VECTOR SINGULAR INTEGRAL OPERATORS ON A LOCAL FIELD

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A theory of vector singular integral operators in the context of the local fields, is established. Applications to maximal functions, a diagonal multiplier theorem of Mihlin-Hörmander type and applications to Besov and Hardy-Sobolev spaces are given.

Introduction. The theory of the vector singular operators with operator valued kernels on Euclidean space was treated systematically by Rubio de Francia, Ruiz and Torrea [6] (see also Garcia-Cuerva and Rubio de Francia [3]). On the other hand, the classical singular integral operators of the Calderón-Zygmund type on finite product of local fields were considered by Phillips and Taibleson [5].

The goal of the present paper is to give a version for local fields of some results of Francia-Ruiz-Torrea [6] that generalize from several perspectives the quoted paper by Phillips-Taibleson.

The contents of the paper is as follows. We begin in §1 some basic notations, definitions and results that we can find in [9]. In §2 we state an inequality of Fefferman-Stein type and, we apply it to obtain an interpolation theorem of Marcinkiewicz-Riviere type. The main results are in §3 where we state the version of the integral singular operator theorem given in [6], for local fields, giving also sequential extensions. Next in §4 we obtain maximal inequalities of F. Zó and Fefferman-Stein type. A diagonal multiplier theorem of Mihlin-Hörmander type (for the Euclidean case see Triebel [11]) that generalize the scalar multiplier theorem of Taibleson [8] is given in §5. Finally, in §6 we give applications of some results obtained in the foregoing sections to Besov and Hardy-Sobolev spaces in local fields.

The extension of all results in this paper for a finite product of local fields will be an immediate consequence of a M. H. Taibleson’s theorem (see [10], pp. 548–549) which states that, if $\mathbb{K}$ is a local field and $d$ is an integer greater than 1, then $\mathbb{K}^d e$, the $d$-dimensional vector space over $\mathbb{K}$, has a field structure, as a local field, which is compatible with the usual vector space norm of $\mathbb{K}^d$.

1. Preliminaries. A local field is any locally compact, non-discrete and totally disconnected field. Let $\mathbb{K}$ be a fixed local field and $dx$ a
Haar measure of the additive group $K^+$ of $K$. The measure of the measurable set $A$ of $K$ with respect to $dx$ we denote for $|A|$. Let $m$ be the modular function for $K^+$, that is, $M(\lambda)|A| = |\lambda A|$ for $\lambda \in K$ and $A$ measurable. We also let $|x| = m(x)$. The sets

$$D = \{x \in K : |x| \leq 1\} \quad \text{and} \quad B = \{x \in K : |x| < 1\}$$

are the ring of integers of $K$ and the unique maximal ideal of $D$, respectively. Let $q = p^c$ ( $p$ prime) be the order of the finite field $D/B$ and $\pi$ a fixed element of maximum absolute value of $B$. The Haar measure $dx$ is normalized such that $|D| = 1$ and thus $|\pi| = |B| = q^{-1}$. We observe that $dx/|x|$ is a Haar measure on the multiplicative group $K^*$ of $K$. We let

$$\mathbb{B}^k = \{x \in K : |x| \leq q^{-k}\}, \quad k \in \mathbb{Z}.$$ 

If $B$ and $R$ are two balls of $K$ such that $B \cap R \neq \emptyset$, then $B \subset R$ or $R \subset B$, For each $k \in \mathbb{Z}$, there is only one sequence $(B_j)_{j \in \mathbb{N}}$ of balls with radius $q^k$ that is a partition of $K$. We fix a character $\chi$ on $K^+$ that is trivial on $D$ but is non-trivial on $B^{-1} = \{x \in K : |x| \leq q\}$. If we take $\chi_y(x) = \chi(x \cdot y)$, then the mapping $y \mapsto \chi_y$ is a topological isomorphism of $K$ onto the group of characters of $K^+$. The Fourier transform of a function $f \in L^1(K)$ is defined by

$$\hat{f}(x) = \int_K f(y)\overline{\chi_x}(y) \, dy,$$

and the inverse Fourier transform of a function $f \in L^\infty_c(K)$ is defined by

$$f^\vee(x) = \int_K f(y)\chi_x(y) \, dy.$$

We denote by $S(K)$ the space of all finite linear combinations of characteristic functions of balls of $K$. The space $S(K)$ is an algebra of continuous functions with compact support that is dense in $L^p(K)$, $1 \leq p < \infty$. We observe that the Fourier transform is a homeomorphism of $S(K)$ onto $S(K)$. The space $S'(K)$ of continuous linear functionals on $S(K)$ is called the space of distributions. We will consider $S'(K)$ with the weak topology.

Let $E$ be a Banach space. The space $l^s(E)$ is the set of all sequences $(c_j)_{j \in \mathbb{Z}}$ of elements of $E$, such that the sequence of its norms is in $l^s$. The space of the quasi-null sequences of elements of $E$, i.e. of the sequences $(c_j)$ such that $c_j = 0$ for $|j| \geq N$, for some $N \geq 0$,
will be denoted by \( l_0^{\infty}(E) \). We denote by \( S(\mathbb{K}, l_0^{\infty}) \) the space of the quasi-null sequences of functions of \( S(\mathbb{K}) \). The space \( S(\mathbb{K}, l_0^{\infty}) \) is dense in the space \( L^p(\mathbb{K}, l^s) \) for \( 1 \leq p, s < \infty \).

The space \( l_r^s(E) \), for \( 1 \leq r \leq \infty \) and \( s \in \mathbb{R} \), will be the set of all sequences \((x_j)_{j \geq 0}\) of elements of \( E \), such that
\[
\|(x_j)_{j \geq 0}\|_{l_r^s(E)} = \|(q^{s_j}\|x_j\|)_{j \geq 0}\|l^r < \infty.
\]

The Hardy-Littlewood maximal function of \( f \in L_{\text{loc}}^1(\mathbb{K}, E) \) is defined by
\[
(3) \quad Mf(x) = \sup_{k \in \mathbb{Z}} q^k \int_{|y-x|\leq q^{-k}} \|f(y)\|_E dy.
\]

The function \( Mf(x) \) is measurable,
\[
(4) \quad \|f(x)\|_E = \lim_{k \to \infty} q^k \int_{|y-x|\leq q^{-k}} \|f(y)\|_E dy,
\]

and
\[
(5) \quad \|f(x)\|_E \leq Mf(x),
\]
for almost all \( x \in \mathbb{K} \). Moreover, \( Mf \) is of the weak type \((1, 1)\) and of the strong type \((p, p)\), \( 1 < p \leq \infty \).

For the details see [9].

2. The BMO(\( E \)) space.

2.1. Definition. Let \( f \in L_{\text{loc}}^1(\mathbb{K}, E) \). The sharp maximal function \( M^#f \) is defined by
\[
M^#f(x) = \sup_{k \in \mathbb{Z}} q^k \int_{|y-x|\leq q^{-k}} \|f(y) - f_k(x)\|_E dy,
\]
where
\[
f_k(x) = q^k \int_{|y-x|\leq q^{-k}} f(y) dy.
\]

2.2. Definition. The space BMO(\( E \)) of the functions of bounded mean oscillation is the set of the functions \( f \in L_{\text{loc}}^1(\mathbb{K}, E) \) such that
\[
(1) \quad \|f\|_* = \|M^#f\|_{\infty} < \infty.
\]

2.3. Remarks. (a) The application \( f \mapsto \|f\|_* \) is a seminorm on BMO(\( E \)) and \( \|f\|_* = 0 \) if and only if \( f \) is constant. We consider the space BMO(\( E \)) like a quotient space with respect to constant functions. (b) We can prove that BMO(\( E \)) is a Banach space analogously to the real case (see [4]). (c) We have \( L^\infty(\mathbb{K}, E) \subset \text{BMO}(E) \),
$L^\infty(\mathbb{K}, E) \neq \text{BMO}(E)$ because the function $f(x) = \log |x|$ if $x \in \mathbb{K}^*$ and $f(0) = 0$ is in BMO($E$) but is not in $L^\infty(\mathbb{K}, E)$.

A classical inequality of Fefferman-Stein also holds in the local field setting.

2.4. **Theorem.** Let $f \in L^1_{\text{loc}}(\mathbb{K}, E)$ such that $Mf \in L^r(\mathbb{K})$ for some $r$ with $0 < r < \infty$. Then for every $p$ with $r \leq p < \infty$, there is a constant $C_p$ depending only on $p$, such that

$$\|M^p f\|_p \leq C_p \|\text{M}^\diamond f\|_p.$$  

The proof of this theorem is an adaptation of the Euclidean case (see [3], Chapter 2, Theorem 3.6). To obtain this adaptation we must remember that the balls of $\mathbb{K}$ have the same properties of the dyadic cubes. We do not need to take dilations of balls, the number 2 that appears in the proof of [3] is the prime number $q$ here, and the functions $\alpha(t)$ and $\beta(t)$ that are considered in [3] are equal in this case.

2.5. **Remark.** The inequality 2.4(1) is not true when $p = \infty$ (see 2.3(c)).

As a consequence of the Fefferman-Stein inequality we obtain an interpolation theorem of Marcinkiewicz-Riviere type, which will be fundamental in the study of the singular integrals.

2.6. **Theorem.** Let $E$ and $F$ be Banach spaces and let $T$ be a linear operator from $L^\infty(\mathbb{K}, E)$ into $L^0(\mathbb{K}, F)$ such that, $T$ has a bounded extension from $L^r(\mathbb{K}, E)$ into $L^r(\mathbb{K}, F)$, for some $r$ with $1 < r < \infty$, and

$$\|Tf\|_* \leq C \|f\|_{L_c^\infty(\mathbb{K}, E)}, \quad f \in L^\infty_c(\mathbb{K}, E).$$

Then $T$ has a bounded extension from $L^p(\mathbb{K}, E)$ into $L^p(\mathbb{K}, F)$, for all $p$ with $r \leq p < \infty$.

3. **Singular integral operators.**

3.1. **Definition.** Let $E$ and $F$ be Banach spaces. A linear operator $T$ defined on $L^\infty_c(\mathbb{K}, E)$, the space of the $E$-valued $L^\infty$-functions with compact support, with values in $L^0(\mathbb{K}, F)$, the space of all $F$-valued strongly measurable functions, is a singular integral operator with an operator valued kernel, if the following two conditions are fulfilled:

SIO 1. $T$ has a bounded extension from $L^r(\mathbb{K}, E)$ into $L^r(\mathbb{K}, F)$, for some $r$ with $1 < r \leq \infty$. 

SIO 2. There is an operator valued kernel $K$, locally integrable from $K \times K \setminus \Delta$ into $L(E, F)$, such that

$$Tf(x) = \int_K K(x, y)f(y)\,dy,$$

for all $f \in L_c^\infty(K, E)$ and for a.e. $x \notin \text{supp} f$.

3.2. **Definition.** Let $T$ be a singular integral operator with a kernel $K$. We say that $K$ satisfies $(H_1)$ if

$$\int_{|x-y'|>|y-y'|} \|K(x, y) - K(x, y')\|_{L(E, F)}\,dx \leq C$$

for all $y \neq y'$, and we say that $K$ satisfies $(H_\infty)$ if

$$\|K(x, y) - K(x, y')\|_{L(E, F)} \leq C \frac{|y-y'|}{|x-y'|^2}$$

for $|x - y'| > |y - y'|$. Moreover, we say that $K$ satisfies $(H'_r)$, for $r = 1$ or $r = \infty$, if $K'(x, y) = K(y, x)$ satisfies $(H_r)$.

3.3. **Remark.** The condition $(H_\infty)$ implies the condition $(H_1)$. In fact, if $|y-y'| = q^l$ and $|x - y'| > |y - y'|$, then

$$\int_{|x-y'|>|y-y'|} \|K(x, y) - K(x, y')\|_{L(E, F)}\,dx = Cq^l \int_{|z| \geq q^{l+1}} \frac{dz}{|z|^2} = Cq^l \sum_{k=l+1}^\infty \int_{|z| = q^k} \frac{dz}{|z|^2} = Cq^{-1}(1 - q^{-1})(1 - q^{-1})^{-1}.$$

Analogously, $(H'_\infty)$ implies $(H'_1)$.

Now we are ready to state the main theorem.

3.4. **Theorem.** Let $T$ be a singular integral operator with kernel $K$, which has a bounded extension from $L^r(K, F)$, for some $r$ with $q < r \leq \infty$. The following hold:

(i) if $K$ satisfies $(H_1)$, then $T$ is of weak type $(1, 1)$ and of strong type $(p, p)$ for $p$ with $q < p \leq r$;

(ii) if $K$ satisfies $(H'_1)$, then $T$ is of strong type $(L^\infty, \text{BMO})$ and of strong type $(p, p)$, for $p$ with $r \leq p < \infty$.

The proof of the above theorem is obtained like the Euclidean case (see [3] or [6]). The crucial part uses a decomposition of the Calderón-Zygmund type (see [9], Chapter 3, results 7.6 and 7.9). Thanks to the decomposition it follows that $T$ is of weak type $(1, 1)$. The Marcinkiewicz interpolation theorem then shows that $T$ is of
strong type \((p, p), 1 < p \leq r\). The proof that \(T\) is of strong type \((L^\infty, \text{BMO})\) is similar to the Euclidean case. Finally, to conclude that \(T\) is of strong type \((p, p)\) for \(r \leq p < \infty\), we need the Marcin- kiewicz-Riviere interpolation Theorem 2.6.

3.5. **Theorem.** Let \((T_j)_{j \in \mathbb{Z}}\) be a sequence of singular integral operators uniformly bounded from \(L^r(\mathbb{K}, E)\) into \(L^r(\mathbb{K}, F)\), for some \(r\) with \(1 < r \leq \infty\). Suppose further that the sequence of associated kernels \((K_j)_{j \in \mathbb{Z}}\) satisfies

\[
\left(1\right) \quad \int |x-y'| > |y-y'| \sup_j \|K_j(x, y) - K_j(x', y')\|_{L(E, F)} \, dx \leq C, \quad y \neq y',
\]

and

\[
\left(2\right) \quad \int |y-x'| < |x-x'| \sup_j \|K_j(x, y) - K_j(x', y')\|_{L(E, F)} \, dy \leq C, \quad x \neq x'.
\]

Then, given \(p\) and \(s\) with \(1 \leq p < \infty\) and \(1 < s < \infty\), there is a constant \(A_{p, s}\) depending only on \(p, s, C\) and \(r\), such that

\[
\left(3\right) \quad \left\{ x : \sum_j \|T_j f_j(x)\|_F^s > \lambda^s \right\} \leq A_{1, s} \lambda^{-1} \|f_j\|_{L^p(l^s(E))}
\]

and

\[
\left(4\right) \quad \|T_j f_j\|_{L^p(l^s(F))} \leq A_{p, s} \|f_j\|_{L^p(l^s(E))}, \quad 1 < p < \infty,
\]

for all \(\lambda > 0\) and \(f = (f_j)_{j \in \mathbb{Z}} \in L^\infty(\mathbb{K}, l^s(E))\). Moreover, the inequality \((4)\) can be extended for all \(f = (f_j)_{j \in \mathbb{Z}} \in L^p(\mathbb{K}, l^s(E))\).

**Proof.** For each positive integer \(m\), let \(\tilde{T}_m\) be the operator from \(L^\infty(\mathbb{K}, l^s(E))\) into \(L^0(\mathbb{K}, l^s(F))\) defined by

\[
\left(5\right) \quad \tilde{T}_m(f_j)_{j \in \mathbb{Z}} = (T_j f_j)_{m \leq j \leq m}, \quad (f_j)_{j \in \mathbb{Z}} \in L^\infty(\mathbb{K}, l^s(E)),
\]

and let \(\tilde{K}_m\) be the kernel from \(\mathbb{K} \times \mathbb{K} \setminus \Delta\) into \(L(l^s(E), l^s(F))\) defined by

\[
\left(6\right) \quad \tilde{K}_m(x, y)(\alpha_j)_{j \in \mathbb{Z}} = (K_j(x, y)\alpha_j)_{-m \leq j \leq m}, \quad (\alpha_j)_{j \in \mathbb{Z}} \in l^s(E).
\]

We observe that the operators \(T_j\) are uniformly bounded from \(L^p(\mathbb{K}, E)\) into \(L^p(\mathbb{K}, F)\) for all \(p, 1 < p < \infty\). Now, we fix
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The operators \( \tilde{T}_m \) are uniformly bounded from \( L^s(\mathbb{K}, l^s(E)) \) into \( L^s(\mathbb{K}, l^s(F)) \) and it is clear that

\[
\tilde{T}_m(f_j)(x) = \int_{\mathbb{K}} \tilde{K}_m(x,y)(f_j(y)) \, dy
\]

for all \( (f_j)_j \in L_c^\infty(\mathbb{K}, l^s(E)) \) and a.a. \( x \notin \text{supp}(f_j)_j \). Since

\[
\|\tilde{K}_m(x,y)\|_{L^p(E), l^r(F))} \leq \sup_{|j| \leq m} \|K_j(x,y)\|_{L(E,F)},
\]

then it follows by (1) and (2) that the kernel \( \tilde{K}_m \) verifies \((H_1)\) and \((H'_1)\). Therefore, by Theorem 3.4, for each \( p \) with \( 1 \leq p < \infty \), there is a constant \( A_{p,s} \) depending only on \( p, s, C \) and \( r \), such that

\[
\left\{ x : \sum_{|j| \leq m} \|T_j f_j(x)\|_F^s > \lambda^s \right\} \leq A_{1,s}\lambda^{-1}\|f_j\|_{L^p(I^r(E))}
\]

and

\[
\|\tilde{T}_m(f_j)\|_{L^p(I^r(F))} \leq A_{p,s}\|f_j\|_{L^p(I^r(E))}, \quad 1 < p < \infty,
\]

for all \( \lambda > 0 \) and \( f = (f_j)_j \in L_c^\infty(\mathbb{K}, l^s(E)) \). Moreover, the inequality (8) can be extended for all \( f = (f_j)_j \in L^p(\mathbb{K}, l^s(E)) \). Then, letting \( m \to \infty \) on both sides of the inequalities (7) and (8) we obtain (3) and (4).

3.6. Corollary. Let \( T \) be a singular integral operator with kernel \( K \) satisfying \((H_1)\) and \((H'_1)\). Then, given \( p \) and \( s \) with \( 1 \leq p < \infty \)
and \( 1 < s < \infty \), there is a constant \( A_{p,s} \) depending only on \( p, s, C \) and \( r \), such that

\[
\left\{ x : \sum_j \|T_j f_j(x)\|_F^s > \lambda^s \right\} \leq A_{1,s}\lambda^{-1}\|f_j\|_{L^p(I^r(E))}
\]

and

\[
\|T f_j\|_{L^p(I^r(F))} \leq A_{p,s}\|f_j\|_{L^p(I^r(E))}, \quad 1 < p < \infty,
\]

for all \( \lambda > 0 \) and \( f = (f_j)_j \in L_c^\infty(\mathbb{K}, l^s(E)) \). Moreover, the inequality (2) can be extended for all \( f = (f_j)_j \in L^p(\mathbb{K}^s(E)) \).

3.7. Remark. In our applications we shall consider singular integral operators of convolution type, that is, with kernels of the type \( K(x,y) = K'(x-y) \) where \( K' \) is locally integrable from \( \mathbb{K}\{0\} \) into \( L(E,F) \).
4. Applications to maximal functions.

4.1. DEFINITION. Let \( \varphi \in L^1(\mathbb{K}) \) and for each \( t \in \mathbb{K}^* \), let \( \varphi_t(x) = |t|^{-1} \varphi(t^{-1}x) \). The maximal operator \( M^\varphi \) is defined by

\[
M^\varphi f(x) = \sup_{t \neq 0} |(\varphi_t * f)(x)|, \quad f \in L_c^\infty(\mathbb{K}).
\]

The Euclidean version of the following theorem is due to F. Zó (see [6] or [12]).

4.2. THEOREM. Let \( \varphi \in C_c(\mathbb{K}) \) such that

\[
(1) \quad \int_{|x| > |y|} \sup_{t \neq 0} |\varphi_t(x-y) - \varphi_t(x)| \, dx \leq C, \quad y \neq 0.
\]

Then, given \( p \) and \( s \) with \( 1 \leq p < \infty \) and \( 1 < s < \infty \), there is a constant \( A_{p,s} \) depending only on \( p, s, C \) and \( \|\varphi\|_1 \), such that

\[
(2) \quad \left\{ x : \sum_j |M^\varphi f_j(x)|^s > \lambda^s \right\} \leq A_{1,s} \lambda^{-1} \|(f_j)_j\|_{L^1(p')}
\]

and

\[
(3) \quad \|(M^\varphi f_j)_j\|_{L^p(p')} \leq A_{p,s} \|(f_j)_j\|_{L^p(p')}, \quad 1 < p < \infty,
\]

for all \( \lambda > 0 \) and \( f = (f_j)_j \in L_c^\infty(\mathbb{K}, l^s) \). Moreover, the inequality (3) can be extended for all \( f = (f_j)_j \in L^p(\mathbb{K}, l^s) \).

Proof. Step 1. Owing to continuity of the function \( t \mapsto (\varphi_t * f)(x) \), it is enough to calculate the supremum, in the definition of \( M^\varphi \), on a countable dense subset \( \{t_j\}_{j \in \mathbb{N}} \) of \( \mathbb{K}^* \), that is,

\[
M^\varphi f(x) = \sup_j |(\varphi_{t_j} * f)(x)|.
\]

Consider the operators \( M^\varphi_m \) defined by

\[
M^\varphi_m f(x) = \sup_{1 \leq j \leq m} |(\varphi_{t_j} * f)(x)|.
\]

We have that \( M^\varphi_m f(x) \uparrow M^\varphi f(x) \) for all \( x \in \mathbb{K} \). Therefore, obtaining estimates for \( M^\varphi_m f \) that do not depend on \( m \), we shall be obtaining also estimates for \( M^\varphi f \).

Step 2. For each positive integer \( m \), let \( T_m \) be the linear operator from \( L_c^\infty(\mathbb{K}) \) into \( L^0(\mathbb{K}, l^\infty) \) defined by

\[
(4) \quad T_m f = (\varphi_{t_j} * f)_{1 \leq j \leq m}, \quad f \in L_c^\infty(\mathbb{K}),
\]
and let $K_m$ be the kernel (of convolution type) from $\mathbb{K}$ into $L(\mathbb{C}, l^\infty)$ defined by

$$K_m(x)\lambda = (\varphi_{t_j}(x)\lambda)_{1 \leq j \leq m}, \quad \lambda \in \mathbb{C}. \tag{5}$$

Since $\|\varphi_t\|_1 = \|\varphi\|_1$ for all $t \neq 0$, we have

$$\|T_m f\|_{L^\infty(l^\infty)} = \text{ess sup} \sup_{x \in \mathbb{K}} |(\varphi_{t_j} * f)(x)| \leq \text{ess sup} \sup_{x \in \mathbb{K}} \|f\|_{\infty} \|\varphi_{t_j}\|_1 = \|\varphi\|_1 \|f\|_{\infty}, \tag{6}$$

i.e., the operator $T_m$ is bounded from $L^\infty(\mathbb{K})$ into $L^\infty(l^\infty)$. On the other hand, we have

$$\int_{\mathbb{K}} \|K_m(x)\|_{L(\mathbb{C}, l^\infty)} \, dx = \int_{\mathbb{K}} \sup_{1 \leq j \leq m} |\varphi_{t_j}(x)| \, dx \leq \sum_{1 \leq j \leq m} \int_{\mathbb{K}} |\varphi_{t_j}(x)| \, dx = m\|\varphi\|_1 < \infty,$$

and

$$T_m f(x) = \left(\int_{\mathbb{K}} \varphi_{t_j}(x-y) f(y) \, dy\right)_{1 \leq j \leq m} = \int_{\mathbb{K}} (\varphi_{t_j}(x-y) f(y))_{1 \leq j \leq m} \, dy = \int_{\mathbb{K}} K_m(x-y) f(y) \, dy,$$

for all $f \in L_c^\infty(\mathbb{K})$ and for a.e. $x \notin \text{supp} f$. Consequently $T_m$ is a singular integral operator of convolution type with kernel $K_m$. Moreover, the kernel $K_m$ satisfies, for all $y \neq 0$,

$$\int_{|x|>|y|} \|K_m(x-y) - K_m(x)\|_{L(\mathbb{C}, l^\infty)} \, dx = \int_{|x|>|y|} \sup_{1 \leq j \leq m} |\varphi_{t_j}(x-y) - \varphi_{t_j}(x)| \, dx \leq \int_{|x|>|y|} \sup_{t \neq 0} |\varphi_{t}(x-y) - \varphi_{t}(x)| \, dx \leq C. \tag{7}$$

**Step 3.** The inequalities (6) and (7) show that the operators $T_m$ and its kernels $K_m$ satisfy uniformly the hypothesis of the Corollary 3.6. Therefore, given $p$ and $s$ with $1 \leq p < \infty$ and $1 < s < \infty$, there is a constant $A_{p,s}$, depending only on $p$, $s$, $C$ and $\|\varphi\|_1$, such that

$$\left\{ x : \sum_j \|T_m f_j(x)\|_{l^\infty} > \lambda^s \right\} \leq A_{1,s} \lambda^{-1} \|f_j\|_{L^1(l^p)} \tag{8}$$
and
\begin{equation}
\|(T_m f_j)_j\|_{L^p(\ell^\infty)} \leq A_{p,s} \|(f_j)_j\|_{L^p(\ell^r)}, \quad 1 < p < \infty,
\end{equation}
for all $\lambda > 0$, $m \in \mathbb{N}$ and $f = (f_j)_j \in L^\infty_c(\mathbb{K}, l^s)$. Moreover, the inequality (9) can be extended for all $f = (f_j)_j \in L^p(\mathbb{K}, l^s)$. Since
\[
\|T_m f_j(x)\|_{\ell^\infty} = M^\phi_m f_j(x),
\]
then, letting $m \to \infty$ on both sides of (8) and (9), we obtain (2) and (3).

From 4.2 we obtain the maximal theorem of Fefferman-Stein (see [2] or [6]) in the context of the local fields.

4.3. **Theorem.** Given $p$ and $s$ with $1 \leq p < \infty$ and $1 < s < \infty$, there is a constant $A_{p,s}$ depending only on $p$ and $s$, such that

\begin{equation}
\left\{ x : \sum_j |M f_j(x)|^s > \lambda^s \right\} \leq A_{1,s} \lambda^{-1} \|(f_j)_j\|_{L^p(\ell^r)}
\end{equation}

and
\begin{equation}
\|(M f_j)_j\|_{L^p(\ell^r)} \leq A_{p,s} \|(f_j)_j\|_{L^p(\ell^r)}, \quad 1 < p < \infty,
\end{equation}
for all $\lambda > 0$ and $f = (f_j)_j \in L^\infty_c(\mathbb{K}, l^s)$. Moreover, the inequality (2) can be extended for all $f = (f_j)_j \in L^p(\mathbb{K}, l^s)$.

**Proof.** Let $\phi$ be the characteristic function of the ball $B^0$. If $|x| > |y|$, then $|t^{-1}(x-y)| = |t^{-1}x|$ and hence $\phi(t^{-1}(x-y)) = \phi(t^{-1}x)$. Therefore
\[
|\phi_t(x-y) - \phi_t(x)| = |t|^{-1} |\phi(t^{-1}(x-y)) - \phi(t^{-1}x)| = 0
\]
and consequently
\begin{equation}
\int_{|x| > |y|} \sup_{t \neq 0} |\phi_t(x-y) - \phi_t(x)| \, dx = 0.
\end{equation}

On the other hand, we have
\[
(|f| * \phi_t)(x) = \int_{\mathbb{K}} |f(x-y)| \phi_t(y) \, dy
\]
\[
= |t|^{-1} \int_{\mathbb{K}} |f(x-y)| \phi(t^{-1}y) \, dy
\]
\[
= |t|^{-1} \int_{|y| \leq |x|} |f(x-y)| \, dy
\]
\[
= |t|^{-1} \int_{|y-x| \leq |x|} |f(y)| \, dy
\]
and hence

\begin{align*}
(4) \quad M^\varphi |f|(x) &= \sup_{t \neq 0} \left| \int f * \varphi_t (x) \right|
= \sup_{t \neq 0} |t|^{-1} \int_{|y-x| \leq |t|} |f(y)| \, dy \\
&= \sup_{k \in \mathbb{Z}} q^k \int_{|y-x| \leq q^{-k}} |f(y)| \, dy = M f(x).
\end{align*}

From (3) it follows that the maximal operator \( M^\varphi \) satisfies the inequalities 4.2(2) and 4.2(3). Then, by (4) we obtain the inequalities (1) and (2) for the Hardy-Littlewood maximal operator.

5. A multiplier theorem on \( L^p(\mathbb{K}, l^2) \)-spaces.

5.1. Lemma. Let \( g \in L^2(\mathbb{K}) \) and \( \alpha > 0 \). Then, there is a constant \( A_\alpha \) depending only on \( \alpha \), such that

\begin{align*}
(1) \quad q^{-\alpha} \int_K |x|^{\alpha} |\hat{g}(x)|^2 \, dx \\
&\leq A_\alpha \int_{K \times K} |g(x+y) - g(x)|^2 |y|^{-(1+\alpha)} \, dx \, dy.
\end{align*}

Proof. See [9], page 220.

5.2. Lemma. Let \( (g_j)_{j \in \mathbb{Z}} \) be a sequence of elements of \( L^2(\mathbb{K}) \) and suppose that there are \( B > 0 \) and \( \varepsilon > 0 \), such that

\begin{align*}
(1) \quad \int_{K \times K} \sum_{j=-\infty}^{+\infty} |g_j(x+y) - g_j(x)|^2 |y|^{-(2+\varepsilon)} \, dx \, dy \leq B^2.
\end{align*}

Then, there is a constant \( A_\varepsilon \) depending only on \( \varepsilon \), such that, for all \( k \in \mathbb{Z} \),

\begin{align*}
(2) \quad \int_{|x| \geq q^k} \sup_j |\hat{g}_j(x)| \, dx \leq A_\varepsilon B q^{-k\varepsilon/2}.
\end{align*}

Proof. It follows from Hölder's Inequality that

\begin{align*}
\int_{|x| \geq q^k} \sup_j |\hat{g}_j(x)| \, dx \\
&\leq \left( \int_K |x|^{(1+\varepsilon)} \sup_j |\hat{g}_j(x)|^2 \, dx \right)^{1/2} \left( \int_{|x| \geq q^k} |x|^{-(1+\varepsilon)} \, dx \right)^{1/2} \\
&= \left( \int_K |x|^{(1+\varepsilon)} \sup_j |\hat{g}_j(x)|^2 \, dx \right)^{1/2} \left( \frac{1 - q^{-1}}{1 - q^{-\varepsilon}} \right)^{1/2} q^{-k\varepsilon/2}.
\end{align*}
Now, setting $\alpha = 1 + \varepsilon$ and applying Lemma 5.1, we obtain
\[
q^{-\alpha} j \int_{\mathbb{R}} |x|^{\alpha} \sup_j |\hat{g}_j(x)|^2 \, dx 
\]
\[
\leq A_{\alpha} \int_{\mathbb{R} \times \mathbb{R}} \sum_{j=-\infty}^{+\infty} |g_j(x + y) - g_j(x)|^2 |y|^{-(2+\varepsilon)} \, dx \, dy \leq A_{\alpha} B^2
\]
and consequently
\[
\int_{|x| \geq q^k} \sup_j |\hat{g}_j(x)| \, dx \leq (A_{\alpha} B^2 q^\alpha)^{1/2} \left( \frac{1 - q^{-1}}{1 - q^{-\varepsilon}} \right)^{1/2} q^{-k\varepsilon/2} = A_{\varepsilon} B q^{-k\varepsilon/2}.
\]

5.3. **Theorem.** Let $(m_j)_{j \in \mathbb{Z}} \in L^\infty(\mathbb{K}, l^2)$ and suppose that there are $B > 0$ and $\varepsilon > 0$, such that, for all $j \in \mathbb{Z}$,
\[
J \int_{|y| < q^j} \int_{|x| = q^j} \sum_{i = -\infty}^{+\infty} |m_i(x + y) - m_i(x)|^2 |y|^{-(2+\varepsilon)} \, dx \, dy \leq B^2 q^{-j}. 
\]
Then, for all $(\varphi_j)_{j \in S(\mathbb{K}, l_0^\infty)}$ and $1 < p, s < \infty$, we have
\[
\|((m_j \varphi_j)^\vee)_{j} \|_{L^p(l')} \leq C \|\varphi_j \|_{L^p(l')},
\]
where $C$ is independent of $(\varphi_j)_j$.

**Proof.** Step 1. Let $\phi_k$ be the characteristic function of the ball $B^k$ and $m^k_j = m_j \phi_k$, $k \in \mathbb{Z}$. Since $(\varphi_j)_j \in S(\mathbb{K}, l_0^\infty)$ has compact support we see that $(m^k_j \varphi_j^\vee)_j = ((m^k_j \varphi_j)^\vee)_j$ for $k$ small enough. Hence, if we wish to show (2), we only need to show that, for all $(\varphi_j)_j \in S(\mathbb{K}, l_0^\infty)$, $k \in \mathbb{Z}$ and $1 < p, s < \infty$, we have
\[
\|((m^k_j \varphi_j)^\vee)_j \|_{L^p(l')} \leq C \|\varphi_j \|_{L^p(l')},
\]
where the constant $C$ is independent of $k$ and $(\varphi_j)_j$.

Step 2. For each $k, j \in \mathbb{Z}$, let $T^k_j$ be the linear operator defined by
\[
T^k_j \varphi = (m^k_j \varphi)^\vee = (m^k_j)^\vee \ast \varphi, \quad \varphi \in S(\mathbb{K}).
\]
For all $k, j \in \mathbb{Z}$ and $\varphi \in S(\mathbb{K})$ we have
\[
\|T^k_j \varphi\|_2 = \|m^k_j \hat{\varphi}\|_2 = \|m^k_j \hat{\varphi}\|_2 
\]
\[
\leq \|m^k_j \|_{\infty} \|\hat{\varphi}\|_2 \leq \|(m_j)_j\|_{L^\infty(l')} \|\varphi\|_2.
\]
Therefore \((T_j^k)_{j \in \mathbb{Z}}\) is a sequence of singular integral operators of convolution type uniformly bounded from \(L^2(\mathbb{R})\) into \(L^2(\mathbb{R})\), with sequence of associated kernels \(((m_j^k)^{\nu})_{j \in \mathbb{Z}}\).

**Step 3.** Let \(m_{jl} = m_j^{-l} - m_j^{1-l}\) for \(j, l \in \mathbb{Z}\). It follows from (1) that

\[
\int_{|y|<q'} \int_{|x|=q'} \sum_{j=-\infty}^{+\infty} |m_{jl}(x+y) - m_{jl}(x)|^2 |y|^{-2(2+\varepsilon)} \, dx \, dy
\]

\[
= \int_{|y|<q'} \int_{|x|=q'} \sum_{j=-\infty}^{+\infty} |m_j(x+y) - m_j(x)|^2 |y|^{-2(2+\varepsilon)} \, dx \, dy
\]

\[
\leq B^2 q^{-\varepsilon l}.
\]

We have also

\[
\int_{|y|\geq q'} \int_{|x|=q'} \sum_{j=-\infty}^{+\infty} |m_{jl}(x+y) - m_{jl}(x)|^2 |y|^{-2(2+\varepsilon)} \, dx \, dy
\]

\[
\leq \int_{|y|\geq q'} \int_{|x|=q'} 2 \sum_{j=-\infty}^{+\infty} (|m_{jl}(x+y)|^2 + |m_{jl}(x)|^2) |y|^{-2(2+\varepsilon)} \, dx \, dy
\]

\[
\leq 4\|m_j\|_{L^\infty(L^2)}^2 (1 - q^{-1})^2 q^l \left( \frac{q^{-(1+\varepsilon)l}}{1 - q^{-(1+\varepsilon)}} \right) = C_1 q^{-\varepsilon l};
\]

\[
\int_{|y|=q'} \int_{|x|<q'} \sum_{j=-\infty}^{+\infty} |m_{jl}(x+y) - m_{jl}(x)|^2 |y|^{-2(2+\varepsilon)} \, dx \, dy
\]

\[
= q^{-2(2+\varepsilon)l} \int_{|y|=q'} \int_{|x|<q'} \sum_{j=-\infty}^{+\infty} |m_j(x+y)|^2 \, dx \, dy
\]

\[
\leq \|m_j\|_{L^\infty(L^2)}^2 (1 - q^{-1}) q^{-1} q^{-\varepsilon l} = C_2 q^{-\varepsilon l};
\]

\[
\int \int \sum_{j=-\infty}^{+\infty} |m_{jl}(x+y) - m_{jl}(x)|^2 |y|^{-2(2+\varepsilon)} \, dx \, dy
\]

\[
\leq \int \int \sum_{j=-\infty}^{+\infty} |m_{jl}(x+y)|^2 |y|^{-2(2+\varepsilon)} \, dx \, dy
\]

\[
\leq \|m_j\|_{L^\infty(L^2)}^2 (1 - q^{-1})^2 q^{-\varepsilon l} (q^{-\varepsilon} / 1 - q^{-\varepsilon}) = C_3 q^{-\varepsilon l}.
\]
Therefore from (6), (7), (8) and (9) we obtain

\[ \int_{\mathbb{R} \times \mathbb{R}} \sum_{j = -\infty}^{+\infty} |m_{jl}(x + y) - m_{jl}(x)|^2 |y|^{-(2 + \varepsilon)} \, dx \, dy \leq C^2 q^{-\varepsilon l}, \]

for all \( l \in \mathbb{Z} \), where the constant \( C \) depends only on \( \| (m_j)_j \|_{L^\infty(l)} \), \( B \) and \( \varepsilon \). Then, it follows by Lemma 5.2 that, for all \( k \in \mathbb{Z} \),

\[ \int_{|x| \geq q^j} \sup_j |(m_{jl})^\vee(x)| \, dx = \int_{|x| \geq q^j} \sup_j |(m_{jl})^\wedge(x)| \, dx \leq A_\varepsilon C q^{-(l+k)/2}. \]

Since \( m_j \phi_{-1} = m_{jl} \), the \( (m_{jl})^\vee(x + y) = (m_{jl})^\vee(x) \) for all \( x, y \in \mathbb{K} \) with \( |y| \leq q^{-l} \) (see [9], page 126). Therefore, for all \( t, j, k \in \mathbb{Z} \) and \( x, y \in \mathbb{K} \) with \( |y| \leq q^i \), we have

\[ |(m_j^k)^\vee(x + y) - (m_j^k)^\vee(x) \leq \sum_{l = -t + 1}^{\infty} |(m_{jl})^\vee(x + y) - (m_{jl})^\vee(x)|. \]

Hence we obtain by (11) that, for all \( t, k \in \mathbb{Z} \),

\[ \int_{|x| > q^i} \sup_j |(m_j^k)^\vee(x + y) - (m_j^k)^\vee(x)| \, dx \leq 2 \sum_{l = -t + 1}^{\infty} \int_{|x| > q^i} \sup_j |(m_{jl})^\vee(x)| \, dx \leq 2A_\varepsilon C(q^{-\varepsilon/2}/1 - q^{-\varepsilon/2}) = C', \]

and consequently for all \( k \in \mathbb{Z} \), we have

\[ \sup_{y \neq 0} \int_{|x| > |y|} \sup_j |(m_j^k)^\vee(x - y) - (m_j^k)^\vee(x)| \, dx = \sup_{t \in \mathbb{Z}} \sup_{|y| \leq q^i} \int_{|x| > q^i} \sup_j |(m_j^k)^\vee(x + y) - (m_j^k)^\vee(x)| \, dx \leq C'. \]

Therefore, the sequences of kernels of convolution type \( ((m_j^k)^\vee)_j \) satisfy uniformly 3.5(1) and 3.5(2). Consequently we obtain (3), which proves the theorem.

6. Applications to Besov and Hardy-Sobolev spaces. In this section we will give some applications of some foregoing results to Besov and Hardy-Sobolev spaces and to spaces of Bessel potentials.

6.1 Let \( A^j = \mathbb{B}^j - \mathbb{B}^{j+1} = \{ x \in \mathbb{K} : |x| = q^{-j} \} \) for \( j \in \mathbb{Z} \). We will consider the sequence \( (\Phi_j)_j \geq 0 \) of elements of \( S(\mathbb{K}) \), where \( \hat{\Phi}_j \) is the
characteristic function of $A^{-j}$ for $j \geq 1$, and $\Phi_0$ is the characteristic function of $\mathbb{D}$.

For each distribution $f \in S'(\mathbb{K})$ and $j \geq 0$ we have that $\Phi_j * f$ is a function (see [9], p. 126). We can easily see that the function $\Phi_j$ satisfies:

1. $\Phi_j * \Phi_j = \Phi_j$ and $\Phi_j * \Phi_i = 0$ for $i \neq j$;
2. $\hat{\Phi}_j(x + y) = \hat{\Phi}_j(x)$ for $|x| > |y|$;
3. $\sum_{j=0}^{\infty} \hat{\Phi}_j = 1$.

6.2. Definitions. Let $s \in \mathbb{R}$ and $1 < p < \infty$. For $1 < r < \infty$, the distribution $f \in S'(\mathbb{K})$ is in $B_{pr}^s(\mathbb{K})$ if

$$\|f\|_{B_{pr}^s} = \|(\Phi_j * f)_{j \geq 0}\|_{\ell^p(L^r)} < \infty.$$  

For $1 < r < \infty$, the distribution $f \in S'(\mathbb{K})$ is in $F_{pr}^s(\mathbb{K})$ if

$$\|f\|_{F_{pr}^s} = \|(\Phi_j * f)_{j \geq 0}\|_{L^p(\ell^r)} < \infty.$$  

6.3. Remark. The sequence $(\Phi_j)_{j \geq 0}$ used in Definition 6.2 and given as in 6.1 is unique. In fact, if $(\psi_j)_{j \geq 0}$ is a sequence of elements of $S(\mathbb{K})$ such that $\sup \psi_j \subset A^{-j}$ for $j \geq 1$, $\supp \psi_0 \subset \mathbb{D}$ and $\sum_j \psi_j = 1$, then $\psi_j$ is the characteristic function of $A^{-j}$ for $j \geq 1$, and $\psi_0$ is the characteristic function of $\mathbb{D}$, that is, $\psi_j = \Phi_j$ for $j \geq 0$.

6.4. Remark. As in the Euclidean case, there is another way to define the spaces $B_{pr}^s(\mathbb{K})$ and $F_{pr}^s(\mathbb{K})$ (see [11]). We can say that the distribution $f$ is in $B_{pr}^s(\mathbb{K})$ ($F_{pr}^s(\mathbb{K})$, respectively) if there is a sequence $(a_j)_{j \geq 0}$ of elements of $S'(\mathbb{K})$ such that $\sum_j a_j$ converges in $S'(\mathbb{K})$ to $f$, $\supp a_j \subset A^{-j}$ for $j \geq 1$, $\supp a_0 \subset \mathbb{D}$ and

$$\|(a_j)_{j \geq 0}\|_{\ell^p(L^r)} < \infty \quad (\|(a_j)_{j \geq 0}\|_{L^p(\ell^r)} < \infty, \text{ respectively}).$$  

But this definition is trivial because there is only one sequence $(a_j)_{j \geq 0}$ for each $f$, namely, the sequence $(\Phi_j * f)_{j \geq 0}$. In fact,

$$\hat{(\Phi_j * f)} = \hat{\Phi}_j \hat{f} = \hat{\Phi}_j \sum_{k=0}^{\infty} \hat{a}_k = \sum_{k=0}^{\infty} \hat{\Phi}_j \hat{a}_k = \hat{\Phi}_j \hat{a}_j = \hat{a}_j,$$

and hence $a_j = \Phi_j * f$ for $j \geq 0$.

If $s \in \mathbb{R}$ and $f \in S'(\mathbb{K})$, the Bessel potential of order $s$ of $f$ is defined by

$$(J^s f) = (\max\{1, |x|\})^s \hat{f}.$$
For $\alpha, \beta \in \mathbb{R}$, the map $f \mapsto J^\alpha f$ is a homeomorphism from $S'(\mathbb{K})$ onto $S'(\mathbb{K})$, $(J^\alpha)^{-1} = J^{-\alpha}$ and $J^{\alpha + \beta} f = J^\alpha (j^\beta f)$ for $f \in S'(\mathbb{K})$ (see [9], p. 137).

The next theorem shows that $J^s$ is an isometry on $F^t_{pr}$ and $B^t_{pr}$.

6.5. **Theorem.** Let $s, t \in \mathbb{R}$ and $1 < p < \infty$. Then

1. $\|J^s f\|_{F^t_{pr}} = \|f\|_{F^t_{pr}}$, $f \in F^t_{pr}(\mathbb{K})$, $1 < r < \infty$;
2. $\|J^s f\|_{B^t_{pr}} = \|f\|_{B^t_{pr}}$, $f \in B^t_{pr}(\mathbb{K})$, $1 \leq r \leq \infty$.

**Proof.** We can easily verify that $J^s \Phi = q^s \Phi$ for $j \geq 0$. Then, for $j \geq 0, s \in \mathbb{R}$ and $f \in S'(\mathbb{K})$ we have

$$J^s (\Phi_j * f) = (J^s \Phi_j) * f = q^s j (\Phi_j * f).$$

For $f \in F^t_{pr}(\mathbb{K})$ and $1 < r < \infty$, it follows from (3) that

$$\|J^s f\|_{F^t_{pr}} = \|(q^s_j \{\Phi_j * f\})_{j \geq 0}\|_{L^p (l^r_{-})}$$
$$= \|(q^s_j \{\Phi_j * f\})_{j \geq 0}\|_{L^p (l^r)}$$
$$= \|f\|_{F^t_{pr}}.$$

Now, for $f \in B^t_{pr}(\mathbb{K})$ and $1 \leq r \leq \infty$, it also follows from (3) that

$$\|J^s f\|_{B^t_{pr}} = \|(q^s_j \{\Phi_j * f\})_{j \geq 0}\|_{l^r_{+}}$$
$$= \|(q^s_j \{\Phi_j * f\})_{j \geq 0}\|_{l^r}$$
$$= \|f\|_{B^t_{pr}}.$$

Now, we will give a theorem of the Littlewood-Paley type. It is a variant of Taibleson's theorem (see [9], pp. 200 and 202), but our proof makes use of vector singular integral operators.

6.6. **Theorem.** For each $1 < p < \infty$, there are constants $A_p$ and $B_p$, depending only on $p$, such that, for all $f \in L^p(\mathbb{K})$ we have

$$A_p \|f\|_p \leq \|(\Phi_j * f)\|_{L^p (l^r_{-})} \leq B_p \|f\|_p.$$

**Proof (Sketch).** Let us consider the operator $T$ from $L^\infty_c(\mathbb{K})$ into $L^0(\mathbb{K}, l^2)$ defined by

$$Tf = (\Phi_j * f)_{j \geq 0},$$

and $S$ from $L^\infty_c(\mathbb{K}, l^2)$ into $L^0(\mathbb{K})$ defined by

$$S(\alpha_j)_{j \geq 0} = \sum_{j=0}^{\infty} \Phi_j * \alpha_j.$$
We can show that
\[ \|Tf\|_{L^2(l^2)} = \|f\|_2 \]
and
\[ \|S(\alpha_j)_{j \geq 0}\|_2 \leq \|(\alpha_j)_{j \geq 0}\|_{L^2(l^2)}. \]
Therefore we can conclude that \( T \) has a bounded extension from \( L^2(\mathbb{K}) \) into \( L^2(\mathbb{K}, l^2) \) and \( S \) has a bounded extension from \( L^2(\mathbb{K}, l^2) \) into \( L^2(\mathbb{K}) \).

Let \( K_1 \) and \( K_2 \) be the kernels defined by
\begin{align*}
(4) \quad K_1(x) \lambda &= (\Phi_j(x) \lambda)_{j \geq 0}, \quad x \in \mathbb{K}, \quad \lambda \in \mathbb{C}; \\
(5) \quad K_2(x)(\lambda_j)_{j \geq 0} &= \sum_{j=0}^{\infty} \Phi_j(x) \lambda_j, \quad x \in \mathbb{K}, \quad (\lambda_j)_{j \geq 0} \in l^2.
\end{align*}
We have that
\[ \|K_2(x)\|_{L(l^2, c)} \leq \|K_1(x)\|_{L(c, l^2)} = \|(\Phi_j(x))_{j \geq 0}\|_{l^2}, \]
therefore, showing that \( x \mapsto \|(\Phi_j(x))_{j \geq 0}\|_{l^2} \) is locally integrable we can conclude that \( K_1 \) and \( K_2 \) are locally integrable. Since
\[ \|K_1(x-y) - K_1(x)\|_{L(c, l^2)} = \|K_2(x-y) - K_2(x)\|_{L(l^2, c)} = 0 \]
for \( |x| > |y| \), we have that \( K_1 \) and \( K_2 \) satisfy the conditions \((H_1)\) and \((H'_1)\) of Theorem 3.4. We can easily verify that
\[ Tf(x) = \int_{\mathbb{K}} K_1(x-y)f(y)\,dy \]
and
\[ S\alpha(x) = \int_{\mathbb{K}} K_2(x-y)\alpha(y)\,dy , \]
for all \( x \in \mathbb{K}, f \in L^\infty_c(\mathbb{K}) \) and \( \alpha \in L^\infty_c(\mathbb{K}, l^2) \). Then, it follows from 3.4 that \( T \) and \( S \) are singular integral operators of the strong type \((p, p)\) for \( 1 < p < \infty \), and consequently we have the inequalities 6.6(1).

In Taibleson \[9\] the space of Bessel potentials \( L^p_s(\mathbb{K}) \) is defined for 
\( s \in \mathbb{R} \) and \( 1 \leq p < \infty \), as the set of all distributions \( f \in S'(\mathbb{K}) \) such that
\[ \|f\|_{L^p_s} = \|J^s f\|_p < \infty. \]

The next theorem is a consequence of Theorem 6.6.
6.7. Theorem. If \( s \in \mathbb{R} \) and \( 1 < p < \infty \), then the spaces \( L^p_s(\mathbb{K}) \) and \( F^s_{p2}(\mathbb{K}) \) are isomorphic.

Proof. If \( f \in S'(\mathbb{K}) \), it follows from 6.6(1) and 6.5(1) that
\[
\|f\|_{L^p_s} = \|J^s f\|_p \approx \|J^s f\|_{F^s_{p2}} = \|f\|_{F^s_{p2}}.
\]

6.8. To close this section we will show that \( B^s_{pr}(\mathbb{K}) \) (\( F^s_{pr}(\mathbb{K}) \), respectively) is a retract of \( l'_s(L^p(\mathbb{K})) \) (\( L^p(\mathbb{K}), l'_r \), respectively). Let us consider mappings \( J \) and \( \mathcal{P} \) given as follows. The mapping \( J \) is defined on the elements of \( S'(\mathbb{K}) \) by
\[
(1) \quad J f = (\Phi_j * f)_{j \geq 0}.
\]
The mapping \( \mathcal{P} \) is defined for sequences \( \alpha = (\alpha_j)_{j \geq 0} \) of elements of \( S'(\mathbb{K}) \) by
\[
(2) \quad \mathcal{P} \alpha = \sum_{j=0}^{\infty} \Phi_j * \alpha_j,
\]
where the convergence is considered in \( S'(\mathbb{K}) \). We are not saying that \( \mathcal{P} \) is defined on all sequences \( \alpha = (\alpha_j)_{j \geq 0} \) of elements of \( S'(\mathbb{K}) \), but only on those sequences for which the series defining \( \mathcal{P} \alpha \) converge in \( S'(\mathbb{K}) \). It follows from the property 6.1(1) that \( \mathcal{P} J f = f \) for all \( f \in B^s_{pr}(\mathbb{K}) \cup F^s_{pr}(\mathbb{K}) \).

6.9. Theorem. The space \( B^s_{pr}(\mathbb{K}) \) is a retract of \( l'_s(L^p(\mathbb{K})) \) and \( F^s_{pr}(\mathbb{K}) \) is a retract of \( L^p(\mathbb{K}), l'_s \), for \( s \in \mathbb{R} \) and \( 1 < p, r < \infty \).

Proof. First we note that
\[
\|f\|_{B^s_{pr}} = \|J f\|_{l'_s(L^p)} \quad \text{and} \quad \|f\|_{F^s_{pr}} = \|J f\|_{L^p(l'_r)}.
\]
Since \( \tilde{\Phi}_j(x+y) = \tilde{\Phi}_j(x) \) for \( |x| > |y| \), it follows that \( \{\tilde{\Phi}_j: j \geq 0\} \) is a family of scalar multipliers uniformly bounded on \( L^p(\mathbb{K}), 1 < p < \infty \) (see [9], p. 218). Thus, using properties of the functions \( \Phi_j \) we obtain for \( \alpha = (\alpha_j)_{j \geq 0} \in S(\mathbb{K}, l'_0) \),
\[
\|\mathcal{P} \alpha\|_{B^s_{pr}} = \|(\Phi_j * \mathcal{P} \alpha)_{j \geq 0}\|_{l'_s(L^p)} \leq C\|(q^s j^r \alpha_{j'} p)_{j \geq 0}\|_{l'_r} = C\|\alpha\|_{l'_s(L^p)}.
\]
On the other hand, since \( \Phi_j(x + y) = \Phi_j(x) \) for \( |x| > |y| \), it follows from 5.3 that \((\hat{\Phi}_j)_{j \geq 0}\) is a multiplier on \( L^p(\mathbb{K}, l^r) \), \( 1 < p, r < \infty \). Consequently, by the properties of the function \( \Phi_j \) we have for \( \alpha = (\alpha_j)_{j \geq 0} \in S(\mathbb{K}, l^\infty) \),

\[
\|\mathcal{P}\alpha\|_{F_{pr}^s} = \|(\Phi_j * \mathcal{P}\alpha)_{j \geq 0}\|_{L^p(l'_s)} = \|(\Phi_j * \alpha_j)_{j \geq 0}\|_{L^p(l'_s)} = \|(q^{-j}\alpha_j)_{j \geq 0}\|_{L^p(l'_s)} \leq C\|(q^{-j}\alpha_j)_{j \geq 0}\|_{L^p(l'_s)} = C\|\alpha\|_{L^p(l'_s)}.
\]

Hence, \( \mathcal{F} \) is bounded from \( B_{pr}^s(\mathbb{K}) \) into \( L^p_s(L^p(\mathbb{K})) \) and from \( F_{pr}^s(\mathbb{K}) \) into \( L^p(\mathbb{K}, l^r_s) \), and \( \mathcal{P} \) is bounded from \( l^s_s(l^p_s(\mathbb{K})) \) into \( B_{pr}^w(\mathbb{K}) \) and from \( L^p(\mathbb{K}, l^s_s) \) into \( F_{pr}^s(\mathbb{K}) \), for \( s \in \mathbb{R} \) and \( 1 < p, r < \infty \).

6.10. Remark. Due to Theorem 6.9 it is possible to obtain interpolation theorems for the spaces \( L^p_s(\mathbb{K}) \), \( B_{pr}^s(\mathbb{K}) \) and \( F_{pr}^s(\mathbb{K}) \) as in the Euclidean case. For instance, we have (see [1], p. 153) that

\[
(L^p_s(\mathbb{K}), L^p_s(\mathbb{K})))_{\theta,r} = B_{pr}^s(\mathbb{K}),
\]

where \( s = (1-\theta)s_0 + \theta s_1 \), \( 0 < \theta < 1 \), \( s_0 \neq s_1 \), \( 1 < p < \infty \), \( 1 \leq r \leq \infty \).

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