RADON-NIKODÝM PROBLEM FOR THE VARIATION OF A VECTOR MEASURE

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We consider the problem of representing the variation $|m|$ of a vector measure $m$ as an integral in the Dinculeanu sense with respect to $M$.

Throughout this paper $(S, \Sigma)$ denotes a measurable space. If $X$ is a Banach space, we write $X^*$ for the dual space and $K_X$ for the closed unit ball of $X$. We use brackets $\langle , \rangle$ for the pairing between a Banach space and its dual. Let $m: \Sigma \to X$ be a vector measure with finite variation $|m|$. Recall that a strongly measurable function $f: S \to X^*$ is said to be integrable in Dinculeanu’s sense if there exists a sequence $\{f_n\}_{n \geq 1}$ of simple functions converging $|m|$-a.e. to $f$ such that

$$\lim_{n, p \to \infty} \int \|f_n - f_p\| \, d|m| = 0,$$

i.e., the function $\|f\|$ is $|m|$-integrable. Further, $D- \int_A f \, dm$ denotes the Dinculeanu integral of the function $f$ with respect to $m$ over the set $A$.

It was proved in [2] that for every $\varepsilon > 0$ there exists an $X^*$-valued strongly measurable function $f$ defined on the set $S$ such that $\|f\| \leq 1 + \varepsilon|m|$-a.e. and $|m|(A) = D- \int_A f \, dm$ for each $A \in \Sigma$. We are interested in the following question: For which Banach spaces may we obtain the preceding equality when we insist that $\|f\| = 1$ a.e. $|m|$?

We begin our investigation by introducing the following property of Banach spaces. The Banach space $X$ has property (DV) if for every equivalent norm on $X$, for every measurable space $(S, \Sigma)$ for every equivalent norm on $X$ and every vector measure $m: \Sigma \to X$ with finite variation $|m|$ there exists a strongly measurable function $f: S \to X^*$ with $\|f\| = 1$ $|m|$-a.e. such that $|m|(A) = D- \int_A f \, dm$ for each $A \in \Sigma$.

**Theorem 1.** If both $X$ and $X^*$ have the Radon-Nikodym Property, then $X$ has property (DV).
Proof. Let \((S, \Sigma)\) be a measurable space and \(m: \Sigma \to X\) be a measure with finite variation \(|m|\). Since \(X\) has RNP, there exists a strongly measurable function \(f: S \to X\) such that \(m(A) = \mathcal{B}\int_A f\,dm\) for each \(A \in \Sigma\). (\(\mathcal{B}\int_A f\,dm\) denotes the Bochner integral of \(f\) with respect to \(m\) over the set \(A\).) For every \(x \in X\) let

\[ G(x) = \{x^* \in K_{X^*}: \|x^*\| = 1\ \text{and} \ \langle x, x^* \rangle = \|x\|\}. \]

Then \(G\) is a set-valued mapping, and \(G(x)\) is non-empty and \(w^*\)-compact for every \(x \in X\). We now see that \(G\) is upper semi-continuous if \(X\) is endowed with the norm topology and \(K_{X^*}\) is endowed with the \(w^*\)-topology. Indeed, let \(H\) be a \(w^*\)-closed subset of \(K_{X^*}\). It suffices to show that

\[ \{x \in X: G(x) \cap H \neq \emptyset\} \]

is norm closed in \(X\). Let \(\|x_n - x\| \to 0\), and suppose that \(G(x_n) \cap H \neq \emptyset\), i.e., for every \(n\) there exists \(x^*_n \in H\) such that \(\|x^*_n\| = 1\) and \(\|x_n\| = \langle x_n, x^*_n \rangle\). Let \(x^*\) be any \(w^*\)-cluster point of \(\{x^*_n\}\). It is not difficult to see that for every \(\varepsilon > 0\) we have \(\|x\| - \langle x, x^* \rangle < \varepsilon\); i.e. the set is norm closed. Following [7, Theorem 8], we see that the set-valued mapping \(G\) has a selector which is of the first Baire class when \(X^*\) is equipped with the norm topology. Then using [1, Lemma 4.11.13] we see that the function \(h: S \to X^*\) defined by \(h = g \circ f\) is strongly measurable. (The preceding lemma and the fact that \(f\) is strongly measurable ensures that \(h\) has essentially separable range; the strong measurability of \(f\) and the fact that \(g\) belongs to the first Baire class ensures that \(h^{-1}(u)\) is an element of the \(|m|\)-completion of \(\Sigma\) for every set \(u\) which is open in the norm topology on \(x^*\).) But for every \(A \in \Sigma\) we have

\[ |m|(A) = \int_A \|f\| \,d|m|. \]

Therefore following [4, Theorem 3.4.II], we have

\[ |m|(A) = \int_A \|f\| \,d|m| = \int_A \langle f(s), h(s) \rangle \,d|m|(s) \]

\[ S = \mathcal{B}\int_A h \,df|m| = \mathcal{B}\int_A h \,dm. \]

Proposition 2. If \(X\) has property (DV), then every subspace \(Y\) of \(X\) has property (DV).
Proof. Let $m : \Sigma \to Y$ be a vector measure with $|m| < \infty$. Since $X$ has property (DV), there exists a strongly measurable function $f : S \to X^*$ with $\|f(x)\| = 1$ $|m|$-a.e. such that $|m|(A) = D- \int_A f dm$ for each $A \in \Sigma$. Define $g : S \to Y^*$ by $g(s) = f(s)|_Y$ (the restriction of $f(s)$ to $Y$). Of course $g$ is strongly measurable and $\|g(s)\| \leq \|f(s)\| = 1$.

For every $A \in \Sigma$ we have $D- \int_A g dm = D- \int_A f dm$ since $m$ takes its values in $Y$. But

$$|m|(A) = D- \int_A f dm = D- \int_A g dm \leq \int_A \|g\| d|m| \leq |m|(A);$$

therefore $\|g(s)\| = 1$ $|m|$-a.e.

Proposition 3. Banach spaces $l_1$ and $c_0$ do not have property (DV).

Proof. Let $(I, \mathcal{B})$ be the unit interval with the Borel $\sigma$-algebra.

(1) For $A \in \mathcal{B}$ define $m$ by $m(A) = (\int_A (1/2^n)r_n(t) dt)^\infty_{n=1}$, where $r_n$ denotes the $n$th Rademacher function. Then $m$ is a vector measure with values in $l_1$ such that $|m| = \lambda$, where $\lambda$ is Lebesgue measure. (It is enough to verify this last equality on intervals of the form $[1/2^i, 1/2^i - 1].$) Suppose there exists a strongly measurable function $f : I \to l_\infty$, $f(t) = (f_n(t))$, such that $\|f(t)\| = 1$ $\lambda$-a.e. and $|m|(A) = D- \int_A f dm$ for each $A$. Because of the definition of $m$, we have

$$|m|(A) = \int_A \sum_{n=1}^\infty f_n(t)(1/2^n)r_n(t) dt.$$

In particular, for $A = [0, 1]$ we have $\sum_{n=1}^\infty f_n(t)(1/2^n)r_n(t) = 1$ $\lambda$-a.e. Further, it is easy to see that $(f_n(t)) = (r_n(t))$ is the unique element of $l_\infty$ which satisfies the preceding equality. But the function $t \to (r_n(t))$ from $I$ to $l_\infty$ is not weakly measurable [9].

(2) For $A \in \mathcal{B}$ define $m$ by $m(A) = (\int_A (n/n + 1)r_n(t) dt)^\infty_{n=1}$. It is easy to verify that $m$ is a vector measure with values in $c_0$ and $|m| = \lambda$. (The last statement follows from the equality $\sup_n (n/n + 1)r_n(t) = 1$.) Assume there exists a strongly measurable function $f : I \to l_1$, $f(t) = (f_n(t))$ with $\|f(t)\| = \sum_{n=1}^\infty |f_n(t)| = 1$ $\lambda$-a.e. such that $|m|(A) = D- \int_A f dm$ for every $A \in \mathcal{B}$. Then for $A = [0, 1]$ we have

$$1 = \int_0^1 \sum_{n=1}^\infty f_n(t)(n/n + 1)r_n(t) dt,$$
i.e., $\sum_{n=1}^{\infty} f_n(t)(n/n + 1)r_n(t) = 1$ $\lambda$-a.e. But this is impossible since for every $n$ we have

$$f_n(t)(n/n + 1)r_n(t) \leq |f_n(t)(n/n + 1)r_n(t)| < |f_n(t)|.$$
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