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FIELD**

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A theory of vector singular integral operators in the context of the local fields, is established. Applications to maximal functions, a diagonal multiplier theorem of Mihlin-Hörmander type and applications to Besov and Hardy-Sobolev spaces are given.

Introduction. The theory of the vector singular operators with operator valued kernels on Euclidean space was treated systematically by Rubio de Francia, Ruiz and Torrea [6] (see also Garcia-Cuerva and Rubio de Francia [3]). On the other hand, the classical singular integral operators of the Calderón-Zygmund type on finite product of local fields were considered by Phillips and Taibleson [5].

The goal of the present paper is to give a version for local fields of some results of Francia-Ruiz-Torrea [6] that generalize from several perspectives the quoted paper by Phillips-Taibleson.

The contents of the paper is as follows. We begin in §1 some basic notations, definitions and results that we can find in [9]. In §2 we state an inequality of Fefferman-Stein type and, we apply it to obtain an interpolation theorem of Marcinkiewicz-Riviere type. The main results are in §3 where we state the version of the integral singular operator theorem given in [6], for local fields, giving also sequential extensions. Next in §4 we obtain maximal inequalities of F. Zó and Fefferman-Stein type. A diagonal multiplier theorem of Mihlin-Hörmander type (for the Euclidean case see Triebel [11]) that generalize the scalar multiplier theorem of Taibleson [8] is given in §5. Finally, in §6 we give applications of some results obtained in the foregoing sections to Besov and Hardy-Sobolev spaces in local fields.

The extension of all results in this paper for a finite product of local fields will be an immediate consequence of a M. H. Taibleson's theorem (see [10], pp. 548–549) which states that, if \mathbb{K} is a local field and d is an integer greater than 1, then $\mathbb{K}^d e$, the d -dimensional vector space over \mathbb{K} , has a field structure, as a local field, which is compatible with the usual vector space norm of \mathbb{K}^d .

1. Preliminaries. A local field is any locally compact, non-discrete and totally disconnected field. Let \mathbb{K} be a fixed local field and dx a

Haar measure of the additive group \mathbb{K}^+ of \mathbb{K} . The measure of the measurable set A of \mathbb{K} with respect to dx we denote for $|A|$. Let m be the modular function for \mathbb{K}^+ , that is, $M(\lambda)|A| = |\lambda A|$ for $\lambda \in \mathbb{K}$ and A measurable. We also let $|x| = m(x)$. The sets

$$\mathbb{D} = \{x \in \mathbb{K}: |x| \leq 1\} \quad \text{and} \quad \mathbb{B} = \{x \in \mathbb{K}: |x| < 1\}$$

are the ring of integers of \mathbb{K} and the unique maximal ideal of \mathbb{D} , respectively. Let $q = p^c$ (p prime) be the order of the finite field \mathbb{D}/\mathbb{B} and π a fixed element of maximum absolute value of \mathbb{B} . The Haar measure dx is normalized such that $|\mathbb{D}| = 1$ and thus $|\pi| = |\mathbb{B}| = q^{-1}$. We observe that $dx/|x|$ is a Haar measure on the multiplicative group \mathbb{K}^* of \mathbb{K} . We let

$$\mathbb{B}^k = \{x \in \mathbb{K}: |x| \leq q^{-k}\}, \quad k \in \mathbb{Z}.$$

If B and R are two balls of \mathbb{K} such that $B \cap R \neq \emptyset$, then $B \subset R$ or $R \subset B$. For each $k \in \mathbb{Z}$, there is only one sequence $(B_j)_{j \in \mathbb{N}}$ of balls with radius q^k that is a partition of \mathbb{K} . We fix a character χ on \mathbb{K}^+ that is trivial on \mathbb{D} but is non-trivial on $\mathbb{B}^{-1} = \{x \in \mathbb{K}: |x| \leq q\}$. If we take $\chi_y(x) = \chi(x \cdot y)$, then the mapping $y \mapsto \chi_y$ is a topological isomorphism of \mathbb{K} onto the group of characters of \mathbb{K}^+ . The Fourier transform of a function $f \in L^1(\mathbb{K})$ is defined by

$$(1) \quad \hat{f}(x) = \int_{\mathbb{K}} f(y) \bar{\chi}_x(y) dy,$$

and the inverse Fourier transform of a function $f \in L_c^\infty(\mathbb{K})$ is defined by

$$(2) \quad f^\vee(x) = \int_{\mathbb{K}} f(y) \chi_x(y) dy.$$

We denote by $S(\mathbb{K})$ the space of all finite linear combinations of characteristic functions of balls of \mathbb{K} . The space $S(\mathbb{K})$ is an algebra of continuous functions with compact support that is dense in $L^p(\mathbb{K})$, $1 \leq p < \infty$. We observe that the Fourier transform is a homeomorphism of $S(\mathbb{K})$ onto $S(\mathbb{K})$. The space $S'(\mathbb{K})$ of continuous linear functionals on $S(\mathbb{K})$ is called the space of distributions. We will consider $S'(\mathbb{K})$ with the weak topology.

Let E be a Banach space. The space $l^s(E)$ is the set of all sequences $(c_j)_{j \in \mathbb{Z}}$ of elements of E , such that the sequence of its norms is in l^s . The space of the quasi-null sequences of elements of E , i.e. of the sequences (c_j) such that $c_j = 0$ for $|j| \geq N$, for some $N \geq 0$,

will be denoted by $l_0^\infty(E)$. We denote by $S(\mathbb{K}, l_0^\infty)$ the space of the quasi-null sequences of functions of $S(\mathbb{K})$. The space $S(\mathbb{K}, l_0^\infty)$ is dense in the space $L^p(\mathbb{K}, l^s)$ for $1 \leq p, s < \infty$.

The space $l'_s(E)$, for $1 \leq r \leq \infty$ and $s \in \mathbb{R}$, will be the set of all sequences $(x_j)_{j \geq 0}$ of elements of E , such that

$$\|(x_j)_{j \geq 0}\|_{l'_s(E)} = \|(q^{sj} \|x_j\|)_{j \geq 0}\|_{l^r} < \infty.$$

The Hardy-Littlewood maximal function of $f \in L^1_{\text{loc}}(\mathbb{K}, E)$ is defined by

$$(3) \quad Mf(x) = \sup_{k \in \mathbb{Z}} q^k \int_{|y-x| \leq q^{-k}} \|f(y)\|_E dy.$$

The function $Mf(x)$ is measurable,

$$(4) \quad \|f(x)\|_E = \lim_{k \rightarrow \infty} q^k \int_{|y-x| \leq q^{-k}} \|f(y)\|_E dy,$$

and

$$(5) \quad \|f(x)\|_E \leq Mf(x),$$

for almost all $x \in \mathbb{K}$. Moreover, Mf is of the weak type $(1, 1)$ and of the strong type (p, p) , $1 < p \leq \infty$.

For the details see [9].

2. The BMO(E) space.

2.1. DEFINITION. Let $f \in L^1_{\text{loc}}(\mathbb{K}, E)$. The sharp maximal function $M^\#f$ is defined by

$$M^\#f(x) = \sup_{k \in \mathbb{Z}} q^k \int_{|y-x| \leq q^{-k}} \|f(y) - f_k(x)\|_E dy,$$

where

$$f_k(x) = q^k \int_{|y-x| \leq q^{-k}} f(y) dy.$$

2.2. DEFINITION. The space BMO(E) of the functions of bounded mean oscillation is the set of the functions $f \in L^1_{\text{loc}}(\mathbb{K}, E)$ such that

$$(1) \quad \|f\|_* = \|M^\#f\|_\infty < \infty.$$

2.3. REMARKS. (a) The application $f \mapsto \|f\|_*$ is a seminorm on BMO(E) and $\|f\|_* = 0$ if and only if f is constant. We consider the space BMO(E) like a quotient space with respect to constant functions. (b) We can prove that BMO(E) is a Banach space analogously to the real case (see [4]). (c) We have $L^\infty(\mathbb{K}, E) \subset \text{BMO}(E)$,

$L^\infty(\mathbb{K}, E) \neq \text{BMO}(E)$ because the function $f(x) = \log|x|$ if $x \in \mathbb{K}^*$ and $f(0) = 0$ is in $\text{BMO}(E)$ but is not in $L^\infty(\mathbb{K}, E)$.

A classical inequality of Fefferman-Stein also holds in the local field setting.

2.4. THEOREM. *Let $f \in L^1_{\text{loc}}(\mathbb{K}, E)$ such that $Mf \in L^r(\mathbb{K})$ for some r with $0 < r < \infty$. Then for every p with $r \leq p < \infty$, there is a constant C_p depending only on p , such that*

$$(1) \quad \|Mf\|_p \leq C_p \|M^\# f\|_p.$$

The proof of this theorem is an adaptation of the Euclidean case (see [3], Chapter 2, Theorem 3.6). To obtain this adaptation we must remember that the balls of \mathbb{K} have the same properties of the dyadic cubes. We do not need to take dilations of balls, the number 2 that appears in the proof of [3] is the prime number q here, and the functions $\alpha(t)$ and $\beta(t)$ that are considered in [3] are equal in this case.

2.5. REMARK. The inequality 2.4(1) is not true when $p = \infty$ (see 2.3(c)).

As a consequence of the Fefferman-Stein inequality we obtain an interpolation theorem of Marcinkiewicz-Riviere type, which will be fundamental in the study of the singular integrals.

2.6. THEOREM. *Let E and F be Banach spaces and let T be a linear operator from $L^\infty(\mathbb{K}, E)$ into $L^0(\mathbb{K}, F)$ such that, T has a bounded extension from $L^r(\mathbb{K}, E)$ into $L^r(\mathbb{K}, F)$, for some r with $1 < r < \infty$, and*

$$(1) \quad \|Tf\|_* \leq C \|f\|_{L^\infty(E)}, \quad f \in L^\infty_c(\mathbb{K}, E).$$

Then T has a bounded extension from $L^p(\mathbb{K}, E)$ into $L^p(\mathbb{K}, F)$, for all p with $r \leq p < \infty$.

3. Singular integral operators.

3.1. DEFINITION. Let E and F be Banach spaces. A linear operator T defined on $L^\infty_c(\mathbb{K}, E)$, the space of the E -valued L^∞ -functions with compact support, with values in $L^0(\mathbb{K}, F)$, the space of all F -valued strongly measurable functions, is a singular integral operator with an operator valued kernel, if the following two conditions are fulfilled:

SIO 1. T has a bounded extension from $L^r(\mathbb{K}, E)$ into $L^r(\mathbb{K}, F)$, for some r with $1 < r \leq \infty$.

SIO 2. There is an operator valued kernel K , locally integrable from $\mathbb{K} \times \mathbb{K} \setminus \Delta$ into $L(E, F)$, such that

$$(1) \quad Tf(x) = \int_{\mathbb{K}} K(x, y)f(y) dy,$$

for all $f \in L_c^\infty(\mathbb{K}, E)$ and for a.e. $x \notin \text{supp } f$.

3.2. DEFINITION. Let T be a singular integral operator with a kernel K . We say that K satisfies (H_1) if

$$(1) \quad \int_{|x-y'| > |y-y'|} \|K(x, y) - K(x, y')\|_{L(E, F)} dx \leq C$$

for all $y \neq y'$, and we say that K satisfies (H_∞) if

$$(2) \quad \|K(x, y) - K(x, y')\|_{L(E, F)} \leq C \frac{|y - y'|}{|x - y'|^2}$$

for $|x - y'| > |y - y'|$. Moreover, we say that K satisfies (H_r') , for $r = 1$ or $r = \infty$, if $K'(x, y) = K(y, x)$ satisfies (H_r) .

3.3. REMARK. The condition (H_∞) implies the condition (H_1) . In fact, if $|y - y'| = q^l$ and $|x - y'| > |y - y'|$, then

$$\begin{aligned} \int_{|x-y'| > |y-y'|} \|K(x, y) - K(x, y')\|_{L(E, F)} dx &= Cq^l \int_{|z| \geq q^{l+1}} \frac{dz}{|z|^2} \\ &= Cq^l \sum_{k=l+1}^{\infty} \int_{|z|=q^k} \frac{dz}{|z|^2} = Cq^{-1}(1 - q^{-1})(1 - q^{-1})^{-1}. \end{aligned}$$

Analogously, (H_∞') implies (H_1') .

Now we are ready to state the main theorem.

3.4. THEOREM. Let T be a singular integral operator with kernel K , which has a bounded extension from $L^r(\mathbb{K}, F)$, for some r with $q < r \leq \infty$. The following hold:

(i) if K satisfies (H_1) , then T is of weak type $(1, 1)$ and of strong type (p, p) for p with $q < p \leq r$;

(ii) if K satisfies (H_1') , then T is of strong type (L^∞, BMO) and of strong type (p, p) , for p with $r \leq p < \infty$.

The proof of the above theorem is obtained like the Euclidean case (see [3] or [6]). The crucial part uses a decomposition of the Calderón-Zygmund type (see [9], Chapter 3, results 7.6 and 7.9). Thanks to the decomposition it follows that T is of weak type $(1, 1)$. The Marcinkiewicz interpolation theorem then shows that T is of

strong type (p, p) , $1 < p \leq r$. The proof that T is of strong type (L^∞, BMO) is similar to the Euclidean case. Finally, to conclude that T is of strong type (p, p) for $r \leq p < \infty$, we need the Marcinkiewicz-Riviere interpolation Theorem 2.6.

3.5. THEOREM. *Let $(T_j)_{j \in \mathbb{Z}}$ be a sequence of singular integral operators uniformly bounded from $L^r(\mathbb{K}, E)$ into $L^r(\mathbb{K}, F)$, for some r with $1 < r \leq \infty$. Suppose further that the sequence of associated kernels $(K_j)_{j \in \mathbb{Z}}$ satisfies*

$$(1) \quad \int_{|x-y'| > |y-y'|} \sup_j \|K_j(x, y) - K_j(x, y')\|_{L(E, F)} dx \leq C, \quad y \neq y',$$

and

$$(2) \quad \int_{|y-x'| < |x-x'|} \sup_j \|K_j(x, y) - K_j(x', y)\|_{L(E, F)} dy \leq C, \quad x \neq x'.$$

Then, given p and s with $1 \leq p < \infty$ and $1 < s < \infty$, there is a constant $A_{p,s}$ depending only on p, s, C and r , such that

$$(3) \quad \left| \left\{ x : \sum_j \|T_j f_j(x)\|_F^s > \lambda^s \right\} \right| \leq A_{1,s} \lambda^{-1} \|(f_j)_j\|_{L^1(l^s(E))}$$

and

$$(4) \quad \|(T_j f_j)_j\|_{L^p(l^s(F))} \leq A_{p,s} \|(f_j)_j\|_{L^p(l^s(E))}, \quad 1 < p < \infty,$$

for all $\lambda > 0$ and $f = (f_j)_j \in L_c^\infty(\mathbb{K}, l^s(E))$. Moreover, the inequality (4) can be extended for all $f = (f_j)_j \in L^p(\mathbb{K}, l^s(E))$.

Proof. For each positive integer m , let \tilde{T}_m be the operator from $L_c^\infty(\mathbb{K}, l^s(E))$ into $L^0(\mathbb{K}, l^s(F))$ defined by

$$(5) \quad \tilde{T}_m(f_j)_j = (T_j f_j)_{m \leq j \leq m}, \quad (f_j)_j \in L^\infty(\mathbb{K}, l^s(E)),$$

and let \tilde{K}_m be the kernel from $\mathbb{K} \times \mathbb{K} \setminus \Delta$ into $L(l^s(E), l^s(F))$ defined by

$$(6) \quad \tilde{K}_m(x, y)(\alpha_j)_j = (K_j(x, y)\alpha_j)_{-m \leq j \leq m}, \quad (\alpha_j)_j \in l^s(E).$$

We observe that the operators T_j are uniformly bounded from $L^p(\mathbb{K}, E)$ into $L^p(\mathbb{K}, F)$ for all p , $1 < p < \infty$. Now, we fix

$s, 1 < s < \infty$. The operators \tilde{T}_m are uniformly bounded from $L^s(\mathbb{K}, l^s(E))$ into $L^s(\mathbb{K}, l^s(F))$ and it is clear that

$$\tilde{T}_m(f_j)_j(x) = \int_{\mathbb{K}} \tilde{K}_m(x, y)(f_j(y))_j dy$$

for all $(f_j)_j \in L_c^\infty(\mathbb{K}, l^s(E))$ and a.a. $x \notin \text{supp}(f_j)_j$. Since

$$\|\tilde{K}_m(x, y)\|_{L(l^s(E), l^s(F))} \leq \sup_{|j| \leq m} \|K_j(x, y)\|_{L(E, F)},$$

then it follows by (1) and (2) that the kernel \tilde{K}_m verifies (H_1) and (H'_1) . Therefore, by Theorem 3.4, for each p with $1 \leq p < \infty$, there is a constant $A_{p,s}$ depending only on p, s, C and r , such that

$$(7) \quad \left| \left\{ x: \sum_{|j| \leq m} \|T_j f_j(x)\|_F^s > \lambda^s \right\} \right| \leq A_{1,s} \lambda^{-1} \|(f_j)_j\|_{L^1(l^s(E))}$$

and

$$(8) \quad \|\tilde{T}_m(f_j)_j\|_{L^p(l^s(F))} \leq A_{p,s} \|(f_j)_j\|_{L^p(l^s(E))}, \quad 1 < p < \infty,$$

for all $\lambda > 0$ and $f = (f_j)_j \in L_c^\infty(\mathbb{K}, l^s(E))$. Moreover, the inequality (8) can be extended for all $f = (f_j)_j \in L^p(\mathbb{K}, l^s(E))$. Then, letting $m \rightarrow \infty$ on both sides of the inequalities (7) and (8) we obtain (3) and (4).

3.6. COROLLARY. *Let T be a singular integral operator with kernel K satisfying (H_1) and (H'_1) . Then, given p and s with $1 \leq p < \infty$ and $1 < s < \infty$, there is a constant $A_{p,s}$ depending only on p, s, C and r , such that*

$$(1) \quad \left| \left\{ x: \sum_j \|T f_j(x)\|_F^s > \lambda^s \right\} \right| \leq A_{1,s} \lambda^{-1} \|(f_j)_j\|_{L^1(l^s(E))}$$

and

$$(2) \quad \|(T f_j)_j\|_{L^p(l^s(F))} \leq A_{p,s} \|(f_j)_j\|_{L^p(l^s(E))}, \quad 1 < p < \infty,$$

for all $\lambda > 0$ and $f = (f_j)_j \in L_c^\infty(\mathbb{K}, l^s(E))$. Moreover, the inequality (2) can be extended for all $f = (f_j)_j \in L^p(\mathbb{K}^s(E))$.

3.7. REMARK. In our applications we shall consider singular integral operators of convolution type, that is, with kernels of the type $K(x, y) = K'(x - y)$ where K' is locally integrable from $\mathbb{K} \setminus \{0\}$ into $L(E, F)$.

4. Applications to maximal functions.

4.1. DEFINITION. Let $\varphi \in L^1(\mathbb{K})$ and for each $t \in \mathbb{K}^*$, let $\varphi_t(x) = |t|^{-1}\varphi(t^{-1}x)$. The maximal operator M^φ is defined by

$$M^\varphi f(x) = \sup_{t \neq 0} |(\varphi_t * f)(x)|, \quad f \in L_c^\infty(\mathbb{K}).$$

The Euclidean version of the following theorem is due to F. Zó (see [6] or [12]).

4.2. THEOREM. Let $\varphi \in C_c(\mathbb{K})$ such that

$$(1) \quad \int_{|x|>|y|} \sup_{t \neq 0} |\varphi_t(x-y) - \varphi_t(x)| dx \leq C, \quad y \neq 0.$$

Then, given p and s with $1 \leq p < \infty$ and $1 < s < \infty$, there is a constant $A_{p,s}$ depending only on p , s , C and $\|\varphi\|_1$, such that

$$(2) \quad \left| \left\{ x : \sum_j |M^\varphi f_j(x)|^s > \lambda^s \right\} \right| \leq A_{1,s} \lambda^{-1} \|(f_j)_j\|_{L^1(l^s)}$$

and

$$(3) \quad \|(M^\varphi f_j)_j\|_{L^p(l^s)} \leq A_{p,s} \|(f_j)_j\|_{L^p(l^s)}, \quad 1 < p < \infty,$$

for all $\lambda > 0$ and $f = (f_j)_j \in L_c^\infty(\mathbb{K}, l^s)$. Moreover, the inequality (3) can be extended for all $f = (f_j)_j \in L^p(\mathbb{K}, l^s)$.

Proof. Step 1. Owing to continuity of the function $t \mapsto (\varphi_t * f)(x)$, it is enough to calculate the supremum, in the definition of M^φ , on a countable dense subset $\{t_j\}_{j \in \mathbb{N}}$ of \mathbb{K}^* , that is,

$$M^\varphi f(x) = \sup_j |(\varphi_{t_j} * f)(x)|.$$

Consider the operators M_m^φ defined by

$$M_m^\varphi f(x) = \sup_{1 \leq j \leq m} |(\varphi_{t_j} * f)(x)|.$$

We have that $M_m^\varphi f(x) \uparrow M^\varphi f(x)$ for all $x \in \mathbb{K}$. Therefore, obtaining estimates for $M_m^\varphi f$ that do not depend on m , we shall be obtaining also estimates for $M^\varphi f$.

Step 2. For each positive integer m , let T_m be the linear operator from $L_c^\infty(\mathbb{K})$ into $L^0(\mathbb{K}, l^\infty)$ defined by

$$(4) \quad T_m f = (\varphi_{t_j} * f)_{1 \leq j \leq m}, \quad f \in L_c^\infty(\mathbb{K}),$$

and let K_m be the kernel (of convolution type) from \mathbb{K} into $L(\mathbb{C}, l^\infty)$ defined by

$$(5) \quad K_m(x)\lambda = (\varphi_{t_j}(x)\lambda)_{1 \leq j \leq m}, \quad \lambda \in \mathbb{C}.$$

Since $\|\varphi_t\|_1 = \|\varphi\|_1$ for all $t \neq 0$, we have

$$(6) \quad \begin{aligned} \|T_m f\|_{L^\infty(l^\infty)} &= \operatorname{ess\,sup}_{x \in \mathbb{K}} \sup_{1 \leq j \leq m} |(\varphi_{t_j} * f)(x)| \\ &\leq \operatorname{ess\,sup}_{x \in \mathbb{K}} \sup_{1 \leq j \leq m} \|f\|_\infty \|\varphi_{t_j}\|_1 = \|\varphi\|_1 \|f\|_\infty, \end{aligned}$$

i.e., the operator T_m is bounded from $L^\infty(\mathbb{K})$ into $L^\infty(l^\infty)$. On the other hand, we have

$$\begin{aligned} \int_{\mathbb{K}} \|K_m(x)\|_{L(\mathbb{C}, l^\infty)} dx &= \int_{\mathbb{K}} \sup_{1 \leq j \leq m} |\varphi_{t_j}(x)| dx \\ &\leq \sum_{1 \leq j \leq m} \int_{\mathbb{K}} |\varphi_{t_j}(x)| dx = m \|\varphi\|_1 < \infty, \end{aligned}$$

and

$$\begin{aligned} T_m f(x) &= \left(\int_{\mathbb{K}} \varphi_{t_j}(x-y) f(y) dy \right)_{1 \leq j \leq m} \\ &= \int_{\mathbb{K}} (\varphi_{t_j}(x-y) f(y))_{1 \leq j \leq m} dy = \int_{\mathbb{K}} K_m(x-y) f(y) dy, \end{aligned}$$

for all $f \in L_c^\infty(\mathbb{K})$ and for a.e. $x \notin \operatorname{supp} f$. Consequently T_m is a singular integral operator of convolution type with kernel K_m . Moreover, the kernel K_m satisfies, for all $y \neq 0$,

$$(7) \quad \begin{aligned} \int_{|x| > |y|} \|K_m(x-y) - K_m(x)\|_{L(\mathbb{C}, l^\infty)} dx \\ &= \int_{|x| < |y|} \sup_{1 \leq j \leq m} |\varphi_{t_j}(x-y) - \varphi_{t_j}(x)| dx \\ &\leq \int_{|x| > |y|} \sup_{t \neq 0} |\varphi_t(x-y) - \varphi_t(x)| dx \leq C. \end{aligned}$$

Step 3. The inequalities (6) and (7) show that the operators T_m and its kernels K_m satisfy uniformly the hypothesis of the Corollary 3.6. Therefore, given p and s with $1 \leq p < \infty$ and $1 < s < \infty$, there is a constant $A_{p,s}$, depending only on p , s , C and $\|\varphi\|_1$, such that

$$(8) \quad \left| \left\{ x : \sum_j \|T_m f_j(x)\|_{l^\infty}^s > \lambda^s \right\} \right| \leq A_{1,s} \lambda^{-1} \|(f_j)_j\|_{L^1(l^s)}$$

and

$$(9) \quad \|(T_m f_j)_j\|_{L^p(l^s(l^\infty))} \leq A_{p,s} \|(f_j)_j\|_{L^p(l^s)}, \quad 1 < p < \infty,$$

for all $\lambda > 0$, $m \in \mathbb{N}$ and $f = (f_j)_j \in L_c^\infty(\mathbb{K}, l^s)$. Moreover, the inequality (9) can be extended for all $f = (f_j)_j \in L^p(\mathbb{K}, l^s)$. Since

$$\|T_m f_j(x)\|_{l^\infty} = M_m^\varphi f_j(x),$$

then, letting $m \rightarrow \infty$ on both sides of (8) and (9), we obtain (2) and (3).

From 4.2 we obtain the maximal theorem of Fefferman-Stein (see [2] or [6]) in the context of the local fields.

4.3. THEOREM. *Given p and s with $1 \leq p < \infty$ and $1 < s < \infty$, there is a constant $A_{p,s}$ depending only on p and s , such that*

$$(1) \quad \left| \left\{ x: \sum_j |M f_j(x)|^s > \lambda^s \right\} \right| \leq A_{1,s} \lambda^{-1} \|(f_j)_j\|_{L^1(l^s)}$$

and

$$(2) \quad \|(M f_j)_j\|_{L^p(l^s)} \leq A_{p,s} \|(f_j)_j\|_{L^p(l^s)}, \quad 1 < p < \infty,$$

for all $\lambda > 0$ and $f = (f_j)_j \in L_c^\infty(\mathbb{K}, l^s)$. Moreover, the inequality (2) can be extended for all $f = (f_j)_j \in L^p(\mathbb{K}, l^s)$.

Proof. Let φ be the characteristic function of the ball \mathbb{B}^0 . If $|x| > |y|$, then $|t^{-1}(x - y)| = |t^{-1}x|$ and hence $\varphi(t^{-1}(x - y)) = \varphi(t^{-1}x)$. Therefore

$$|\varphi_t(x - y) - \varphi_t(x)| = |t|^{-1} |\varphi(t^{-1}(x - y)) - \varphi(t^{-1}x)| = 0$$

and consequently

$$(3) \quad \int_{|x| > |y|} \sup_{t \neq 0} |\varphi_t(x - y) - \varphi_t(x)| dx = 0.$$

On the other hand, we have

$$\begin{aligned} (|f| * \varphi_t)(x) &= \int_{\mathbb{K}} |f(x - y)| \varphi_t(y) dy \\ &= |t|^{-1} \int_{\mathbb{K}} |f(x - y)| \varphi(t^{-1}y) dy \\ &= |t|^{-1} \int_{|y| \leq |t|} |f(x - y)| dy \\ &= |t|^{-1} \int_{|y-x| \leq |t|} |f(y)| dy \end{aligned}$$

and hence

$$\begin{aligned}
 (4) \quad M^\varphi |f|(x) &= \sup_{t \neq 0} (|f| * \varphi_t)(x) \\
 &= \sup_{t \neq 0} |t|^{-1} \int_{|y-x| \leq |t|} |f(y)| dy \\
 &= \sup_{k \in \mathbb{Z}} q^k \int_{|y-x| \leq q^{-k}} |f(y)| dy = Mf(x).
 \end{aligned}$$

From (3) it follows that the maximal operator M^φ satisfies the inequalities 4.2(2) and 4.2(3). Then, by (4) we obtain the inequalities (1) and (2) for the Hardy-Littlewood maximal operator.

5. A multiplier theorem on $L^p(\mathbb{K}, l^s)$ -spaces.

5.1. **LEMMA.** *Let $g \in L^2(\mathbb{K})$ and $\alpha > 0$. Then, there is a constant A_α depending only on α , such that*

$$\begin{aligned}
 (1) \quad q^{-\alpha} \int_{\mathbb{K}} |x|^\alpha |\hat{g}(x)|^2 dx \\
 \leq A_\alpha \iint_{\mathbb{K} \times \mathbb{K}} |g(x+y) - g(x)|^2 |y|^{-(1+\alpha)} dx dy.
 \end{aligned}$$

Proof. See [9], page 220.

5.2. **LEMMA.** *Let $(g_j)_{j \in \mathbb{Z}}$ be a sequence of elements of $L^2(\mathbb{K})$ and suppose that there are $B > 0$ and $\varepsilon > 0$, such that*

$$(1) \quad \iint_{\mathbb{K} \times \mathbb{K}} \sum_{j=-\infty}^{+\infty} |g_j(x+y) - g_j(x)|^2 |y|^{-(2+\varepsilon)} dx dy \leq B^2.$$

Then, there is a constant A_ε depending only on ε , such that, for all $k \in \mathbb{Z}$,

$$(2) \quad \int_{|x| \geq q^k} \sup_j |\hat{g}_j(x)| dx \leq A_\varepsilon B q^{-k\varepsilon/2}.$$

Proof. It follows from Hölder's Inequality that

$$\begin{aligned}
 &\int_{|x| \geq q^k} \sup_j |\hat{g}_j(x)| dx \\
 &\leq \left(\int_{\mathbb{K}} |x|^{(1+\varepsilon)} \sup_j |\hat{g}_j(x)|^2 dx \right)^{1/2} \left(\int_{|x| \geq q^k} |x|^{-(1+\varepsilon)} dx \right)^{1/2} \\
 &= \left(\int_{\mathbb{K}} |x|^{(1+\varepsilon)} \sup_j |\hat{g}_j(x)|^2 dx \right)^{1/2} \left(\frac{1 - q^{-1}}{1 - q^{-\varepsilon}} \right)^{1/2} q^{-k\varepsilon/2}.
 \end{aligned}$$

Now, setting $\alpha = 1 + \varepsilon$ and applying Lemma 5.1, we obtain

$$\begin{aligned} q^{-\alpha j} \int_{\mathbb{K}} |x|^\alpha \sup_j |\hat{g}_j(x)|^2 dx \\ \leq A_\alpha \iint_{\mathbb{K} \times \mathbb{K}} \sum_{j=-\infty}^{+\infty} |g_j(x+y) - g_j(x)|^2 |y|^{-(2+\varepsilon)} dx dy \leq A_\alpha B^2 \end{aligned}$$

and consequently

$$\begin{aligned} \int_{|x| \geq q^k} \sup_j |\hat{g}_j(x)| dx &\leq (A_\alpha B^2 q^\alpha)^{1/2} \left(\frac{1 - q^{-1}}{1 - q^{-\varepsilon}} \right)^{1/2} q^{-k\varepsilon/2} \\ &= A_\varepsilon B q^{-k\varepsilon/2}. \end{aligned}$$

5.3. THEOREM. *Let $(m_j)_{j \in \mathbb{Z}} \in L^\infty(\mathbb{K}, l^2)$ and suppose that there are $B > 0$ and $\varepsilon > 0$, such that, for all $j \in \mathbb{Z}$,*

$$(1) \quad \int_{|y| < q^j} \int_{|x|=q^j} \sum_{i=-\infty}^{+\infty} |m_i(x+y) - m_i(x)|^2 |y|^{-(2+\varepsilon)} dx dy \leq B^2 q^{-\varepsilon j}.$$

Then, for all $(\varphi_j)_j \in S(\mathbb{K}, l_0^\infty)$ and $1 < p, s < \infty$, we have

$$(2) \quad \|((m_j \hat{\varphi}_j)^\vee)_j\|_{L^p(l^s)} \leq C \|(\varphi_j)_j\|_{L^p(l^s)},$$

where C is independent of $(\varphi_j)_j$.

Proof. Step 1. Let ϕ_k be the characteristic function of the ball \mathbb{B}^k and $m_j^k = m_j \phi_k$, $k \in \mathbb{Z}$. Since $(\varphi_j)_j \in S(\mathbb{K}, l_0^\infty)$ has compact support we see that $((m_j^k \hat{\varphi}_j)^\vee)_j = ((m_j \hat{\varphi}_j)^\vee)_j$ for k small enough. Hence, if we wish to show (2), we only need to show that, for all $(\varphi_j)_j \in S(\mathbb{K}, l_0^\infty)$, $k \in \mathbb{Z}$ and $1 < p, s < \infty$, we have

$$(3) \quad \|((m_j^k \hat{\varphi}_j)^\vee)_j\|_{L^p(l^s)} \leq C \|(\varphi_j)_j\|_{L^p(l^s)},$$

where the constant C is independent of k and $(\varphi_j)_j$.

Step 2. For each $k, j \in \mathbb{Z}$, let T_j^k be the linear operator defined by

$$(4) \quad T_j^k \varphi = (m_j^k \hat{\varphi})^\vee = (m_j^k)^\vee * \varphi, \quad \varphi \in S(\mathbb{K}).$$

For all $k, j \in \mathbb{Z}$ and $\varphi \in S(\mathbb{K})$ we have

$$\begin{aligned} (5) \quad \|T_j^k \varphi\|_2 &= \|(m_j^k \hat{\varphi})^\vee\|_2 = \|m_j^k \hat{\varphi}\|_2 \\ &\leq \|m_j^k\|_\infty \|\hat{\varphi}\|_2 \leq \|(m_j)_j\|_{L^\infty(l^2)} \|\varphi\|_2. \end{aligned}$$

Therefore $(T_j^k)_{j \in \mathbb{Z}}$ is a sequence of singular integral operators of convolution type uniformly bounded from $L^2(\mathbb{K})$ into $L^2(\mathbb{K})$, with sequence of associated kernels $((m_j^k)^\vee)_{j \in \mathbb{Z}}$.

Step 3. Let $m_{jl} = m_j^{-l} - m_j^{1-l}$ for $j, l \in \mathbb{Z}$. It follows from (1) that

$$\begin{aligned}
 (6) \quad & \int_{|y| < q'} \int_{|x|=q'} \sum_{j=-\infty}^{+\infty} |m_{jl}(x+y) - m_{jl}(x)|^2 |y|^{-(2+\varepsilon)} dx dy \\
 &= \int_{|y| < q'} \int_{|x|=q'} \sum_{j=-\infty}^{+\infty} |m_j(x+y) - m_j(x)|^2 |y|^{-(2+\varepsilon)} dx dy \\
 &\leq B^2 q^{-\varepsilon l}.
 \end{aligned}$$

We have also

$$\begin{aligned}
 (7) \quad & \int_{|y| \geq q'} \int_{|x|=q'} \sum_{j=-\infty}^{+\infty} |m_{jl}(x+y) - m_{jl}(x)|^2 |y|^{-(2+\varepsilon)} dx dy \\
 &\leq \int_{|y| \geq q'} \int_{|x|=q'} 2 \sum_{j=-\infty}^{+\infty} (|m_{jl}(x+y)|^2 + |m_{jl}(x)|^2) |y|^{-(2+\varepsilon)} dx dy \\
 &\leq 4 \|(m_j)_j\|_{L^\infty(\mathbb{K}^2)}^2 (1 - q^{-1})^2 q^l \left(\frac{q^{-(1+\varepsilon)l}}{1 - q^{-(1+\varepsilon)}} \right) = C_1 q^{-l\varepsilon};
 \end{aligned}$$

$$\begin{aligned}
 (8) \quad & \int_{|y|=q'} \int_{|x| < q'} \sum_{j=-\infty}^{+\infty} |m_{jl}(x+y) - m_{jl}(x)|^2 |y|^{-(2+\varepsilon)} dx dy \\
 &= q^{-(2+\varepsilon)l} \int_{|y|=q'} \int_{|x| < q'} \sum_{j=-\infty}^{+\infty} |m_j(x+y)|^2 dx dy \\
 &\leq \|(m_j)_j\|_{L^\infty(\mathbb{K}^2)}^2 (1 - q^{-1}) q^{-1} q^{-\varepsilon l} = C_2 q^{-\varepsilon l};
 \end{aligned}$$

$$\begin{aligned}
 (9) \quad & \iint_{|x|=|y| > q'} \sum_{j=-\infty}^{+\infty} |m_{jl}(x+y) - m_{jl}(x)|^2 |y|^{-2+\varepsilon} dx dy \\
 &\leq \iint_{|x|=|y| > q'} \sum_{j=-\infty}^{+\infty} |m_{jl}(x+y)|^2 |y|^{-2+\varepsilon} dx dy \\
 &\leq \|(m_j)_j\|_{L^\infty(\mathbb{K}^2)}^2 (1 - q^{-1})^2 q^{-\varepsilon l} (q^{-\varepsilon} / 1 - q^{-\varepsilon}) = C_3 q^{-\varepsilon l}.
 \end{aligned}$$

Therefore from (6), (7), (8) and (9) we obtain

$$(10) \quad \iint_{\mathbb{K} \times \mathbb{K}} \sum_{j=-\infty}^{+\infty} |m_{jl}(x+y) - m_{jl}(x)|^2 |y|^{-(2+\varepsilon)} dx dy \leq C^2 q^{-\varepsilon l},$$

for all $l \in \mathbb{Z}$, where the constant C depends only on $\|(m_j)_j\|_{L^\infty(l^2)}$, B and ε . Then, it follows by Lemma 5.2 that, for all $k \in \mathbb{Z}$,

$$(11) \quad \int_{|x| \geq q^k} \sup_j |(m_{jl})^\vee(x)| dx = \int_{|x| \geq q^k} \sup_j |(m_{jl})^\wedge(x)| dx \leq A_\varepsilon C q^{-(l+k)/2}.$$

Since $m_{jl}\phi_{-1} = m_{jl}$, the $(m_{jl})^\vee(x+y) = (m_{jl})^\vee(x)$ for all $x, y \in \mathbb{K}$ with $|y| \leq q^{-l}$ (see [9], page 126). Therefore, for all $t, j, k \in \mathbb{Z}$ and $x, y \in \mathbb{K}$ with $|y| \leq q^t$, we have

$$|(m_j^k)^\vee(x+y) - (m_j^k)^\vee(x)| \leq \sum_{l=-t+1}^{\infty} |(m_{jl})^\vee(x+y) - (m_{jl})^\vee(x)|.$$

Hence we obtain by (11) that, for all $t, k \in \mathbb{Z}$,

$$(12) \quad \int_{|x| > q^t} \sup_j |(m_j^k)^\vee(x+y) - (m_j^k)^\vee(x)| dx \leq 2 \sum_{l=-t+1}^{\infty} \int_{|x| > q^t} \sup_j |(m_{jl})^\vee(x)| dx \leq 2A_\varepsilon C (q^{-\varepsilon/2}/1 - q^{-\varepsilon/2}) = C',$$

and consequently for all $k \in \mathbb{Z}$, we have

$$\begin{aligned} & \sup_{y \neq 0} \int_{|x| > |y|} \sup_j |(m_j^k)^\vee(x-y) - (m_j^k)^\vee(x)| dx \\ &= \sup_{t \in \mathbb{Z}} \sup_{|y| \leq q^t} \int_{|x| > q^t} \sup_j |(m_j^k)^\vee(x+y) - (m_j^k)^\vee(x)| dx \leq C'. \end{aligned}$$

Therefore, the sequences of kernels of convolution type $((m_j^k)^\vee)_{j \in \mathbb{Z}}$ satisfy uniformly 3.5(1) and 3.5(2). Consequently we obtain (3), which proves the theorem.

6. Applications to Besov and Hardy-Sobolev spaces. In this section we will give some applications of some foregoing results to Besov and Hardy-Sobolev spaces and to spaces of Bessel potentials.

6.1 Let $A^j = \mathbb{B}^j - \mathbb{B}^{j+1} = \{x \in \mathbb{K} : |x| = q^{-j}\}$ for $j \in \mathbb{Z}$. We will consider the sequence $(\Phi_j)_{j \geq 0}$ of elements of $S(\mathbb{K})$, where $\widehat{\Phi}_j$ is the

characteristic function of A^{-j} for $j \geq 1$, and $\widehat{\Phi}_0$ is the characteristic function of \mathbb{D} .

For each distribution $f \in S'(\mathbb{K})$ and $j \geq 0$ we have that $\Phi_j * f$ is a function (see [9], p. 126). We can easily see that the function Φ_j satisfies:

- (1) $\Phi_j * \Phi_j = \Phi_j$ and $\Phi_j * \Phi_i = 0$ for $i \neq j$;
- (2) $\widehat{\Phi}_j(x + y) = \widehat{\Phi}_j(x)$ for $|x| > |y|$;
- (3)
$$\sum_{j=0}^{\infty} \widehat{\Phi}_j = 1.$$

6.2. DEFINITIONS. Let $s \in \mathbb{R}$ and $1 < p < \infty$. For $1 \leq r \leq \infty$, the distribution $f \in S'(\mathbb{K})$ is in $B_{pr}^s(\mathbb{K})$ if

$$\|f\|_{B_{pr}^s} = \|(\Phi_j * f)_{j \geq 0}\|_{l'_s(L^p)} < \infty.$$

For $1 < r < \infty$, the distribution $f \in S'(\mathbb{K})$ is in $F_{pr}^s(\mathbb{K})$ if

$$\|f\|_{F_{pr}^s} = \|(\Phi_j * f)_{j \geq 0}\|_{L^p(l'_s)} < \infty.$$

6.3. REMARK. The sequence $(\Phi_j)_{j \geq 0}$ used in Definition 6.2 and given as in 6.1 is unique. In fact, if $(\psi_j)_{j \geq 0}$ is a sequence of elements of $S(\mathbb{K})$ such that $\text{supp } \psi_j \subset A^{-j}$ for $j \geq 1$, $\text{supp } \psi_0 \subset \mathbb{D}$ and $\sum_j \psi_j = 1$, then ψ_j is the characteristic function of A^{-j} for $j \geq 1$, and ψ_0 is the characteristic function of \mathbb{D} , that is, $\psi_j = \Phi_j$ for $j \geq 0$.

6.4. REMARK. As in the Euclidean case, there is another way to define the spaces $B_{pr}^s(\mathbb{K})$ and $F_{pr}^s(\mathbb{K})$ (see [11]). We can say that the distribution f is in $B_{pr}^s(\mathbb{K})$ ($F_{pr}^s(\mathbb{K})$, respectively) if there is a sequence $(a_j)_{j \geq 0}$ of elements of $S'(\mathbb{K})$ such that $\sum_j a_j$ converges in $S'(\mathbb{K})$ to f , $\text{supp } \hat{a}_j \subset A^{-j}$ for $j \geq 1$, $\text{supp } \hat{a}_0 \subset \mathbb{D}$ and

$$\|(a_j)_{j \geq 0}\|_{l'_s(L^p)} < \infty \quad (\|(a_j)_{j \geq 0}\|_{L^p(l'_s)} < \infty, \text{ respectively}).$$

But this definition is trivial because there is only one sequence $(a_j)_{j \geq 0}$ for each f , namely, the sequence $(\Phi_j * f)_{j \geq 0}$. In fact,

$$(\Phi_j * f)^\wedge = \widehat{\Phi}_j \hat{f} = \widehat{\Phi}_j \sum_{k=0}^{\infty} \hat{a}_k = \sum_{k=0}^{\infty} \widehat{\Phi}_j \hat{a}_k = \widehat{\Phi}_j \hat{a}_j = \hat{a}_j,$$

and hence $a_j = \Phi_j * f$ for $j \geq 0$.

If $s \in \mathbb{R}$ and $f \in S'(\mathbb{K})$, the Bessel potential of order s of f is defined by

$$(J^s f)^\wedge = (\max\{1, |x|\})^s \hat{f}.$$

For $\alpha, \beta \in \mathbb{R}$, the map $f \mapsto J^\alpha f$ is a homeomorphism from $S'(\mathbb{K})$ onto $S'(\mathbb{K})$, $(J^\alpha)^{-1} = J^{-\alpha}$ and $J^{\alpha+\beta} f = J^\alpha(j^\beta f)$ for $f \in S'(\mathbb{K})$ (see [9], p. 137).

The next theorem shows that J^s is an isometry on F_{pr}^t and B_{pr}^t .

6.5. THEOREM. *Let $s, t \in \mathbb{R}$ and $1 < p < \infty$. Then*

$$(1) \quad \|J^s f\|_{F_{pr}^{t-s}} = \|f\|_{F_{pr}^t}, \quad f \in F_{pr}^t(\mathbb{K}), \quad 1 < r < \infty;$$

$$(2) \quad \|J^s f\|_{B_{pr}^{t-s}} = \|f\|_{B_{pr}^t}, \quad f \in B_{pr}^t(\mathbb{K}), \quad 1 \leq r \leq \infty.$$

Proof. We can easily verify that $J^s \Phi_j = q^{sj} \Phi_j$ for $j \geq 0$. Then, for $j \geq 0$, $s \in \mathbb{R}$ and $f \in S'(\mathbb{K})$ we have

$$(3) \quad J^s(\Phi_j * f) = (J^s \Phi_j) * f = q^{sj}(\Phi_j * f).$$

For $f \in F_{pr}^t(\mathbb{K})$ and $1 < r < \infty$, it follows from (3) that

$$\begin{aligned} \|J^s f\|_{F_{pr}^{t-s}} &= \|(q^{sj} \{\Phi_j * f\})_{j \geq 0}\|_{L^p(l_{-s}^p)} \\ &= \|(q^{tj} \{\Phi_j * f\})_{j \geq 0}\|_{L^p(l^p)} \\ &= \|f\|_{F_{pr}^t}. \end{aligned}$$

Now, for $f \in B_{pr}^t(\mathbb{K})$ and $1 \leq r \leq \infty$, it also follows from (3) that

$$\begin{aligned} \|J^s f\|_{B_{pr}^{t-s}} &= \|(q^{sj} \{\Phi_j * f\})_{j \geq 0}\|_{l_{-s}^p(L^p)} \\ &= \|(q^{tj} \{\Phi_j * f\})_{j \geq 0}\|_{l^p(L^p)} \\ &= \|f\|_{B_{pr}^t}. \end{aligned}$$

Now, we will give a theorem of the Littlewood-Paley type. It is a variant of Taibleson's theorem (see [9], pp. 200 and 202), but our proof makes use of vector singular integral operators.

6.6. THEOREM. *For each $1 < p < \infty$, there are constants A_p and B_p , depending only on p , such that, for all $f \in L^p(\mathbb{K})$ we have*

$$(1) \quad A_p \|f\|_p \leq \|(\Phi_j * f)_{j \geq 0}\|_{L^p(l^2)} \leq B_p \|f\|_p.$$

Proof (Sketch). Let us consider the operator T from $L_c^\infty(\mathbb{K})$ into $L^0(\mathbb{K}, l^2)$ defined by

$$(2) \quad Tf = (\Phi_j * f)_{j \geq 0},$$

and S from $L_c^\infty(\mathbb{K}, l^2)$ into $L^0(\mathbb{K})$ defined by

$$(3) \quad S(\alpha_j)_{j \geq 0} = \sum_{j=0}^{\infty} \Phi_j * \alpha_j.$$

We can show that

$$\|Tf\|_{L^2(l^2)} = \|f\|_2$$

and

$$\|S(\alpha_j)_{j \geq 0}\|_2 \leq \|(\alpha_j)_{j \geq 0}\|_{L^2(l^2)}.$$

Therefore we can conclude that T has a bounded extension from $L^2(\mathbb{K})$ into $L^2(\mathbb{K}, l^2)$ and S has a bounded extension from $L^2(\mathbb{K}, l^2)$ into $L^2(\mathbb{K})$.

Let K_1 and K_2 be the kernels defined by

$$(4) \quad K_1(x)\lambda = (\Phi_j(x)\lambda)_{j \geq 0}, \quad x \in \mathbb{K}, \quad \lambda \in \mathbb{C};$$

$$(5) \quad K_2(x)(\lambda_j)_{j \geq 0} = \sum_{j=0}^{\infty} \Phi_j(x)\lambda_j, \quad x \in \mathbb{K}, \quad (\lambda_j)_{j \geq 0} \in l^2.$$

We have that

$$\|K_2(x)\|_{L(l^2, \mathbb{C})} \leq \|K_1(x)\|_{L(\mathbb{C}, l^2)} = \|(\Phi_j(x))_{j \geq 0}\|_{l^2},$$

therefore, showing that $x \mapsto \|(\Phi_j(x))_{j \geq 0}\|_{l^2}$ is locally integrable we can conclude that K_1 and K_2 are locally integrable. Since

$$\|K_1(x-y) - K_1(x)\|_{L(\mathbb{C}, l^2)} = \|K_2(x-y) - K_2(x)\|_{L(l^2, \mathbb{C})} = 0$$

for $|x| > |y|$, we have that K_1 and K_2 satisfy the conditions (H_1) and (H'_1) of Theorem 3.4. We can easily verify that

$$Tf(x) = \int_{\mathbb{K}} K_1(x-y)f(y) dy$$

and

$$S\alpha(x) = \int_{\mathbb{K}} K_2(x-y)\alpha(y) dy,$$

for all $x \in \mathbb{K}$, $f \in L_c^\infty(\mathbb{K})$ and $\alpha \in L_c^\infty(\mathbb{K}, l^2)$. Then, it follows from 3.4 that T and S are singular integral operators of the strong type (p, p) for $1 < p < \infty$, and consequently we have the inequalities 6.6(1).

In Taibleson [9] the space of Bessel potentials $L_s^p(\mathbb{K})$ is defined for $s \in \mathbb{R}$ and $1 \leq p < \infty$, as the set of all distributions $f \in S'(\mathbb{K})$ such that

$$\|f\|_{L_s^p} = \|J^s f\|_p < \infty.$$

The next theorem is a consequence of Theorem 6.6.

6.7. THEOREM. *If $s \in \mathbb{R}$ and $1 < p < \infty$, then the spaces $L^p_s(\mathbb{K})$ and $F^s_{p_2}(\mathbb{K})$ are isomorphic.*

Proof. If $f \in S'(\mathbb{K})$, it follows from 6.6(1) and 6.5(1) that

$$\|f\|_{L^p_s} = \|J^s f\|_p \approx \|J^s f\|_{F^0_{p_2}} = \|f\|_{F^s_{p_2}}.$$

6.8. To close this section we will show that $B^s_{pr}(\mathbb{K})$ ($F^s_{pr}(\mathbb{K})$, respectively) is a retract of $l^r_s(L^p(\mathbb{K}))$ ($L^p(\mathbb{K}, l^r_s)$, respectively). Let us consider mappings \mathcal{S} and \mathcal{P} given as follows. The mapping \mathcal{S} is defined on the elements of $S'(\mathbb{K})$ by

$$(1) \quad \mathcal{S}f = (\Phi_j * f)_{j \geq 0}.$$

The mapping \mathcal{P} is defined for sequences $\alpha = (\alpha_j)_{j \geq 0}$ of elements of $S'(\mathbb{K})$ by

$$(2) \quad \mathcal{P}\alpha = \sum_{j=0}^{\infty} \Phi_j * \alpha_j,$$

where the convergence is considered in $S'(\mathbb{K})$. We are not saying that \mathcal{P} is defined on all sequences $\alpha = (\alpha_j)_{j \geq 0}$ of elements of $S'(\mathbb{K})$, but only on those sequences for which the series defining $\mathcal{P}\alpha$ converge in $S'(\mathbb{K})$. It follows from the property 6.1(1) that $\mathcal{P}\mathcal{S}f = f$ for all $f \in B^s_{pr}(\mathbb{K}) \cup F^s_{pr}(\mathbb{K})$.

6.9. THEOREM. *The space $B^s_{pr}(\mathbb{K})$ is a retract of $l^r_s(L^p(\mathbb{K}))$ and $F^s_{pr}(\mathbb{K})$ is a retract of $L^p(\mathbb{K}, l^r_s)$, for $s \in \mathbb{R}$ and $1 < p, r < \infty$.*

Proof. First we note that

$$\|f\|_{B^s_{pr}} = \|\mathcal{S}f\|_{l^r_s(L^p)} \quad \text{and} \quad \|f\|_{F^s_{pr}} = \|\mathcal{S}f\|_{L^p(l^r_s)}.$$

Since $\widehat{\Phi}_j(x+y) = \widehat{\Phi}_j(x)$ for $|x| > |y|$, it follows that $\{\widehat{\Phi}_j : j \geq 0\}$ is a family of scalar multipliers uniformly bounded on $L^p(\mathbb{K})$, $1 < p < \infty$ (see [9], p. 218). Thus, using properties of the functions Φ_j we obtain for $\alpha = (\alpha_j)_{j \geq 0} \in S(\mathbb{K}, l^{\infty}_0)$,

$$\begin{aligned} \|\mathcal{P}\alpha\|_{B^s_{pr}} &= \|(\Phi_j * \mathcal{P}\alpha)_{j \geq 0}\|_{l^r_s(L^p)} \\ &= \|(\Phi_j * \alpha_j)_{j \geq 0}\|_{l^r_s(L^p)} \\ &= \|(q^{sj} \|\Phi_j * \alpha_j\|_p)_{j \geq 0}\|_{l^r} \\ &\leq C \|(q^{sj} \|\alpha_j\|_p)_{j \geq 0}\|_{l^r} = C \|\alpha\|_{l^r_s(L^p)}. \end{aligned}$$

On the other hand, since $\widehat{\Phi}_j(x+y) = \widehat{\Phi}_j(x)$ for $|x| > |y|$, it follows from 5.3 that $(\widehat{\Phi}_j)_{j \geq 0}$ is a multiplier on $L^p(\mathbb{K}, l^r)$, $1 < p, r < \infty$. Consequently, by the properties of the function Φ_j we have for $\alpha = (\alpha_j)_{j \geq 0} \in S(\mathbb{K}, I_0^\infty)$,

$$\begin{aligned} \|\mathcal{P}\alpha\|_{F_{pr}^s} &= \|(\Phi_j * \mathcal{P}\alpha)_{j \geq 0}\|_{L^p(l_s^r)} \\ &= \|(\Phi_j * \alpha_j)_{j \geq 0}\|_{L^p(l_s^r)} \\ &= \|(\Phi_j * \{q^{sj}\alpha_j\})_{j \geq 0}\|_{L^p(l_s^r)} \\ &\leq C\|(q^{sj}\alpha_j)_{j \geq 0}\|_{L^p(l_s^r)} = C\|\alpha\|_{L^p(l_s^r)}. \end{aligned}$$

Hence, \mathcal{S} is bounded from $B_{pr}^s(\mathbb{K})$ into $L_s^r(L^p(\mathbb{K}))$ and from $F_{pr}^s(\mathbb{K})$ into $L^p(\mathbb{K}, l_s^r)$, and \mathcal{P} is bounded from $l_s^r(L^p(\mathbb{K}))$ into $B_{pr}^w(\mathbb{K})$ and from $L^p(\mathbb{K}, l_s^r)$ into $F_{pr}^s(\mathbb{K})$, for $s \in \mathbb{R}$ and $1 < p, r < \infty$.

6.10. REMARK. Due to Theorem 6.9 it is possible to obtain interpolation theorems for the spaces $L_s^p(\mathbb{K})$, $B_{pr}^s(\mathbb{K})$ and $F_{pr}^s(\mathbb{K})$ as in the Euclidean case. For instance, we have (see [1], p. 153) that

$$(L_{s_0}^p(\mathbb{K}), L_{s_1}^p(\mathbb{K}))_{\theta, r} = B_{pr}^s(\mathbb{K}),$$

where $s = (1-\theta)s_0 + \theta s_1$, $0 < \theta < 1$, $s_0 \neq s_1$, $1 < p < \infty$, $1 \leq r \leq \infty$.

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