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## **RADON-NIKODÝM PROBLEM FOR THE VARIATION OF A VECTOR MEASURE**

LILIANA JANICKA

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We consider the problem of representing the variation  $|m|$  of a vector measure  $m$  as an integral in the Dinculeanu sense with respect to  $M$ .

Throughout this paper  $(S, \Sigma)$  denotes a measurable space. If  $X$  is a Banach space, we write  $X^*$  for the dual space and  $K_X$  for the closed unit ball of  $X$ . We use brackets  $\langle \cdot, \cdot \rangle$  for the pairing between a Banach space and its dual. Let  $m: \Sigma \rightarrow X$  be a vector measure with finite variation  $|m|$ . Recall that a strongly measurable function  $f: S \rightarrow X^*$  is said to be integrable in Dinculeanu's sense if there exists a sequence  $\{f_n\}_{n \geq 1}$  of simple functions converging  $|m|$ -a.e. to  $f$  such that

$$\lim_{n, p \rightarrow \infty} \int \|f_n - f_p\| d|m| = 0,$$

i.e., the function  $\|f\|$  is  $|m|$ -integrable. Further,  $D\text{-}\int_A f dm$  denotes the Dinculeanu integral of the function  $f$  with respect to  $m$  over the set  $A$ .

It was proved in [2] that for every  $\varepsilon > 0$  there exists an  $X^*$ -valued strongly measurable function  $f$  defined on the set  $S$  such that  $\|f\| \leq 1 + \varepsilon |m|$ -a.e. and  $|m|(A) = D\text{-}\int_A f dm$  for each  $A \in \Sigma$ . We are interested in the following question: For which Banach spaces may we obtain the preceding equality when we insist that  $\|f\| = 1$  a.e.  $|m|$ ?

We begin our investigation by introducing the following property of Banach spaces. The Banach space  $X$  has property (DV) if for every equivalent norm on  $x$ , for every measurable space  $(S, \Sigma)$  for every equivalent norm on  $X$  and every vector measure  $m: \Sigma \rightarrow X$  with finite variation  $|m|$  there exists a strongly measurable function  $f: S \rightarrow X^*$  with  $\|f\| = 1$   $|m|$ -a.e. such that  $|m|(A) = D\text{-}\int_A f dm$  for each  $A \in \Sigma$ .

**THEOREM 1.** *If both  $X$  and  $X^*$  have the Radon-Nikodym Property, then  $X$  has property (DV).*

*Proof.* Let  $(S, \Sigma)$  be a measurable space and  $m: \Sigma \rightarrow X$  be a measure with finite variation  $|m|$ . Since  $X$  has RNP, there exists a strongly measurable function  $f: S \rightarrow X$  such that  $m(A) = \mathbf{B}\text{-}\int_A f dm$  for each  $A \in \Sigma$ . ( $\mathbf{B}\text{-}\int_A f dm$  denotes the Bochner integral of  $f$  with respect to  $m$  over the set  $A$ .) For every  $x \in X$  let

$$G(x) = \{x^* \in K_{X^*} : \|x^*\| = 1 \text{ and } \langle x, x^* \rangle = \|x\|\}.$$

Then  $G$  is a set-valued mapping, and  $G(x)$  is non-empty and  $w^*$ -compact for every  $x \in X$ . We now see that  $G$  is upper semi-continuous if  $X$  is endowed with the norm topology and  $K_{X^*}$  is endowed with the  $w^*$ -topology. Indeed, let  $H$  be a  $w^*$ -closed subset of  $K_{X^*}$ . It suffices to show that

$$\{x \in X : G(x) \cap H \neq \emptyset\}$$

is norm closed in  $X$ . Let  $\|x_n - x\| \rightarrow 0$ , and suppose that  $G(x_n) \cap H \neq \emptyset$ , i.e., for every  $n$  there exists  $x_n^* \in H$  such that  $\|x_n^*\| = 1$  and  $\|x_n\| = \langle x_n, x_n^* \rangle$ . Let  $x^*$  be any  $w^*$ -cluster point of  $\{x_n^*\}$ . It is not difficult to see that for every  $\varepsilon > 0$  we have  $|\|x\| - \langle x, x^* \rangle| < \varepsilon$ ; i.e. the set is norm closed. Following [7, Theorem 8], we see that the set-valued mapping  $G$  has a selector which is of the first Baire class when  $X^*$  is equipped with the norm topology. Then using [1, Lemma 4.11.13] we see that the function  $h: S \rightarrow X^*$  defined by  $h = g \circ f$  is strongly measurable. (The preceding lemma and the fact that  $f$  is strongly measurable ensures that  $h$  has essentially separable range; the strong measurability of  $f$  and the fact that  $g$  belongs to the first Baire class ensures that  $h^{-1}(u)$  is an element of the  $|m|$ -completion of  $\Sigma$  for every set  $u$  which is open in the norm topology on  $X^*$ .) But for every  $A \in \Sigma$  we have

$$|m|(A) = \int_A \|f\| d|m|.$$

Therefore following [4, Theorem 3.4.II], we have

$$\begin{aligned} |m|(A) &= \int_A \|f\| d|m| = \int_A \langle f(s), h(s) \rangle d|m|(s) \\ \mathbb{S} &= \mathbf{D}\text{-}\int_A h df|m| = \mathbf{D}\text{-}\int_A h dm. \end{aligned}$$

**PROPOSITION 2.** *If  $X$  has property (DV), then every subspace  $Y$  of  $X$  has property (DV).*

*Proof.* Let  $m: \Sigma \rightarrow Y$  be a vector measure with  $|m| < \infty$ . Since  $X$  has property (DV), there exists a strongly measurable function  $f: S \rightarrow X^*$  with  $\|f(x)\| = 1$   $|m|$ -a.e. such that  $|m|(A) = D\text{-}\int_A f dm$  for each  $A \in \Sigma$ . Define  $g: S \rightarrow Y^*$  by  $g(s) = f(s)|_{Y^*}$  (the restriction of  $f(s)$  to  $Y$ ). Of course  $g$  is strongly measurable and  $\|g(s)\| \leq \|f(s)\| = 1$ . For every  $A \in \Sigma$  we have  $D\text{-}\int_A g dm = D\text{-}\int_A f dm$  since  $m$  takes its values in  $Y$ . But

$$|m|(A) = D\text{-}\int_A f dm = D\text{-}\int_A g dm \leq \int_A \|g\| d|m| \leq |m|(A);$$

therefore  $\|g(s)\| = 1$   $|m|$ -a.e.

**PROPOSITION 3.** *Banach spaces  $l_1$  and  $c_0$  do not have property (DV).*

*Proof.* Let  $(I, \mathcal{B})$  be the unit interval with the Borel  $\sigma$ -algebra.

(1) For  $A \in \mathcal{B}$  define  $m$  by  $m(A) = (\int_A (1/2^n)r_n(t) dt)_{n=1}^\infty$ , where  $r_n$  denotes the  $n$ th Rademacher function. Then  $m$  is a vector measure with values in  $l_1$  such that  $|m| = \lambda$ , where  $\lambda$  is Lebesgue measure. (It is enough to verify this last equality on intervals of the form  $[1/2^i, 1/2^{i-1})$ .) Suppose there exists a strongly measurable function  $f: I \rightarrow l_\infty$ ,  $f(t) = (f_n(t))$ , such that  $\|f(t)\| = 1$   $\lambda$ -a.e. and  $|m|(A) = D\text{-}\int_A f dm$  for each  $A$ . Because of the definition of  $m$ , we have

$$|m|(A) = \int_A \sum_{n=1}^\infty f_n(t)(1/2^n)r_n(t) dt.$$

In particular, for  $A = [0, 1]$  we have  $\sum_{n=1}^\infty f_n(t)(1/2^n)r_n(t) = 1$   $\lambda$ -a.e. Further, it is easy to see that  $(f_n(t)) = (r_n(t))$  is the unique element of  $l_\infty$  which satisfies the preceding equality. But the function  $t \rightarrow (r_n(t))$  from  $I$  to  $l_\infty$  is not weakly measurable [9].

(2) For  $A \in \mathcal{B}$  define  $m$  by  $m(A) = (\int_A (n/n + 1)r_n(t) dt)_{n=1}^\infty$ . It is easy to verify that  $m$  is a vector measure with values in  $c_0$  and  $|m| = \lambda$ . (The last statement follows from the equality  $\sup_n (n/n + 1)r_n(t) = 1$ .) Assume there exists a strongly measurable function  $f: I \rightarrow l_1$ ,  $f(t) = (f_n(t))$  with  $\|f(t)\| = \sum_{n=1}^\infty |f_n(t)| = 1$   $\lambda$ -a.e. such that  $|m|(A) = D\text{-}\int_A f dm$  for every  $A \in \mathcal{B}$ . Then for  $A = [0, 1]$  we have

$$1 = \int_0^1 \sum_{n=1}^\infty f_n(t)(n/n + 1)r_n(t) dt,$$

i.e.,  $\sum_{n=1}^{\infty} f_n(t)(n/n+1)r_n(t) = 1$   $\lambda$ -a.e. But this is impossible since for every  $n$  we have

$$f_n(t)(n/n+1)r_n(t) \leq |f_n(t)(n/n+1)r_n(t)| < |f_n(t)|.$$

**REMARK 1.** Propositions 2 and 3 show that none of the assumptions in Theorem 1 can be omitted. Namely,  $l_1$  has RNP,  $c_0$  does not have RNP, and  $c_0$  does not have (DV). Similarly,  $l_1$  has RNP,  $l_\infty$  does not have RNP, and  $l_1$  does not have (DV).

**REMARK 2.** Since  $c_0$  does not have property (DV) and  $l_1$  has RNP, we note that (1) and (2) of the theorem in [3] are, in fact, not equivalent. The difficulty with the proof of this equivalence occurs when the author concludes that the  $w^*$ -cluster point of a sequence of strongly measurable functions is  $w^*$ -measurable. Indeed, it is well known that every pointwise cluster point of the sequence of Rademacher functions is not Lebesgue measurable. We note that there is also a difficulty with the proof that (3)  $\Rightarrow$  (1) in [3]. The author makes strong use of this Lemma 1 in this proof, and in the proof of Lemma 1 he concludes that if  $X^*$  is not separable, then  $\bigcap \ker\{x_j^*\} \neq \{\theta\}$  when the intersection is taken over a countable set of indices. However, if  $X = l_1$ , then  $X^*$  is not separable, but it does have a countable total subset. In fact, we note that this formulation of Lemma 1 is incorrect. To see this, let  $X$  be separable and let  $B$  be a countable subset of smooth points of the unit sphere which is dense in the unit sphere (Mazur's theorem provides us with the set  $B$ ). If there exist nets  $\{x_\alpha\}_{\alpha < \Omega} \subset B$  and  $\{x_\alpha^*\}_{\alpha < \Omega} \subset S(X^*)$ , with  $\langle x_\alpha, x_\alpha^* \rangle = 1$  and  $\|x_\alpha - x_\beta\| > 0$  as required in Lemma 1 of [3], then we contradict the smoothness of  $x_\alpha$  for some  $\alpha$ . Further, Theorem 5.6 of [8] shows that Lemma 2 is also incorrect as stated.

We are able to deduce a weaker version of Debieve's conjecture, however. Using the fact that  $X^*$  has the weak RNP whenever  $l_1$  does not embed in  $X$  [6]—and the results of this paper—we obtain the following result.

**COROLLARY.** *If  $X$  has property (DV), then  $X^*$  has the weak RNP.*

Unfortunately, we are not able to decide if  $X^*$  must have RNP whenever  $X$  has property (DV).

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## REFERENCES

- [1] C. Constantinescu, *Spaces of measures*, de Gruyter Studies in Mathematics 4, New York, 1984.
- [2] C. Debieve, *On a Raon-Nikodym Problem for vector-valued measures*, Pacific J. Math., **107** (1983), 335–339.
- [3] C. Debieve, *On Banach spaces having a Radon-Nikodym dual*, Pacific J. Math., **120** (1985), 327–330.
- [4] N. Dinculeanu, *Vector Measures*, Pergamon Press, Berlin, 1966.
- [5] R. Holmes, *Geometrical Functional Analysis and its Applications*, Graduate Texts in Mathematics, No. 24, Springer-Verlag.
- [6] L. Janicka, *Some measure-theoretic characterization of Banach spaces containing  $l_1$* , Bull. Acad. Polon. Sci., **27** (1979), 561–565.
- [7] J. E. Jayne and C. A. Rogers, *Borel selectors for upper semi-continuous set-valued maps*, Acta Math., **155** (1985), 41–79.
- [8] J. C. Oxtoby, *Measure and Category*, Springer-Verlag, New York, 1971.
- [9] W. Sierpinski, *Fonctions additives non completement additives et fonctions non mesurables*, Fund. Math., **30** (1938), 96–99.

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