss^{-1}] \cap S_\alpha \). Then \( us \sigma e_\alpha s \). Since \( S_\alpha / \sigma \) is a group we have \( u \sigma e_\alpha \), i.e. \( u^n = e_\alpha \) for some \( n \in \mathbb{N} \). Hence \( s = u^n s = e_\alpha s \).

In general, if \( s \in S_\alpha \) and \( u \in [ss^{-1}] \cap S_\beta \), then \( s = us \in S_\alpha \cap S_{\beta \alpha} \), so \( \beta \geq \alpha \). Thus, when \( \alpha \) is maximal in \( Y \) we have that \([ss^{-1}] \subseteq S_\alpha \) for all \( s \in S_\alpha \). It follows that \( e_\alpha S_\alpha = S_\alpha \), so that \( S_\alpha \) is a group by Lemma 2.3.

2.6. Lemma. Let \( s = \bigcup_{\alpha \in Y} S_\alpha \) be as in Lemma 2.3. Then \([ss^{-1}] \cap \{e_\alpha | \alpha \in Y\} \neq \emptyset \) for all \( s \in S \). In particular \( l^1(S) \) is unital.

Proof. First note that, if \( u \in [ss^{-1}] \), then \([uu^{-1}] \subseteq [ss^{-1}] \). Let \( s \in S \) and let \( S_{\alpha_0} \) be the archimedean component of \( s \). Put \( u_0 = s \) and choose successively \( u_k \in [u_{k-1} u_{k-1}^{-1}] \). Let \( S_{\alpha_k} \) be the archimedean component of \( u_k \). As noted in the proof of Lemma 2.5 we have \( \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq \cdots \). Since \( \text{card} Y < \infty \), we eventually have \( S_{\alpha_k} = S_{\alpha_{k+1}} \), whence \([u_k u_k^{-1}] \cap S_{\alpha_k} \neq \emptyset \), so that \( e_{\alpha_k} \in [u_k u_k^{-1}] \) by Lemma 2.5. As observed in the beginning of the proof \( e_{\alpha_k} \in [ss^{-1}] \).

From [5, Theorem 7.5] it follows that \( l^1(S) \) has a unit. \( \square \)

We are now able to prove:

2.7. Theorem. Let \( S \) be a commutative semigroup. Then \( l^1(S) \) is amenable if and only if \( S \) is a finite semilattice of commutative groups.

Proof. The sufficiency has been noted in the introduction. Hence we assume that \( l^1(S) \) is amenable. Let \( s = \bigcup_{\alpha \in Y} S_\alpha \) be the decomposition as in Lemma 2.3. By Lemma 2.5 the theorem is true if \( \text{card} Y = 1 \). We proceed by induction on \( n = \text{card} Y \). Assume that \( n \geq 2 \) and that the theorem is true for semigroups which are semilattices of archimedean semigroups with cardinality of the semilattice strictly less than \( n \). Let \( \alpha_0 \) be the minimal element in \( Y \).

Let \( \beta \in Y \setminus \{\alpha_0\} \), and define \( T_\beta = \bigcup_{\alpha \geq \beta} S_\alpha \). As in the proof of Lemma 2.2, we see that \( l^1(T_\beta) \) is amenable. Thus, by the induction hypothesis, we have that \( S_{\alpha} \) is a group for \( \alpha \in Y \setminus \{\alpha_0\} \). We finish the induction step by proving that \( S_{\alpha_0} = e_{\alpha_0} S_{\alpha_0} \). To this end, define a congruence \( \sim \) on \( S \) by

\[
 s \sim t \iff S s = S t \quad (s, t \in S).
\]

Note that, if \( s \sim t \), then \( s \in St \), since \([ss^{-1}] \neq \emptyset \). Using that \( S_\alpha \) is a group for \( \alpha \neq \alpha_0 \), we see that \( S/\sim \cong \bigcup_{\alpha \neq \alpha_0} \{e_\alpha\} \cup S_{\alpha_0}/\sim \). Hence \( l^1(S_{\alpha_0}/\sim) \) is (isomorphic to) a closed ideal of finite codimension in the
amenable Banach algebra $l^1(S/\sim)$, and therefore $l^1(S_{\alpha_0}/\sim)$ is itself amenable [2, Theorem 4.1]. From Lemma 2.5 we get that $S_{\alpha_0}/\sim$ is a group. In particular we have for all $s \in S_{g\alpha_0}$ that $s \sim e_{\alpha_0}s$, so, by the note above, $S_{\alpha_0} \subseteq e_{\alpha_0}S_{\alpha_0}$. The induction step is hereby completed. □

Acknowledgment. I wish to acknowledge a stimulating correspondence with Professor J. Duncan on the subject. I wish to thank Dr. K. B. Laursen for a careful reading of the manuscript.

References


Received August 29, 1988.

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The Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924 (ISSN 0030-8730) is published monthly. Second-class postage paid at Carmel Valley, California 93924, and additional mailing offices. Postmaster: send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

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