ON SOME TOTALLY ERGODIC FUNCTIONS

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Dedicated to Dagmara Klim and Nina Tomaszewska

We study some classes of totally ergodic functions on locally compact Abelian groups. Among other things, we establish the following result: If $R$ is a locally compact commutative ring, $\mathbb{R}$ is the additive group of $R$, $\chi$ is a continuous character of $\mathbb{R}$, and $p$ is the function from $\mathbb{R}^n$ ($n \in \mathbb{N}$) into $\mathbb{R}$ induced by a polynomial of $n$ variables with coefficients in $R$, then the function $\chi \circ p$ either is a trigonometric polynomial on $\mathbb{R}^n$ or all of its Fourier-Bohr coefficients with respect to any Banach mean on $L^\infty(\mathbb{R}^n)$ vanish.

1. Introduction. Let $G$ be a locally compact Abelian group, $\lambda_G$ be the Haar measure in $G$, and $L^\infty(G)$ be the space of all classes of complex-valued $\lambda_G$-measurable $\lambda_G$-essentially bounded functions on $G$ endowed with the $\lambda_G$-essential supremum norm.

A linear continuous functional $m$ on $L^\infty(G)$ is called a Banach mean on $L^\infty(G)$ if it satisfies the following conditions:

(i) $m(1) = 1 = \|m\|$, 
(ii) $m(T_a f) = m(f)$ for each $a \in G$ and each $f \in L^\infty(G)$, where $T_a f(b) = f(a + b)$ for any $b \in G$.

When $G$ is finite, there is precisely one Banach mean on $L^\infty(G)$.

When $G$ is infinite, then the set of all Banach means on $L^\infty(G)$ has at least the cardinality of the continuum (cf. [6, Propositions 22.26 and 22.41]).

Let $\hat{G}$ be the dual group of $G$. Given $f \in L^\infty(G)$, $\chi \in \hat{G}$, and a Banach mean $m$ on $L^\infty(G)$, let $\mathcal{F}_m f(\chi)$ stand for the Fourier-Bohr coefficient of $f$ at $\chi$ with respect to $m$, defined to be $m(f\chi)$.

A function $f$ in $L^\infty(G)$ is said to be ergodic if its mean value $m(f)$ is independent of the choice of the Banach mean $m$ on $L^\infty(G)$. A function $f$ in $L^\infty(G)$ is said to be totally ergodic if, for every $\chi \in \hat{G}$, the function $f\chi$ is ergodic (cf. [7, 8]). Let $E(G)$ be the space of all ergodic functions in $L^\infty(G)$, $TE(G)$ be the space of all totally ergodic functions in $L^\infty(G)$, and $TE_0(G)$ be the subspace of $TE(G)$ consisting of those $f \in L^\infty(G)$ for which $\mathcal{F}_m f(\chi) = 0$ for any $\chi \in \hat{G}$ and any Banach mean $m$ on $L^\infty(G)$. Let $P(G)$ be the space of all
functions in \( L^\infty(G) \) which, to within modification on a \( \lambda_G \)-null set, are trigonometric polynomials on \( G \). It is readily verified that

\[
P(G) \subset TE(G)
\]

and that

\[
P(G) \cap TE_0(G) = \{0\}.
\]

The chief aim of the present paper is to show that certain subsets of \( L^\infty(G) \), determined by conditions formulated with use of some coboundary operator, are contained in \( P(G) \cup TE_0(G) \). One consequence of the main result about those subsets reads as follows: If \( R \) is a locally compact commutative ring, \( \mathcal{R} \) is the additive group of \( R \), \( \chi \) is an element of \( \hat{\mathcal{R}} \), and \( p \) is the function from \( \mathcal{R}^n \) \((n \in \mathbb{N})\) into \( \mathcal{R} \) induced by a polynomial of \( n \) variables with coefficients in \( R \), then the function \( \chi \circ p \) is an element either of \( P(\mathcal{R}^n) \) or of \( TE_0(\mathcal{R}^n) \).

2. Preliminaries. Given a set \( A \), \( \#A \) denotes the cardinality of \( A \). If \( A \) is subset of a larger set, then \( 1_A \) stands for the characteristic function of \( A \).

Given \( a \in G \) and a subset \( A \) of \( G \), let

\[
a + A = \{b \in G : b - a \in A\}.
\]

A complex-valued function \( f \) on \( G \) with values of unit modulus will be called unitary. A function in \( L^\infty(G) \) which, to within modification on a \( \lambda_G \)-null set, is unitary will be called almost unitary. We denote by \( U(G) \) the set of all almost unitary functions in \( L^\infty(G) \), and write \( U_0(G) \) for \( U(G) \cap P(G) \).

Let \( f \) be function in \( U(G) \). For each \( a \in G \), put

\[
\delta_a f = f \cdot T_a f
\]

and, for any \( a_1, \ldots, a_n \in G \), set inductively

\[
\delta_{a_1 \cdots a_n} f = \delta_{a_n} (\delta_{a_1 \cdots a_{n-1}} f).
\]

For each \( 1 \leq p < +\infty \), let \( L^p(G) \) be the \( p \)th Lebesgue space based on \( \lambda_G \).

Given \( f \in L^1(G) \), let \( \mathcal{F} f \) denote the Fourier transform of \( f \), defined by

\[
\mathcal{F} f(\chi) = \int_G f(a)(a, -\chi) d\lambda_G(a) \quad (\chi \in \hat{G});
\]

here \( (a, -\chi) \) stands for the value of the character \(-\chi\) at \( a \). Let \( \sigma(f) \) denote the spectrum of \( f \), that is, the support of \( \mathcal{F} f \).
If \( f \in P(G) \) is \( \lambda_G \)-essentially equal to a trigonometric polynomial \( \sum_{\chi \in \hat{G}} a_{\chi} \chi \), then the set \( \{ \chi \in \hat{G} : a_{\chi} \neq 0 \} \) will also be denoted as \( \sigma(f) \) and referred to as the spectrum of \( f \).

For each \( n \in \mathbb{N} \), let
\[
P_n(G) = \{ f \in P(G) : \# \sigma(f) \leq n \}
\]
and
\[
U_n(G) = \{ f \in U(G) : \delta_{a_1, \ldots, a_n} f \in P(G) \text{ for } a_1, \ldots, a_n \in G \}.
\]
For each \( m \in \mathbb{N} \), let
\[
U_{0,m}(G) = U(G) \cap P_m(G)
\]
and, for any \( n, m \in \mathbb{N} \), let
\[
U_{n,m}(G) = \{ f \in U(G) : \delta_{a_1, \ldots, a_n} f \in P_m(G) \text{ for } a_1, \ldots, a_n \in G \}.
\]

Given a probability triple \((\Omega, \mathcal{B}, \mathbb{P})\) and a \( \sigma \)-subalgebra \( \mathcal{A} \) of \( \mathcal{B} \), we write \( E^\mathcal{A} \) for the conditional expectation operator relative to \( \mathcal{A} \).

For a subset \( A \) of a vector space, the linear span of \( A \) is denoted by \( \text{span} \ A \).

For a subset \( A \) of a set \( B \) with a topology, we denote by \( \overline{A} \) the closure of \( A \) in \( B \).

3. A characterization of \( U_0(G) \). In this section, we give a characterization of the set \( U_0(G) \) for an arbitrary locally compact Abelian group \( G \). We start with the following.

**Proposition 3.1.** Let \( G \) be a locally compact Abelian group such that \( \hat{G} \) is torsion-free. Then
\[
U_0(G) = U_{0,1}(G).
\]

**Proof.** Clearly, it suffices to show that \( U_0(G) \subset U_{0,1}(G) \).

Let \( f \) be a function in \( U_0(G) \) and let \( \sum_{i=1}^{n} a_i \chi_i \) be the trigonometric polynomial on \( G \) \( \lambda_G \)-essentially equal to \( f \), with \( \sigma(f) = \{ \chi_i : 1 \leq i \leq n \} \). Suppose that \( n \geq 2 \). Let \( \Gamma \) be the subgroup of \( \hat{G} \) generated by \( \sigma(f) \). Of course, \( \Gamma \) is countable and torsion-free. Hence there exists a monomorphism \( h \) from \( \Gamma \) into the group of reals (cf. [9, Theorem 8.1.2]). Changing, if necessary, the enumeration of the elements of \( \sigma(f) \), we may assume that \( h(\chi_i) < h(\chi_j) \) whenever \( 1 \leq i < j \leq n \).

Since
\[
h(\chi_n \chi_1) = h(\chi_n) - h(\chi_1) > h(\chi_i) - h(\chi_j) = h(\chi_i \chi_j)
\]
whenever \((i, j) \neq (n, 1) (1 \leq i \leq n, 1 \leq j \leq n)\), it follows that the Fourier coefficient of \(\sum_{i,j=1}^{n} a_i \bar{a}_j \chi_i \bar{\chi}_j\) at \(\chi_n \bar{\chi}_1\) is equal to \(a_n a_1\).

Moreover, since

\[ h(\chi_n \bar{\chi}_1) > h(\chi_n \bar{\chi}_n) = 0, \]

we see that \(\chi_n \bar{\chi}_1\) is a non-trivial character of \(G\). But

\[ \sum_{i,j=1}^{n} a_i \bar{a}_j \chi_i \bar{\chi}_j = \left| \sum_{i=1}^{n} a_i \chi_i \right|^2 = 1, \]

so uniqueness of the Fourier expansion implies that \(a_n a_1 = 0\). This contradiction shows that \(\sigma(f)\) is a singleton.

The proof is complete.

Passing to the characterization of \(U_0(G)\) in the general case, we first show that the problem reduces to characterizing \(U_0(G)\) for a compact Abelian group \(G\) such that the component of 0 in \(G\) (which is a closed subgroup of \(G\)) has finite index.

With \(G\) an arbitrary locally compact Abelian group, let \(f\) be an element of \(U_0(G)\). Denote by \((\widehat{G})_d\) the group \(\widehat{G}\) furnished with the discrete topology. Let \(\Gamma\) be the subgroup of \((\widehat{G})_d\) generated by \(\sigma(f)\), \(\text{Per}(\Gamma)\) be the subgroup of \(\Gamma\) consisting of all elements of finite order, and \(H\) be the component of 0 in \(\widehat{\Gamma}\). Then the dual of \(\widehat{\Gamma}/H\) coincides with \(\text{Per}(\Gamma)\) (cf. [5, Corollary 24.20]). Since \(\Gamma\) is finitely generated, it follows that \(\text{Per}(\Gamma)\) is finite and hence \(H\) has finite index. Let \(\alpha\) be the canonical homomorphism from \(\Gamma\) into \(\widehat{G}\). Then the dual homomorphism \(\hat{\alpha}\), defined by

\[ (\hat{\alpha}(g), \chi) = (g, \alpha(\chi)) \quad (g \in G, \chi \in \Gamma), \]

maps \(G\) onto a dense subgroup of \(\widehat{\Gamma}\). Moreover, there exists a unique \(p\) in \(U_0(\widehat{\Gamma})\) such that \(f = p \circ \hat{\alpha}\). Thus it is clear that the passage from \(f\) to \(p\) yields the desired reduction.

Now we may and do assume that \(G\) is a compact Abelian group such that the component \(H\) of 0 in \(G\) has finite index. Let \(\{a_i : 1 \leq i \leq n\}\) be a subset of \(G\) such that the sets \(a_i + H\) \((1 \leq i \leq n)\) form the collection of all cosets of \(H\) in \(G\). We claim that

\[ U_0(G) = \{f \in \mathbb{T}^G : f(a_i + g) = c_i \chi_i(g) \text{ for } g \in H, \quad c_i \in \mathbb{T}, \chi_i \in \widehat{H} (1 \leq i \leq n)\}, \]

where \(\mathbb{T}\) denotes the circle group.

Indeed, if we let \(A\) denote the right-hand side set, the containment of \(U_0(G)\) in \(A\) follows from Proposition 3.1 and the fact that the
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dual of a connected locally compact Abelian group is torsion-free (cf. [5, Corollary 24.19]). Conversely, if \( f \in A \), then
\[
\text{span}\{T_a f : a \in G\} \subset \text{span}\{\chi_i 1_{a,+H} : 1 \leq i \leq n, 1 \leq j \leq n\},
\]
so \( \text{span}\{T_a : a \in G\} \) is finite dimensional, and hence \( f \) is a trigonometric polynomial on \( G \) (cf. [9, Theorem 7.8.3]). Thus \( A \subset U_0(G) \) and the claim follows.

4. The main results. The starting point of our main considerations is the following.

**Theorem 4.1.** Let \( G \) be a compact Abelian group. Then
\[
U_n(G) = U_0(G)
\]
for each \( n \in \mathbb{N} \).

**Proof.** Clearly, it suffices to prove that \( U_n(G) \subset U_0(G) \) for each \( n \in \mathbb{N} \). A simple induction argument shows that in fact it suffices to establish the containment of \( U_1(G) \) in \( U_0(G) \).

Given \( f \in U_1(G) \), let \( \Sigma \) be the subgroup of \( \hat{G} \) generated by \( \sigma(f) \). Clearly, \( \Sigma \) is countable. Let \( (\sigma_n)_{n \in \mathbb{N}} \) be a family of finite subsets of \( \Sigma \) such that \( \sigma_n \subset \sigma_{n+1} \) for each \( n \in \mathbb{N} \), and
\[
\Sigma = \bigcup_{n=1}^{\infty} \sigma_n.
\]
Given \( n \in \mathbb{N} \), let
\[
F_n = \{a \in G : \sigma(\delta_a f) \subset \sigma_n\}.
\]
Each \( F_n \) is clearly closed. Since, for each \( a \in G \), \( \sigma(\delta_a f) \) is a finite subset of \( \Sigma \), it follows that
\[
G = \bigcup_{n=1}^{\infty} F_n.
\]
By Baire’s theorem, there exist an open subset \( V \) of \( G \) and a positive integer \( m \) such that \( V \subset F_m \). By the compactness of \( G \), there exists a finite subset \( \{a_i : 1 \leq i \leq k\} \) of \( G \) such that
\[
G = \bigcup_{i=1}^{\infty} (a_i + V).
\]
For each \( a \in G \), if \( 1 \leq i \leq k \) and \( v \in V \) are such that \( a = a_i + v \), then
\[
T_a f = T_{a_i} (\delta_v f) \cdot T_{a_i} f.
\]
Thus
\[ \text{span}\{T_a f : a \in G\} \subset \text{span}\{\chi T_a f : \chi \in \sigma_m, 1 \leq i \leq k\}. \]
Consequently, \( \text{span}\{T_a f : a \in G\} \) is finite dimensional, and hence \( f \) is in \( U_0(G) \).

The proof is complete.

**Lemma 4.2.** Let \( G \) be a compact Abelian group. Let \( f \) be an almost unitary function in \( G \), \( S \) be a dense subset of \( G \), and \( n \) and \( m \) be positive integers such that \( \#\sigma(\delta_{s_1} \cdots \delta_{s_n} f) \leq m \) for any \( s_1, \ldots, s_n \in S \). Then \( f \in U_0(G) \).

**Proof.** Suppose that for some \( a_1, \ldots, a_n \in G \), the spectrum of \( \delta_{a_1} \cdots \delta_{a_n} f \) contains \( m + 1 \) distinct elements \( \chi_1, \ldots, \chi_{m+1} \). Then, in view of the continuity of the functions
\[ G^n \ni (b_1, \ldots, b_n) \rightarrow \mathcal{F} \delta_{b_1} \cdots \delta_{b_n} f(\chi_i) \quad (1 \leq i \leq m + 1) \]
and the denseness of \( S \) in \( G \), there exist \( s_1, \ldots, s_n \in S \) such that
\[ \{\chi_1, \ldots, \chi_{m+1}\} \subset \sigma(\delta_{a_1} \cdots \delta_{a_n} f), \]
a contradiction. Thus \( f \in U_{m,n}(G) \), and hence, by the preceding theorem, \( f \in U_0(G) \).

The proof is complete.

The next theorem is the main result of this section.

**Theorem 4.3.** Let \( G \) be a locally compact Abelian group. Then
\[ U_{n,m}(G) \subset U_0(G) \cup TE_0(G) \]
for each \( n \in \mathbb{N} \cup \{0\} \) and each \( m \in \mathbb{N} \).

**Proof.** We shall proceed by induction on \( n \) with \( m \) arbitrarily fixed.

The case \( n = 0 \) is obvious.

Assume the assertion for \( n-1 \). Suppose that \( f \in U_{n,m}(G) \setminus TE_0(G) \).
Then there exist \( \chi \in \hat{G} \) and a Banach mean \( m \) on \( L^\infty(G) \) such that
\( \mathcal{F}m f(\chi) \neq 0 \). Let \( h = f \chi \). Then, clearly, \( m(h) \neq 0 \). Moreover, for each \( a \in G \), \( \delta_a h \in U_{n-1,m}(G) \), and hence, by the inductive hypothesis, either \( \delta_a h \in U_0(G) \) or \( \delta_a h \in TE_0(G) \). Since, for each \( a \in G \),
\[ \delta_{-a} h = T_{-a} \delta_a h \]
and, for any \( a, b \in G \),
\[ \delta_{a+b} h = \delta_a h \cdot T_a \delta_b h, \]
it follows that
\[ G_0 = \{ a \in G : \delta_a h \in U_0(G) \} \]
is a subgroup of \( G \). We claim that the index of \( G_0 \) is finite.

Suppose, on the contrary, that there exists an infinite subset \( \{ a_n : n \in \mathbb{N} \} \) of \( G \) such that \( a_n - a_m \not\in G_0 \) whenever \( n \neq m \). Then, if \( n \neq m \), then \( \delta_{a_n - a_m} h \) is in \( TE_0(G) \), and hence
\[
m(\delta_{a_n} h \cdot \delta_{a_m} h) = m(T_{a_n} \delta_{a_n - a_m} h) = m(\delta_{a_n - a_m} h) = 0.
\]
We see that the image of \( \{ \delta_a h : n \in \mathbb{N} \} \) by the canonical mapping from \( L^\infty(G) \) onto the pre-Hilbert space
\[
H_m(G) = L^\infty(G)/\{ f \in L^\infty(G) : m(|f|^2) = 0 \}
\]
is an orthonormal set. For each \( n \in \mathbb{N} \), we have
\[
m(h) = m(T_{a_n} h) = m(h \cdot \delta_{a_n} h).
\]
Thus the Fourier coefficients of the image of \( h \) in \( H_m(G) \) relative to the image of \( \{ \delta_a h : n \in \mathbb{N} \} \) in \( H_m(G) \) are equal to \( m(h) \), and hence, by Bessel's inequality, \( m(h) = 0 \). This contradiction establishes the claim.

Let \( bG \) be the Bohr compactification of \( G \) and \( \alpha : G \to bG \) be the canonical monomorphism from \( G \) into \( bG \). For each \( \chi \in \hat{G} \), let \( \tilde{\chi} \) be the continuous character of \( bG \) such that \( \tilde{\chi} \circ \alpha = \chi \). As is known, the Fourier transformation sets up a one-to-one correspondence between \( L^2(bG) \) and \( l^2(\hat{G}) (= L^2((\hat{G})_d)) \). Since by Bessel's inequality, the function
\[
\hat{G} \ni \chi \to \mathcal{F}_m f(\chi) \in \mathbb{C}
\]
is in \( l^2(\hat{G}) \), there exists a unique element \( X \) in \( L^2(bG) \) such that
\[
(4.1) \quad \mathcal{F} X(\tilde{\chi}) = \mathcal{F}_m f(\chi) \quad (\chi \in \hat{G}).
\]
Since
\[
G_0 = \{ a \in G : \delta_a f \in U_0(G) \},
\]
it follows that for each \( a \in G_0 \), there exists a unique unitary trigonometric polynomial \( P_a \) on \( bG \) such that
\[
(4.2) \quad \delta_a f = P_a \circ \alpha \quad \lambda_G\text{-a.e.}
\]
If, for each \( a \in G_0 \), we let
\[
P_a = \sum_{\gamma \in \hat{G}} b_{\alpha, \gamma} \tilde{\chi},
\]
then, in view of (4.1), for each \( \chi \in \hat{G} \),
\[
\mathcal{F}T_{a(a)}X(\tilde{\chi}) = (\alpha(a), \tilde{\chi})\mathcal{F}X(\tilde{\chi}) = (a, \chi)\mathcal{F}_{m}f(\chi)
\]
\[
= \mathcal{F}_{m}T_{a}f(\chi) = \mathcal{F}_{m}(f\delta_{a}f)(\chi)
\]
\[
= \sum_{\gamma \in \hat{G}} b_{a,}\gamma \mathcal{F}_{m}f(\tilde{\chi} - \tilde{\gamma})
\]
\[
= \sum_{\gamma \in \hat{G}} b_{a,}\gamma \mathcal{F}X(\tilde{\chi} - \tilde{\gamma}) = \mathcal{F}(XP_{a})(\tilde{\chi})
\]
whence
\[
(4.3) \quad T_{\alpha(a)}X = XP_{a} \quad \lambda_{bG}-a.e.
\]

Let \( \{a_{i}: 1 \leq k\} \) be a subset of \( G \) such that the sets \( a_{i} + G_{0} \) \((1 \leq i \leq k)\) form the collection of all cosets of \( G_{0} \) in \( G \). Since
\[
bG \supset \bigcup_{i=1}^{k}(\alpha(a_{i}) + \alpha(G_{0})) \supset \bigcup_{i=1}^{k} \alpha(a_{i} + G_{0}) = bG,
\]
the closures being taken in \( bG \), it follows that the index of \( \overline{\alpha(G_{0})} \) in \( bG \) is no greater than \( k \). Thus \( \overline{\alpha(G_{0})} \) is an open subgroup of \( bG \) and, in particular, the Haar measure in \( \overline{\alpha(G_{0})} \) is, to within normalization, the restriction to \( \overline{\alpha(G_{0})} \) of the Haar measure in \( bG \).

Let \( \{b_{j}: 1 \leq j \leq l\} \) be a subset of \( \{a_{i}: 1 \leq i \leq k\} \) such that the sets \( \alpha(b_{j}) + \alpha(G_{0}) \) \((1 \leq j \leq l)\) form the collection of all cosets of \( \overline{\alpha(G_{0})} \) in \( bG \). For each \( 1 \leq j \leq l \), let \( X_{j} \) denote the restriction of \( T_{\alpha(b_{j})}X \) to \( \overline{\alpha(G_{0})} \). In view of (4.3), for each \( 1 \leq j \leq l \) and each \( a \in G_{0} \),
\[
T_{\alpha(a)}|X_{j}| = |X_{j}| \quad \lambda_{\overline{\alpha(G_{0})}}-a.e.
\]
Applying the Fourier transformation to both sides of the latter equality, we readily find that for \( 1 \leq j \leq l \), \( |X_{j}| \) is \( \lambda_{\overline{\alpha(G_{0})}} \)-essentially constant. Choose \( j_{0} \) so that
\[
X_{j_{0}} \neq 0 \quad \lambda_{\overline{\alpha(G_{0})}}-a.e.
\]
and set
\[
Y = |X_{j_{0}}|^{-1}X_{j_{0}}.
\]
For each \( a \in G_{0} \), let \( R_{a} \) denote the restriction of \( T_{\alpha(b_{j_{0}})}P_{a} \) to \( \overline{\alpha(G_{0})} \).
Since, by (4.3), for each \( a \in G_{0} \),
\[
(4.4) \quad \delta_{\alpha(a)}Y = R_{a} \quad \lambda_{\overline{\alpha(G_{0})}}-a.e.,
\]
it follows from Lemma 4.2 that \( Y \in U_{0}(\alpha(G_{0})) \). Hence in particular
the set
\[ \Gamma = \{ \gamma \in (\alpha(G_0))_\gamma^\circ : \gamma = \gamma_1 \gamma_2 \text{ for } \gamma_1, \gamma_2 \in \sigma(Y) \} \]
is finite.

Let \( (\alpha(G_0))_\perp \) be the annihilator of \( \alpha(G_0) \) in \( (bG)_\gamma^\circ \), that is, the set
\[ \{ \gamma \in (bG)_\gamma^\circ : (\alpha(a), \gamma) = 1 \text{ for } a \in G_0 \} \]
Being the dual of the quotient group \( bG/\alpha(G_0) \), the group \( (\alpha(G_0))_\perp \) is finite. Let \( \pi \) be the canonical homomorphism from \( (bG)^\circ \) onto \( (\alpha(G_0))_\perp \). Since the kernel of \( \pi \) coincides with \( (\alpha(G_0))_\perp \), we see that the set \( \pi^{-1}(\Gamma) \) is finite, and hence the set
\[ \Xi = \{ \chi \in \hat{G} : \chi = \gamma \circ \alpha \text{ for } \gamma \in \pi^{-1}(\Gamma) \} \]
is also finite.

In view of (4.4), \( \Gamma \) contains the spectra of all the \( R_a \) \( (a \in G_0) \). Consequently, \( \pi^{-1}(\Gamma) \) contains the spectra of all the \( T_{a,b_0}P_a \) \( (a \in G_0) \), and hence the spectra of all the \( P_a \) \( (a \in G_0) \). Now Eq. (4.2) implies that
\[ \{ T_a f : a \in G_0 \} \subset \text{span}\{ \chi f : \chi \in \Xi \} \]
whence
\[ \{ T_a f : a \in G \} \subset \text{span}\{ \chi T_a f : \chi \in \Xi, 1 \leq i \leq k \}. \]
We see that \( \text{span}\{ T_a f : a \in G \} \) is finite dimensional, and so \( f \) is in \( U_0(G) \).

The proof is complete.

5. Applications. Let \( R \) be a locally compact commutative ring, \( R \) be the additive group of \( R \), \( \chi \) be an element of \( \hat{R} \), and \( p \) be the function from \( R^n \) \( (n \in \mathbb{N}) \) into \( R \) induced by a polynomial
\[ \sum_{|\alpha| \leq k} a_\alpha x^\alpha \quad (\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n, \ |\alpha| = \alpha_1 + \cdots + \alpha_n) \]
of \( n \) variables with coefficients in \( R \), of degree \( k \). Then, of course, \( \chi \circ p \) is in \( U_{k,1}(R^n) \). Applying Theorem 4.3 to \( \chi \circ p \), we obtain the following.

**Theorem 5.1.** The function \( \chi \circ p \) is an element either of \( U_0(R^n) \) or of \( TE_0(R^n) \).

Let
\[ R^{(2)} = \{ r \in R : r = st \text{ for } s, t \in \mathbb{R} \}. \]
**Theorem 5.2.** Suppose that $R = R^{(2)}$, that $\mathcal{R}$ is torsion-free, that $k$ is not less than 2, and that for some $\alpha$ with $|\alpha| = k$, the character $r \rightarrow \chi(a_\alpha r)$ of $\mathcal{R}$ is non-trivial. Then $\chi \circ p$ is in $TE_0(\mathcal{R}^n)$.

**Proof.** Suppose, on the contrary, that $\chi \circ p$ is not in $TE_0(\mathcal{R}^n)$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index with $|\alpha| = k$ such that the character $r \rightarrow \chi(a_\alpha r)$ of $\mathcal{R}$ is non-trivial. Given $r_1, \ldots, r_k \in R$, put

$$a_1 = (r_1, 0, \ldots, 0),$$

$$a_{\alpha_1} = (r_{\alpha_1}, 0, \ldots, 0),$$

$$a_{\alpha_1+1} = (0, r_{\alpha_1+1}, \ldots, 0),$$

$$a_{\alpha_1+\alpha_2} = (0, r_{\alpha_1+\alpha_2}, \ldots, 0),$$

$$a_k = (0, \ldots, 0, r_k).$$

A straightforward calculation shows that

$$\delta_{a_1, \ldots, a_k}(\chi \circ p) = \chi(\alpha!a_\alpha r_1 \cdots r_k) = (\alpha! = \alpha_r! \cdots \alpha_n!).$$

Now Proposition 3.1, Theorem 4.3, and the fact that $\mathcal{R}$ is torsion-free imply that $\chi \circ p$ is in $U_{0,1}(\mathcal{R}^n)$. Since $k \geq 2$, it follows that $\delta_{a_1, \ldots, a_k}(\chi \circ p) = 1$ and, consequently, that $\chi(\alpha!a_\alpha r_1 \cdots r_k) = 1$ for any $r_1, \ldots, r_k \in R$. Taking into account that $R = R^{(2)}$ and that $\mathcal{R}$ is torsion-free, we infer that $\chi(a_\alpha r) = 1$ for each $r \in R$, a contradiction.

The proof is complete.

As an immediate consequence of Theorem 5.2, we get the following generalization of a result of [1]:

**Theorem 5.3.** Let $K$ be a locally compact commutative field, $\mathcal{H}$ be the additive group of $K$, $\chi$ be an element of $\mathcal{H}$, and $p$ be the function from $\mathcal{H}^n (n \in \mathbb{N})$ into $\mathcal{H}$ induced by a polynomial of $n$ variables with coefficients in $K$, of degree not less than 2. Then $\chi \circ p$ is in $TE_0(\mathcal{H}^n)$.

6. A counter-example. In this section we show that Theorem 4.3 fails in general if in the statement the set $U_{n,m}(G)$ is replaced by the set $U_n(G)$.

For each $n \in \mathbb{N}$, let $G_n$ be a non-zero finite Abelian group with a pair number of elements. Let $G$ be the direct sum of the $G_n$ ($n \in \mathbb{N}$), and $\Sigma$ be the direct product of the $G_n$ ($n \in \mathbb{N}$). Endow $G$ with the discrete topology, and $\Sigma$ with the product topology (of course, each $G_n$ is given the discrete topology). For each $n \in \mathbb{N}$, let $\pi_n$ be the canonical projection from $G$ onto $G_n$, and $\rho_n$ be the canonical
projection from $\Sigma$ onto $G_n$. Let $\alpha$ be the canonical monomorphism from $G$ into $\Sigma$. Given $n \in \mathbb{N}$, let $e_n$ be a function from $G_n$ onto \{-1, 1\} such that

$$\sum_{g \in G_n} e_n(g) = 0,$$

and put

$$f_n = e_n \circ \rho_n.$$ 

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of real numbers such that

$$\sum_{n=1}^{\infty} |a_n|^2 < +\infty, \quad \sum_{n=1}^{\infty} |a_n| = +\infty,$$

and $|a_n| < \pi/4$ for each $n \in \mathbb{N}$. Given $\sigma \in \Sigma$ and $a \in G$, set

$$A(\sigma, a) = \exp \left[ i \sum_{n=1}^{\infty} a_n (f_n(\sigma) - f_n(\sigma + \alpha(a))) \right].$$

To see that the above definition makes sense, note that given $a \in G$, there exists $m \in \mathbb{N}$ such that $\pi_n(a) = 0$ whenever $n > m$, and so, for each $\sigma \in \Sigma$,

$$(6.1) \quad A(\sigma, a) = \exp \left[ i \sum_{n=1}^{m} (f_n(\sigma) - f_n(\sigma + \alpha(a))) \right].$$

One verifies at once that the mapping $A: (\sigma, a) \rightarrow A(\sigma, a)$ is a Borel unitary function on $\Sigma \times G$ satisfying

$$A(\sigma, a + b) = A(\sigma, a)A(\sigma + \alpha(a), b)$$

for all $\sigma \in \Sigma$ and all $a, b \in G$. $A$ is an example of what is called a cocycle on $\Sigma$ (cf. [2, 3, 4]).

Since, clearly, $(f_n)_{n \in \mathbb{N}}$ is a Bernoulli sequence on the probability triple $(\Sigma, \mathcal{B}(\Sigma), \lambda_\Sigma)$, where $\mathcal{B}(\Sigma)$ stands for the Borel $\sigma$-algebra of $\Sigma$ and $\lambda_\Sigma$ is the normalized Haar measure in $\Sigma$, it follows that the series $\sum_{n=1}^{\infty} a_n f_n(\sigma)$ converges for $\lambda_\Sigma$-almost all $\sigma$ in $\Sigma$. Let $Z$ be a real Borel function on $\Sigma$ $\lambda_\Sigma$-almost everywhere equal to the sum of the above series. On putting

$$(6.2) \quad Y = \exp(iZ),$$

we see that given $a \in G$, the identity

$$(6.3) \quad A(\sigma, a) = Y(\sigma)Y(\sigma + \alpha(a))$$

holds for $\lambda_\Sigma$-almost all $\sigma$ in $\Sigma$. The existence of a representation of $A$ as above is usually expressed as saying that $A$ is a coboundary.
Each function of the form \( a \rightarrow A(\sigma, a) \) \((\sigma \in \Sigma)\) is called a trajectory of \( A \). In view of (5.1), for each \( a \in G \), the function \( \sigma \rightarrow A(\sigma, a) \) is a unitary trigonometric polynomial on \( \Sigma \). Hence, for each \( \sigma \in \Sigma \) and each \( b \in G \), the function \( a \rightarrow A(\sigma + \alpha(a), b) \) is a unitary trigonometric polynomial on \( G \). Taking into account the identity

\[
A(\sigma, a)A(\sigma, a + b) = A(\sigma + \alpha(a), b) \quad (\sigma \in \Sigma, \ a, b \in G),
\]

we thus see that each trajectory of \( A \) is in \( U_1(G) \). On the other hand, a modification of an argument used in the proof to [2, Theorem 2.4] shows that if some trajectory of \( A \) is totally ergodic, then \( A \) is a so-called \( c \)-coboundary, that is, there exists a unitary continuous function \( X \) on \( \Sigma \) such that

\[
A(\sigma, a) = X(\sigma)\overline{X(\sigma + \alpha(a))} \quad (\sigma \in \Sigma, \ a \in G).
\]

(6.4)

for each \( \sigma \in \Sigma \) and each \( a \in G \). Below we shall show that \( A \) is not a \( c \)-coboundary. Consequently, each trajectory of \( A \) will provide an example of an element of \( U_1(G) \) that is not in \( U_0(G) \cup T\mathcal{E}_0(G) \).

To show that \( A \) is not a \( c \)-coboundary, suppose, contrariwise, that there exists a unitary continuous function \( X \) on \( \Sigma \) satisfying (6.4). Then in view of (6.3), given \( a \in G \), the identity

\[
Y(\sigma + \alpha(a))\overline{X(\sigma + \alpha(a))} = Y(\sigma)\overline{X(\sigma)}
\]

holds for \( \lambda_\Sigma \)-almost all \( \sigma \) in \( \Sigma \). Applying the Fourier transformation to both sides of the latter equality, we see that there exist \( c \in \mathbb{T} \) such that

\[
Y(\sigma) = cX(\sigma)
\]

(6.5)

for \( \lambda_\Sigma \)-almost all \( \sigma \) in \( \Sigma \).

Let \( M \) be a positive number such that

\[
|z| \leq M|e^z - 1|
\]

(6.6)

for each complex number \( z \) with \( |z| < \pi \). Since \( \Sigma \) is compact, it follows that \( X \) is uniformly continuous, and so there exists \( k \in \mathbb{N} \) such that if \( \sigma \) is in

\[
U_k = \{\theta \in \Sigma: \pi_n(\theta) = 0 \text{ for } n < k\},
\]

then

\[
\|T_\sigma X - X\|_\infty < \pi/2M,
\]

(6.7)
where $\| \cdot \|_{\infty}$ denotes the supremum norm. For each $n \geq k$, let $\mathcal{A}_n$ be the $\sigma$-subalgebra of $\mathcal{B}(\Sigma)$ generated by the $f_j$ with $k \leq j \leq n$. Then, by (6.7), for each $n \geq k$ and each $\sigma \in U_k$,

$$\|E_{s_n}^\sigma T_\sigma X - E_{s_n}^\sigma X\|_{\infty} < \pi/2M$$

whence, in view of (6.2), (6.3), and (6.5),

$$\left\| \exp \left[ i \sum_{j=k}^{m} a_j(T_\sigma f_j - f_j) \right] - 1 \right\|_{\infty} < \pi/2M. \quad (6.8)$$

Proceeding by induction on $n$, we show now that for each $n \geq k$ and each $\sigma \in U_k$,

$$\left\| \sum_{j=k}^{n} a_j(T_\sigma f_j - f_j) \right\|_{\infty} < \pi/2. \quad (6.9)$$

For $n = k$, the inequality follows from the estimates

$$\|a_k(T_\sigma f_k - f_k)\|_{\infty} \leq 2|a_k| < \pi/2.$$

Assume the validity of the inequality for $n - 1 \geq k$. Then

$$\left\| \sum_{j=k}^{n} a_j(T_\sigma f_j - f_j) \right\|_{\infty} < \pi/2 + 2|a_n| < \pi$$

and now (6.9) results from (6.6) and (6.8).

Choose $\theta$ in $\Sigma$ so that the series $\sum_{j=1}^{\infty} a_j f_j(\theta)$ converges. Then, in view of (6.9), for each $n \geq k$ and each $\sigma \in U_k$,

$$\left| \sum_{j=k}^{n} a_j f_j(\sigma + \theta) \right| < \pi/2 + \sup \left\{ \left| \sum_{j=k}^{m} a_j f_j(\theta) \right| : m \geq k \right\}.$$

On the other hand, it is easily seen that for each $n \geq k$,

$$\sup \left\{ \left| \sum_{j=k}^{n} a_j f_j(\sigma + \theta) \right| : \sigma \in U_k \right\} = \sum_{j=k}^{n} |a_j|.$$

The last two relations show that $(a_n)_{n \in \mathbb{N}}$ is summable, a contradiction. Thus $A$ is not a $c$-coboundary, as was to be shown.

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