RADON-NIKODÝM PROBLEM FOR THE VARIATION OF A VECTOR MEASURE

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We consider the problem of representing the variation $|m|$ of a vector measure $m$ as an integral in the Dinculeanu sense with respect to $M$.

Throughout this paper $(S, \Sigma)$ denotes a measurable space. If $X$ is a Banach space, we write $X^*$ for the dual space and $K_X$ for the closed unit ball of $X$. We use brackets $\langle \ , \ \rangle$ for the pairing between a Banach space and its dual. Let $m: \Sigma \rightarrow X$ be a vector measure with finite variation $|m|$. Recall that a strongly measurable function $f: S \rightarrow X^*$ is said to be integrable in Dinculeanu's sense if there exists a sequence $\{f_n\}_{n \geq 1}$ of simple functions converging $|m|$-a.e. to $f$ such that

$$\lim_{n, p \rightarrow \infty} \int \|f_n - f_p\| \, d|m| = 0,$$

i.e., the function $\|f\|$ is $|m|$-integrable. Further, $D-\int_A f \, dm$ denotes the Dinculeanu integral of the function $f$ with respect to $m$ over the set $A$.

It was proved in [2] that for every $\varepsilon > 0$ there exists an $X^*$-valued strongly measurable function $f$ defined on the set $S$ such that $\|f\| \leq 1 + \varepsilon|m|$-a.e. and $|m|(A) = D-\int_A f \, dm$ for each $A \in \Sigma$. We are interested in the following question: For which Banach spaces may we obtain the preceding equality when we insist that $\|f\| = 1$ a.e. $|m|$?

We begin our investigation by introducing the following property of Banach spaces. The Banach space $X$ has property (DV) if for every equivalent norm on $x$, for every measurable space $(S, \Sigma)$ for every equivalent norm on $X$ and every vector measure $m: \Sigma \rightarrow X$ with finite variation $|m|$ there exists a strongly measurable function $f: S \rightarrow X^*$ with $\|f\| = 1$ $|m|$-a.e. such that $|m|(A) = D-\int_A f \, dm$ for each $A \in \Sigma$.

**Theorem 1.** If both $X$ and $X^*$ have the Radon-Nikodym Property, then $X$ has property (DV).
Proof. Let \( (S, \Sigma) \) be a measurable space and \( m: \Sigma \to X \) be a measure with finite variation \(|m|\). Since \( X \) has RNP, there exists a strongly measurable function \( f: S \to X \) such that \( m(A) = \text{B-}\int_A f \, dm \) for each \( A \in \Sigma \). (\( \text{B-}\int_A f \, dm \) denotes the Bochner integral of \( f \) with respect to \( m \) over the set \( A \).) For every \( x \in X \) let

\[
G(x) = \{ x^* \in K_{X^*} : \|x^*\| = 1 \text{ and } \langle x, x^* \rangle = \|x\| \}.
\]

Then \( G \) is a set-valued mapping, and \( G(x) \) is non-empty and \( w^* \)-compact for every \( x \in X \). We now see that \( G \) is upper semi-continuous if \( X \) is endowed with the norm topology and \( K_{X^*} \) is endowed with the \( w^* \)-topology. Indeed, let \( H \) be a \( w^* \)-closed subset of \( K_{X^*} \). It suffices to show that

\[
\{ x \in X : G(x) \cap H \neq \emptyset \}
\]

is norm closed in \( X \). Let \( \|x_n - x\| \to 0 \), and suppose that \( G(x_n) \cap H \neq \emptyset \), i.e., for every \( n \) there exists \( x_n^* \in H \) such that \( \|x_n^*\| = 1 \) and \( \|x_n\| = \langle x_n, x_n^* \rangle \). Let \( x^* \) be any \( w^* \)-cluster point of \( \{ x_n^* \} \). It is not difficult to see that for every \( \varepsilon > 0 \) we have \( \|x\| - \langle x, x^* \rangle < \varepsilon \); i.e. the set is norm closed. Following \([7, \text{Theorem 8}]\), we see that the set-valued mapping \( G \) has a selector which is of the first Baire class when \( X^* \) is equipped with the norm topology. Then using \([1, \text{Lemma 4.11.13}]\) we see that the function \( h: S \to X^* \) defined by \( h = g \circ f \) is strongly measurable. (The preceding lemma and the fact that \( f \) is strongly measurable ensures that \( h \) has essentially separable range; the strong measurability of \( f \) and the fact that \( g \) belongs to the first Baire class ensures that \( h^{-1}(u) \) is an element of the \(|m|\)-completion of \( \Sigma \) for every set \( u \) which is open in the norm topology on \( x^* \).) But for every \( A \in \Sigma \) we have

\[
|m|(A) = \int_A \|f\| \, dm.
\]

Therefore following \([4, \text{Theorem 3.4.II}]\), we have

\[
|m|(A) = \int_A \|f\| \, dm = \int_A \langle f(s), h(s) \rangle \, dm(s)
\]

\[
S = \text{D-} \int_A h \, df \, dm = \text{D-} \int_A h \, dm.
\]

**Proposition 2.** If \( X \) has property (DV), then every subspace \( Y \) of \( X \) has property (DV).
Proof. Let \( m : \Sigma \rightarrow Y \) be a vector measure with \( |m| < \infty \). Since \( X \) has property (DV), there exists a strongly measurable function \( f : S \rightarrow X^* \) with \( \|f(x)\| = 1 \) \( |m| \)-a.e. such that \( |m|(A) = \text{D-} \int_A f \, dm \) for each \( A \in \Sigma \). Define \( g : S \rightarrow Y^* \) by \( g(s) = f(s)|_{Y^*} \) (the restriction of \( f(s) \) to \( Y \)). Of course \( g \) is strongly measurable and \( \|g(s)\| \leq \|f(s)\| = 1 \). For every \( A \in \Sigma \) we have \( \text{D-} \int_A g \, dm = \text{D-} \int_A f \, dm \) since \( m \) takes its values in \( Y \). But

\[
|m|(A) = \int_A f \, dm = \int_A g \, dm \leq \int_A \|g\| \, d|m| \leq |m|(A);
\]

therefore \( \|g(s)\| = 1 \) \( |m| \)-a.e.

**Proposition 3.** Banach spaces \( l_1 \) and \( c_0 \) do not have property (DV).

Proof. Let \((I, \mathcal{B})\) be the unit interval with the Borel \( \sigma \)-algebra.

(1) For \( A \in \mathcal{B} \) define \( m \) by \( m(A) = (\int_A (1/2^n) r_n(t) \, dt)_{n=1}^{\infty} \), where \( r_n \) denotes the \( n \)th Rademacher function. Then \( m \) is a vector measure with values in \( l_1 \) such that \( |m| = \lambda \), where \( \lambda \) is Lebesgue measure. (It is enough to verify this last equality on intervals of the form \( [1/2^i, 1/2^{i-1}] \).) Suppose there exists a strongly measurable function \( f : I \rightarrow l_\infty \), \( f(t) = (f_n(t)) \), such that \( \|f(t)\| = 1 \) \( \lambda \)-a.e. and \( |m|(A) = \text{D-} \int_A f \, dm \) for each \( A \). Because of the definition of \( m \), we have

\[
|m|(A) = \int_A \sum_{n=1}^{\infty} f_n(t)(1/2^n) r_n(t) \, dt.
\]

In particular, for \( A = [0, 1] \) we have \( \sum_{n=1}^{\infty} f_n(t)(1/2^n) r_n(t) = 1 \) \( \lambda \)-a.e. Further, it is easy to see that \( (f_n(t)) = (r_n(t)) \) is the unique element of \( l_\infty \) which satisfies the preceding equality. But the function \( t \rightarrow (r_n(t)) \) from \( I \) to \( l_\infty \) is not weakly measurable [9].

(2) For \( A \in \mathcal{B} \) define \( m \) by \( m(A) = (\int_A (n/n+1) r_n(t) \, dt)_{n=1}^{\infty} \). It is easy to verify that \( m \) is a vector measure with values in \( c_0 \) and \( |m| = \lambda \). (The last statement follows from the equality \( \sup_n (n/n+1) r_n(t) = 1 \).) Assume there exists a strongly measurable function \( f : I \rightarrow l_1 \), \( f(t) = (f_n(t)) \) with \( \|f(t)\| = \sum_{n=1}^{\infty} |f_n(t)| = 1 \) \( \lambda \)-a.e. such that \( |m|(A) = \text{D-} \int_A f \, dm \) for every \( A \in \mathcal{B} \). Then for \( A = [0, 1] \) we have

\[
1 = \int_0^1 \sum_{n=1}^{\infty} f_n(t)(n/n+1) r_n(t) \, dt,
\]
i.e., \( \sum_{n=1}^{\infty} f_n(t)(n/n + 1)r_n(t) = 1 \) \( \lambda \)-a.e. But this is impossible since for every \( n \) we have
\[
f_n(t)(n/n + 1)r_n(t) \leq |f_n(t)(n/n + 1)r_n(t)| < |f_n(t)|.
\]

**Remark 1.** Propositions 2 and 3 show that none of the assumptions in Theorem 1 can be omitted. Namely, \( l_1 \) has RNP, \( c_0 \) does not have RNP, and \( c_0 \) does not have (DV). Similarly, \( l_1 \) has RNP, \( l_\infty \) does not have RNP, and \( l_1 \) does not have (DV).

**Remark 2.** Since \( c_0 \) does not have property (DV) and \( l_1 \) has RNP, we note that (1) and (2) of the theorem in [3] are, in fact, not equivalent. The difficulty with the proof of this equivalence occurs when the author concludes that the \( w^* \)-cluster point of a sequence of strongly measurable functions is \( w^* \)-measurable. Indeed, it is well known that every pointwise cluster point of the sequence of Rademacher functions is not Lebesgue measurable. We note that there is also a difficulty with the proof that (3) \( \Rightarrow \) (1) in [3]. The author makes strong use of this Lemma 1 in this proof, and in the proof of Lemma 1 he concludes that if \( X^* \) is not separable, then \( \bigcap \ker \{x_j^*\} \neq \{\theta\} \) when the intersection is taken over a countable set of indices. However, if \( X = l_1 \), then \( X^* \) is not separable, but it does have a countable total subset. In fact, we note that this formulation of Lemma 1 is incorrect. To see this, let \( X \) be separable and let \( B \) be a countable subset of smooth points of the unit sphere which is dense in the unit sphere (Mazur’s theorem provides us with the set \( B \)). If there exist nets \( \{x_\alpha\}_{\alpha<\Omega} \subset B \) and \( \{x_\alpha^*\}_{\alpha<\Omega} \subset S(X^*) \), with \( \langle x_\alpha, x_\alpha^* \rangle = 1 \) and \( \|x_\alpha - x_\beta\| > 0 \) as required in Lemma 1 of [3], then we contradict the smoothness of \( x_\alpha \) for some \( \alpha \). Further, Theorem 5.6 of [8] shows that Lemma 2 is also incorrect as stated.

We are able to deduce a weaker version of Debieve’s conjecture, however. Using the fact that \( X^* \) has the weak RNP whenever \( l_1 \) does not embed in \( X \) [6]—and the results of this paper—we obtain the following result.

**Corollary.** If \( X \) has property (DV), then \( X^* \) has the weak RNP.

Unfortunately, we are not able to decide if \( X^* \) must have RNP whenever \( X \) has property (DV).

**Acknowledgment.** I am very grateful to Professor C. Ryll-Nardzewski and to my colleagues participating in his seminar for helpful comments.
References


Received September 19, 1988.

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Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924 (ISSN 0030-8730) is published monthly. Second-class postage paid at Carmel Valley, California 93924, and additional mailing offices. Postmaster: send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

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