MASS OF RAYS ON COMPLETE OPEN SURFACES

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The total curvature of a complete open surface describes certain properties of the Riemannian structure which defines it. We study relationships between the total curvature and the mass of rays on a finitely connected complete open surface and obtain some integral formulas.

0. Introduction. Throughout this paper let $M$ be a connected, finitely connected, oriented, complete and noncompact Riemannian 2-manifold without boundary. The total curvature $c(M)$ of $M$ is defined to be an improper integral over $M$ of Gaussian curvature $G$ with respect to the area element $dM$ of $M$. A well-known theorem due to Cohn-Vossen [1] states that if $M$ admits total curvature, then $2\pi \chi(M) - c(M) \geq 0$, where $\chi(M)$ is the Euler characteristic of $M$. Clearly $c(M)$ depends on the choice of Riemannian metric. This phenomenon gives rise to the idea that the value $2\pi \chi(M) - c(M)$ should describe certain properties of Riemannian metric which defines it.

A ray (respectively, a straight line) on $M$ is by definition a unit speed geodesic parametrized on $[0, \infty)$ (respectively, on $\mathbb{R}$) every subarc of which realizes distance between its terminal points. For a point $p \in M$ let $S_p(1)$ be the unit circle centered at the origin of the tangent space $M_p$ to $M$ at $p$. Let $A(p)$ be the set of all unit vectors tangent to rays emanating from $p$. $A(p)$ is closed in $S_p(1)$. Let $\mathfrak{m}$ be the natural measure on $S_p(1)$ induced from the Riemannian metric. A relation between the mass of rays and the total curvature was first investigated by Maeda in [6], [7]. He proved that if $M$ is homeomorphic to $\mathbb{R}^2$ and if $G \geq 0$, then $\mathfrak{m} \circ A \geq 2\pi - c(M)$, and in particular $\inf_M \mathfrak{m} \circ A = 2\pi - c(M)$. These results were extended by Shiga in [10], [11] to Riemannian planes whose Gaussian curvatures change sign, and later by Oguchi [9] to finitely connected $M$ with one endpoint. In connection with an isoperimetric problem discussed by Fiala [3] and Hartman [4], the first-named author proved in [14] that if $M$ has one end and if $2\pi \chi(M) - c(M) < 2\pi$, then for every monotone increasing sequence $\{K_j\}$ of compact sets with $\bigcup K_j = M$,
The proof of this equation essentially depends on the fact that $M$ admits no straight lines. This property is guaranteed by the assumptions on the total curvature and the uniqueness of endpoint of $M$.

It should also be noted that all results mentioned above are obtained under the assumption that $M$ has one endpoint. In the case where $M$ has more than one endpoint (and this is the case where we are interested in this paper), it will be natural to consider that each endpoint shares the value $2\pi \chi(M) - c(M)$ in the following sense. Let $M$ have $k$ endpoints and let $K \subset M$ be a compact set with the property that $M \setminus \text{Int}(K)$ consists of $k$ tubes $U_1, \ldots, U_k$ such that each $U_i$ is homeomorphic to $S^1 \times [0, \infty)$ and that each $\partial U_i$ is a piecewise smooth simply closed curve. Then the Gauss-Bonnet theorem states that $c(K) + \sum_{i=1}^{k} \kappa(\partial U_i) = 2\pi \chi(M)$, where $c(K) = \int_K G \, dM$ and $\kappa(\partial U_i)$ denotes the curvature integral over the boundary curve $\partial U_i$. For each $i = 1, \ldots, k$ the value

$$s_i(M) := \kappa(\partial U_i) - c(U_i)$$

is nonnegative and independent of the choice of tube. Moreover

$$\sum_{i=1}^{k} s_i(M) = 2\pi \chi(M) - c(M).$$

For details see [15]. Thus one observes that each endpoint corresponding to $U_i$ shares the value $2\pi \chi(M) - c(M)$.

With these notations our main results will be stated as follows.

**Theorem A.** Assume that $M$ admits total curvature and has $k$ endpoints. If $s_i(M) \leq 2\pi$ holds for each $i = 1, \ldots, k$, then for every monotone increasing sequence $\{K_j\}$ of compact sets with $\bigcup K_j = M$,

$$\min_{1 \leq i \leq k} s_i(M) \leq \liminf_{j \to \infty} \frac{\int_{K_j} \mathcal{M} \circ A \, dM}{\int_{K_j} dM} \leq \limsup_{j \to \infty} \frac{\int_{K_j} \mathcal{M} \circ A \, dM}{\int_{K_j} dM} \leq \max_{1 \leq i \leq k} s_i(M).$$

**Theorem B.** Assume that $M$ admits total curvature and has $k$ endpoints. Let $\mathcal{C}$ be a simply closed smooth curve in $M$ and let $B(t) := \{x \in M; d(x, \mathcal{C}) \leq t\}$ and $S(t) := \{x \in M; d(x, \mathcal{C}) = t\}$,
where \( d \) is the distance function induced from Riemannian metric. If \( s_i(M) \leq 2\pi \) holds for each \( i = 1, \ldots, k \), then

\[
\lim_{t \to \infty} \frac{\int_{B(t)} M \circ A \, dM}{\int_{B(t)} \alpha \, dM} = \begin{cases} 
\frac{\sum_{i=1}^{k} s_i^2(M)}{2\pi \chi(M) - c(M)} & \text{if } 2\pi \chi(M) - c(M) > 0, \\
0 & \text{if } 2\pi \chi(M) - c(M) = 0.
\end{cases}
\]

**Remark 1.** Shiohama first proved an inequality in Theorem B under the stronger assumption that \( s_i(M) < 2\pi \). But subsequent improvement on the asymptotic behavior of \( M \circ A \) was obtained by Shioya and Tanaka. It turns out that the existence of straight lines on \( M \) is no objection at all. Tanaka's proof for the asymptotic behavior of \( M \circ A \) by assuming \( s_i(M) = 2\pi \) will be provided in Lemma 1.1. Shioya has extended this result to the case where \( +\infty \geq s_i(M) \geq 2\pi \). This result will be published independently because the proof is fascinating and of independent interest in itself.

**Remark 2.** Theorem B does not hold for any monotone increasing sequence \( \{K_j\} \) of compact sets with \( \bigcup K_j = M \). For example, consider a surface \( M \) of revolution in \( \mathbb{R}^3 \): Let \( f: \mathbb{R} \to (0, \infty) \) be a positive smooth function satisfying \( f(t) = 1 \) for \( t \leq -1 \), \( f(t) = (t \cdot \tan \theta + 1) \) for \( t \geq 1 \), where \( \theta \) is a constant in \( (0, \pi/2) \). \( M \) is defined as

\[
M = \{(x, y, z) \in \mathbb{R}^3; y^2 + z^2 = f(x)^2, x \in \mathbb{R}\}.
\]

Then \( s_1(M) \) and \( s_2(M) \) are 0 and \( 2\pi \sin \theta \) and \( 2\pi \chi(M) - c(M) = 2\pi \sin \theta \). For any given \( \varepsilon > 0 \), there exists a positive number \( t_\varepsilon \) such that if \( p \in M \) satisfies \( x(p) < -t_\varepsilon \), then \( M \circ A(p) < \varepsilon \), and such that if \( x(p) > t_\varepsilon \), then \( M \circ A(p) \in (s_2(M) - \varepsilon, s_2(M) + \varepsilon) \). For an arbitrary fixed number \( \alpha > 0 \) choose a monotone increasing sequence \( \{K_j^\alpha\} \) of compact sets of \( M \) with \( \bigcup K_j^\alpha = M \) such that

\[
\text{Area}\{p \in K_j^\alpha; x(p) > 0\}/\text{Area}\{p \in K_j^\alpha; x(p) < 0\} = \alpha.
\]

Then, computation will show that

\[
\lim_{j \to \infty} \frac{\int_{K_j^\alpha} M \circ A \, dM}{\int_{K_j^\alpha} \, dM} = \frac{s_1(M) + \alpha s_2(M)}{\alpha + 1} = \frac{(2\pi \chi(M) - c(M))\alpha}{\alpha + 1}.
\]

Since \( \alpha > 0 \) is arbitrary, this example will suggest the validity of Theorem A.
1. Preliminaries. Let $K \subset M$ be a compact set with the property that $M \setminus \text{Int}(K)$ consists of $k$ tubes $U_1, \ldots, U_k$ such that each $\partial U_i$ is a piecewise smooth closed curve. For a point $p \in M \setminus \text{Int}(K)$ taken sufficiently away from $K$, $A(p)$ is divided into two subsets $A_K(p)$ and $A'_K(p)$ as follows: For $u \in A(p)$ set $\gamma_u(t) := \exp_p tu$, $t \geq 0$.

$$
A_K(p) := \{u \in A(p); \gamma_u([0, \infty)) \cap K \neq \emptyset\},
$$

$$
A'_K(p) := \{u \in A(p); \gamma_u([0, \infty)) \cap \text{Int}(K) = \emptyset\}.
$$

Both $A_K(p)$ and $A'_K(p)$ are closed in $S_p(l)$. It follows from minimizing property of rays emanating from $p$ that $A_K(p) \cap A'_K(p)$ consists of at most two elements. Therefore

$$
\mathcal{M} \circ A(p) = \mathcal{M} \circ A_K(p) + \mathcal{M} \circ A'_K(p).
$$

It was proved in §§2 and 3 in [14] that if $0 < s_i(M) < 2\pi$, then for any given $\varepsilon > 0$ there exists an $R(\varepsilon)$ such that for every $p \in U_i$ with $d(p, K) > R(\varepsilon)$

$$
(\ast) \quad s_i(M) - \varepsilon \leq \mathcal{M} \circ A'_K(p) \leq s_i(M) + \varepsilon.
$$

A crucial step of the proof of Theorems A and B is to obtain the asymptotic behavior of $\mathcal{M} \circ A$. What is left for this purpose is to prove for all $i = 1, \ldots, k$ and for all $p \in U_i$ with $d(p, K) > R(\varepsilon)$,

$$
(\ast\ast) \quad \mathcal{M} \circ A_K(p) < \varepsilon
$$

and the following

**Lemma 1.1 (Tanaka).** Assume that $s_i(M) = 2\pi$. Then there exists a compact set $K$ with the property that for any $\varepsilon > 0$ there exists an $R_i(\varepsilon) > 0$ such that if $p \in U_i$ satisfies $d(p, K) > R_i(\varepsilon)$, then

$$
\mathcal{M} \circ A'_K(p) > 2\pi - \varepsilon.
$$

Making use of a slightly extended version of an idea developed in the proof of Theorem C in [12], $(\ast\ast)$ is proved for a more general closed subinterval $S_p(D(p))$ of $S_p(1)$ which contains $A_K(p)$. For $p \in U_i$ and for $u, v \in A_K(p)$ let $D_{u,v}(p)$ be the disk domain in $U_i$ bounded by the subarcs of $\gamma_u$ and $\gamma_v$ between $p = \gamma_u(0) = \gamma_v(0)$ and their first intersections with $K$ and a subarc of $\partial U_i$ between them. Let $D(p)$ be the maximal disk domain among $\{D_{u,v}(p): u, v \in A_K(p)\}$ and $S_p(D(p)) \subset S_p(1)$ the set of all unit vectors at $p$ tangent to $D(p)$. Define an angle

$$
\theta_K(p) := \mathcal{M}(S_p(D(p))).
$$

Then the proof of $(\ast\ast)$ is a direct consequence of the following.
Lemma 1.2 (Shioya). Let $K \subset M$ be as above and assume that $s_i(M) \leq +\infty$ holds for all $i = 1, \ldots, k$. For any $\varepsilon > 0$ there exists an $R(\varepsilon) > 0$ such that if $p \in M \setminus K$ satisfies $d(p, K) > R(\varepsilon)$, then

$$\theta_K(p) < \varepsilon.$$

2. Proof of Theorems A and B by assuming Lemmas 1.1 and 1.2. First of all consider the case where the total area of $M$ is bounded. Then a slight modification of Lemma 3.1 in [14] implies that there exist $k$ distinct Busemann functions on $M$, each of which corresponds to an endpoint of $M$. A Busemann function is differentiable except a set of measure zero since it is Lipschitz continuous. This fact means that there exists a measure zero set $E$ on $M$ such that $A(p)$ for every $p \in M \setminus E$ consists of exactly $k$ elements. Furthermore one has $2\pi \chi(M) - c(M) = 0$ if the total area of $M$ is bounded (see Theorem 12 in [5] and Corollary of Theorem A in [13]). Therefore the proof of theorems in this case is complete.

Assume that the total area of $M$ is unbounded. Let

$$R(\varepsilon) := \max_{1 \leq i \leq k} R_i(\varepsilon).$$

Let $a$ be the area of closed $R(\varepsilon)$-ball around $K$ and $b$ the integral of $\mathcal{M} \circ A$ over this closed ball. It follows from (*), Lemmas 1.1 and 1.2 that for all sufficiently large $j$

$$b + \left( \min_{1 \leq i \leq k} s_i(M) - \varepsilon \right) \left\{ \int_{K_j} dM - a \right\}$$

$$\leq \frac{\int_{K_j} \mathcal{M} \circ A dM}{\int_{K_j} dM} \leq \frac{b + \left( \max_{1 \leq i \leq k} s_i(M) + \varepsilon \right) \left\{ \int_{K_j} dM - a \right\}}{\int_{K_j} dM}.$$

The proof of Theorem A is complete since $\varepsilon$ is any and the total area of $M$ is unbounded.

For the proof of Theorem B one applies the Fiala-Hartman type isoperimetric inequality which was refined by Shiohama in [12] and [13]. Fix a compact set $K$ containing $C$ as in Lemmas 1.1 and 1.2. For every $i = 1, \ldots, k$ and for sufficiently large $t > 0$ let $L_i(t)$ and $A_i(t)$ be the length of $S(t) \cap U_i$ and the area of $B(t) \cap U_i$. Because $M$ admits total curvature $S(t) \cap U_i$ is homeomorphic to a circle for all large $t$ (see Theorem B in [13]), and is piecewise smooth for almost all $t$. Note that $A_i(t) - A_i(t') = \int_t^{t'} L_i(u) du$. For every $i = 1, \ldots, k$

$$\lim_{t \to \infty} \frac{L_i(t)}{t} = \lim_{t \to \infty} \frac{2A_i(t)}{t^2} = s_i(M).$$
By choosing $R(\varepsilon)$ sufficiently large so as to fulfil

$$s_i(M) - \varepsilon < \frac{L_i(t)}{t} < s_i(M) + \varepsilon$$

for all $i = 1, \ldots, k$ and for all $t > R(\varepsilon)$, one obtains

$$\frac{b + \sum_{i=1}^{k}(s_i(M) - 2\varepsilon)(s_i(M) - \varepsilon)(t^2 - R(\varepsilon)^2)/2}{\sum_{i=1}^{k}(s_i(M) + \varepsilon)(t^2 - R(\varepsilon)^2)/2 + a} \leq \frac{\int_{B(t)} M \circ A \, dM}{\int_{B(t)} dM} \leq \frac{b + \sum_{i=1}^{k}(s_i(M) + 2\varepsilon)(s_i(M) + \varepsilon)(t^2 - R(\varepsilon)^2)/2}{\sum_{i=1}^{k}(s_i(M) - \varepsilon)(t^2 - R(\varepsilon)^2)/2 + a}.$$

This completes the proof of Theorem B.

3. Proof of Lemmas. A general formula for the mass of rays emanating from a point $p \in M$ is obtained by using an idea developed by Shiga in [10]. This is stated as

\[ (***) \quad M \circ A(p) = 2\pi \chi(M) - c(M \setminus F_p), \]

where $F_p := \{\exp_p tu; u \in A(p), t \geq 0\}$. This formula plays an essential role for the proof of Lemma 1.1.

For the proof of (***), fix a point $p \in M$ and let $T > 0$ be a sufficiently large number such that $S(p, T) := \{x \in M; d(p, x) = T\}$ consists of $k$ piecewise smooth closed curves $C_1, \ldots, C_k$ in $U_1, \ldots, U_k$ and such that the break points $x_{i,1}, \ldots, x_{i,m(i)}$ of $C_i$ are joined to $p$ by exactly two distinct minimizing geodesics $\alpha_{i,1}, \alpha_{i,1}^-, \ldots, \alpha_{i,m(i)}^-, \alpha_{i,m(i)}^+$ with $\alpha_{i,m(i)}^-(0) = \alpha_{i,m(i)}^+(0) = p$, $\alpha_{i,m(i)}^-(T) = \alpha_{i,m(i)}^+(T) = x_{i,m}$ and $x_{i,m}$ is not conjugate to $p$ along $\alpha_{i,m}^-$ and $\alpha_{i,m}^+$. This is possible whenever $T$ is taken to be a sufficiently large non-exceptional value (see [4], [13]). Let $F_{i,m}$ ($i = 1, \ldots, k$, $1 \leq m \leq m(i)$) be a disk domain surrounded by $\alpha_{i,m}^+([0, T])$, the smooth subarc of $S(p, T)$ with terminal points $x_{i,m}$ and $x_{i,m+1}$ and $\alpha_{i,m+1}^-[0, T]$, and $\theta_{i,m}$ the angle between $-\alpha_{i,m}^-(T)$ and $-\alpha_{i,m+1}^+(T)$. If $\kappa_{i,m}$ is the curvature integral of the subarc on $\partial F_{i,m} \cap S(p, T)$, then

$$c(F_{i,m}) = M(S_p(F_{i,m})) - \kappa_{i,m}.$$ 

If $B(p, T)$ is the closed $T$-ball around $p$, then

$$c(B(p, T)) + \sum_{i=1}^{k} \sum_{m=1}^{m(i)} \kappa_{i,m} - \sum_{i=1}^{k} \sum_{m=1}^{m(i)} \theta_{i,m} = 2\pi \chi(M).$$
It follows from construction that \( \bigcup_i \bigcup_m S_p(F_i, m) \) is monotone decreasing with \( T \) and converges to \( A(p) \) as \( T \to \infty \). The proof of (***) is complete since \( \lim_{T \to \infty} \sum_{i=1}^k \sum_{m=1}^{m(i)} \theta_i, m = 0 \) (see Theorem C, [12]) and \( \lim_{T \to \infty} c(B(p, T) \setminus \bigcup_i \bigcup_m F_i, m) = c(M \setminus F_p) \).

**Proof of Lemma 1.1.** For a compact set \( C \) such that \( M \setminus C \) consists of \( k \) tubes, we choose a \( K \) containing \( C \) such that every minimizing geodesic joining points in \( C \) does not meet \( \partial K \). Let \( M_i \) be a complete open 2-manifold having one end with the properties that there exists an isometric embedding \( \iota \) of \( K \cup U_i \) into \( M_i \) and that \( M_i \setminus \iota(\partial K \cup U_i) \) consists of \( k - 1 \) disks. From construction it follows that \( 2\pi \chi(M_i) - c(M_i) = s_i(M) \) and \( \chi(M_i) = \chi(M) + (k - 1) \). Without loss of generality one may identify points in \( U_i \) with those images in \( M_i \) as well as other objects. For \( p \in U_i \) let \( A_i(p), A_{K,i}(p) \) and \( A'_{K,i}(p) \) be the set of all unit vectors tangent to rays on \( M_i \) from \( p \) with the same properties as defined in \( M \). Then \( A'_{K,i}(p) = A_k(p) \) follows from the choice of \( K \). There is no strict relationship between \( A_{K,i}(p) \) and \( A_k(p) \). But both of them will be estimated in Lemma 1.2. Since \( \mathcal{M} \circ A(p) = (\mathcal{M} \circ A(k) - \mathcal{M} \circ A_{K,i}(p)) + \mathcal{M} \circ A_i(p) \) and the first term in the right-hand side turns out to be small by Lemma 1.2, one only needs to show that \( \mathcal{M} \circ A_i(p) > 2\pi - \varepsilon \) if \( p \) is taken sufficiently away from \( K \) in \( M_i \).

From now on one identifies \( M_i \) with \( M \). For any \( \varepsilon > 0 \) let \( K_\varepsilon \subset M \) be a compact set containing \( K \) such that
\[
\int_{M \setminus K_\varepsilon} |G| dM < \varepsilon.
\]
By means of (***) it suffices for the proof of Lemma 1.1 to show \( c(M \setminus F_p) < c(M) + 5\varepsilon \) for \( p \in M \) with \( d(p, K) > R(\varepsilon) \). It follows from finite connectivity of \( M \) that there are at most finitely many non-overlapping sectors \( V_1(p), \ldots, V_l(p) \) in \( M \) with the following properties: (1) \( V_j(p) \cap K_\varepsilon \neq \emptyset \), (2) \( \partial V_j(p) \) consists of two rays emanating from \( p \), (3) \( V_j(p) \) is homeomorphic to a closed half-plane, and (4) every ray emanating from \( p \) is contained in some \( V_j(p) \) if it intersects \( K_\varepsilon \). \( V_j(p) \) has the property that if \( V_j'(p) \subset V_j(p) \) is a subsector such that there is no ray emanating from \( p \) and passing through a point on \( \text{Int}(V_j'(p)) \), then \( c(V_j'(p)) = \mathcal{M}(S_p(V_j'(p))) \). Let \( \{p_n\} \) be a divergent sequence of points in \( M \setminus K_\varepsilon \) such that \( \{V_j(p_n)\} \) for each \( j = 1, \ldots, l \) has a limit \( V_j \) as \( n \to \infty \). This \( V_j \) is a strip if it has a nonempty interior. If \( V_j \subset V_j \) is a substrip such that there exists no straight line contained entirely in \( \text{Int}(V_j) \), then \( c(V_j) = 0 \).
Set $V = V_1 \cup \cdots \cup V_l$. $c(M \setminus F_{p_n}) \leq c(K) - c(K \cap F_{p_n}) \pm \varepsilon$ and \{c(K \cap F_{p_n})\}_n$ tends to $c(K \cap V)$ as $n \to \infty$. Thus for all sufficiently large numbers $n$, $c(M \setminus F_{p_n}) \leq c(M \setminus V) + 4\varepsilon$. Since $V_j$ is a strip, a result of Cohn-Vossen (see Satz 3, [2]) implies that $c(V_j) \leq 0$ for all $j = 1, \ldots, l$. This implies that $c(M \setminus V_j) \leq 2\pi \chi(M \setminus V_j) - 4\pi$. But since $\chi(M \setminus V_j) = \chi(M) + 1$ the above inequality reduces to $c(M \setminus V_j) \leq 2\pi \chi(M) - 2\pi$. It follows from the assumption for $c(M)$ that $c(M \setminus V_j) \leq c(M)$, and in particular $c(V_j) = 0$ for all $j = 1, \ldots, l$. Therefore $c(M \setminus F_{p_n}) \leq c(M \setminus V) + 4\varepsilon \leq c(M) + 5\varepsilon$. This together with (***) proves Lemma 1.1.

**Proof of Lemma 1.2.** A contradiction will be derived by supposing that there exists a divergent sequence $\{p_n\}$ of points such that $\theta_K(p_n) \geq \varepsilon_0$ holds for all $n$ and for some $\varepsilon_0 > 0$. Without loss of generality we may consider that $\{p_n\}$ is contained in a tube $U$.

To derive a contradiction consider the universal Riemannian covering $\tilde{U}$ of $U$ whose covering projection is denoted by $\pi$. Let $\tau: [0, \infty) \to M$ be a ray emanating from a point on $\partial U$ such that $\tau([0, \infty))$ is contained entirely in $U$. Cut open $U$ along $\tau([0, \infty))$ and let $\tilde{U}_1, \tilde{U}_0, \tilde{U}_1, \ldots$ be the fundamental domains of $U$ lying in this order in $\tilde{U}$. Let $\tilde{\tau}_i: [0, \infty) \to \tilde{U}$ be the lifted ray of $\tau$ such that its image lies in $\partial \tilde{U}_{i-1} \cap \partial \tilde{U}_i$ and $\tilde{W} := \tilde{U}_0 \cup \tilde{U}_1 \cup \tilde{U}_2$. Then $\partial \tilde{W}$ consists of two rays $\tilde{\tau}_0([0, \infty))$, $\tilde{\tau}_3([0, \infty))$ and a subarc of $\partial \tilde{U}$ whose terminal points are $\tilde{\tau}_0(0)$ and $\tilde{\tau}_3(0)$.

The intersection of the two minimizing segments on $\partial D(p_n)$ with $\partial U$ will be denoted by $x_n$ and $y_n$. Set $D_n = D(p_n)$ and let $\hat{p}_n := \pi^{-1}(p_n) \cap \tilde{U}_1$ and $\tilde{D}_n \subset \tilde{U}$ the lift up of $D_n$ satisfying $\hat{p}_n \in \partial \tilde{D}_n$. Let $\hat{x}_n := \pi^{-1}(x_n) \cap \partial \tilde{D}_n$ and $\hat{y}_n := \pi^{-1}(y_n) \cap \partial \tilde{D}_n$. It follows from minimizing property of rays that the lifted minimizing geodesics joining $\hat{p}_n$ to $\hat{x}_n$ and $\hat{p}_n$ to $\hat{y}_n$ intersect $\pi^{-1}(\tau)$ at most at one point. This fact means that these geodesics are in $\tilde{W}$, and in particular, $\hat{x}_n$ and $\hat{y}_n$ are on $\partial \tilde{W} \cap \partial \tilde{U}$. By choosing a subsequence, if necessary, one may consider that $\{\hat{x}_n\}$, $\{\hat{y}_n\}$ and $\{\tilde{D}_n\}$ converge to $\hat{x}$, $\hat{y}$ and to an unbounded domain $\tilde{D}$ in $\tilde{W}$. Two cases occur in the convergence of $\{\tilde{D}_n\}$. In the first case, assume that $\{\hat{p}_n\}$ is contained in the closure of $\tilde{D}$. Then one may consider that $\{\tilde{D}_n\}$ is monotone increasing and $\bigcup \tilde{D}_n = \tilde{D}$. A slight modification of Theorem C in [12] implies that $\{\theta_K(p_n)\}$ converges to 0, a contradiction. In the second case, assume that $\{\hat{p}_n\}$ is not contained in the closure of $\tilde{D}$. Without loss of generality one may consider that the lifted minimizing geodesic joining
\( \hat{p}_n \) to \( \hat{x}_n \) intersects \( \partial \tilde{D} \) at a point \( \hat{r}_n \). Set \( \tilde{E}_n := \tilde{D}_n \setminus \tilde{D} \) and let \( \alpha_n \in (0, \pi) \) be the angle at \( \hat{r}_n \) of the corner of \( \tilde{D}_n \cap \tilde{D} \). By construction, \( \{\hat{r}_n\} \) contains a divergent subsequence. Then Cohn-Vossen's argument (see §5, [2]) implies that \( \{\alpha_n\} \) has a limit 0. Let \( K_\varepsilon \subset \mathcal{M} \) be a compact set so as to satisfy

\[
\int_{\mathcal{M} \setminus K_\varepsilon} G_+ dM < \varepsilon.
\]

Then the area of \( \pi^{-1}(K_\varepsilon \cap U) \cap \tilde{E}_n \) tends to zero as \( n \to \infty \) and the curvature integral over \( \tilde{E}_n \setminus \pi^{-1}(K_\varepsilon \cap U) \) is bounded above by \( \varepsilon \).

These facts together with the Gauss-Bonnet theorem for \( \tilde{E}_n \) imply that \( \{\theta_K(p_n)\} \) contains a subsequence converging to 0 as \( n \to \infty \), a contradiction. This completes the proof of Lemma 1.2.

**References**


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