ON THE PROJECTIVE NORMALITY OF SOME VARIETIES OF DEGREE 5

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We give some sufficient conditions for projective normality of complete non-singular varieties of degree five. And we prove that every complete non-singular surfaces of degree five embedded by a complete linear system is projectively normal.

Introduction. Let $X$ be a complete non-singular variety over an algebraically closed field, and let $L$ be an ample line bundle on $X$. The classification of some $(X, L)$ is found in Fugita’s papers (Fujita [1], [2], [3], [4]). In this paper, we consider the projective normality of $(X, L)$ and the defining equations. This problem is trivial in the case of $(D^n) = 1, 2$ where $n = \dim X$ and $\mathcal{L} = \mathcal{O}(D)$. If $(D^n) = 3$, then $(X, L)$ is projectively normal and the ideal is generated by degree 2 and 3 (X.X.X. [11]). If $(D^n) = 4$, then $(X, L)$ is projectively normal and the ideal is generated by degree 2 and 3 (Swinnerton-Dyer [10]). So we consider the case of $(D^n) = 5$. In this paper we give some sufficient conditions for projective normality of varieties of degree 5 and give the generator of the defining ideal. The main part of this paper is the case of $(D^n) = 5$ and $\Delta(X, L) = 2$ (other cases are clearly obtained by Fujita’s theory). This is a non-degenerate and non-singular variety of codimension 2 in some projective space $\mathbb{P}^N$. On the other hand, the following conjecture is known as a conjecture of Hartshorne.

Conjecture (cf. Hartshorne [6]). If $X \subset \mathbb{P}^N$ is a non-singular closed subvariety and $\dim X > 2N/3$, then $X$ is a complete intersection.

If this conjecture is true, then we obtain that every non-degenerate and non-singular variety which is degree 5 and codimension 2 is not contained in $\mathbb{P}^N$ for $N \geq 7$. As every non-singular variety is projectively normal if it is a complete intersection, therefore the results in this paper are recognized as a step to prove the above conjecture. Throughout this paper, variety means a complete non-singular variety.
Notations.

\((D_1 \cdots D_n)\): The intersection number of divisors \(D_1, \ldots, D_n\) on a variety \(X\) where \(n = \dim X\).

\(O_X\): The structure sheaf of a variety \(X\).

\(L_Y\): The restriction of a line bundle \(L\) to a subscheme \(Y\).

\(H^i(X, \mathcal{F})\): The \(i\)th cohomology group of a sheaf \(\mathcal{F}\).

\(h^i(X, \mathcal{F})\): The dimension of \(H^i(X, \mathcal{F})\) as a vector space.

\(|D|\): The complete linear system defined by a divisor \(D\).

\(\phi|_D\): The rational map defined by \(|D|\).

\(\mathcal{L}\): The invertible sheaf associated to a line bundle \(L\).

\(\mathcal{O}(D)\): The invertible sheaf associated to a divisor \(D\).

\(\mathbb{P}(E)\): The projective bundle defined by a vector bundle \(E\).

\(K_X\): The canonical divisor on a non-singular variety \(X\).

\(\mathcal{O}_X(k)\): The sheaf \(\mathcal{O}_X \otimes \mathcal{O}_X(k)\) for a projective variety \(X\) embedded in \(\mathbb{P}^n\).

1. Preliminary. We give several theorems from Fujita's theory.

**Definition ([2])**. Let \(X\) be a non-singular variety and let \(L\) be an ample line bundle. We define a \(\Delta\)-genus of \((X, L)\) by

\[
\Delta(X, L) = (D^n) + n - h^0(X, L)
\]

where \(n = \dim X\) and \(L = \mathcal{O}(D)\).

The above pair \((X, L)\) is called a polarized non-singular variety.

**Definition ([8])**. Let \((X, L)\) be a polarized non-singular variety. We say that \(L\) is normally generated if

\[
H^0(X, \mathcal{L})^\otimes k \to H^0(X, \mathcal{L}^\otimes k)
\]

is surjective for any positive integer \(k\). And in this case, we call \((X, L)\) projectively normal.

**Definition ([2])**. Let \((X, L)\) be a polarized non-singular variety and set \(L = \mathcal{O}(D)\). Let \(V\) be a reduced irreducible non-singular member of \(|D|\) (if there exists). We call \(V\) a regular member if

\[
H^0(X, \mathcal{L}) \to H^0(V, \mathcal{L}_V)
\]

is surjective.

**Definition ([2])**. Let \((X, L)\) be a polarized non-singular variety. We define \(g(X, L)\) by

\[
2g(X, L) - 2 = ((K_X + (n - 1)D).D^{n-1})
\]

where \(L = \mathcal{O}(D)\) and \(n = \dim X\). We call this \(g(X, L)\) a sectional genus of \((X, L)\).
If $L$ is very ample, then this $g(X, L)$ is the genus of the generic curve section of $X$ in the projective embedding defined by $L$.

**Theorem A ([2]).** Let $(X, L)$ be a polarized non-singular variety. If $V$ is a reduced irreducible non-singular member of $|D|$ where $\mathcal{L} = \mathcal{O}(D)$, then $\Delta(V, L_V) \leq \Delta(X, L)$. Moreover the following conditions are equivalent:

(a) $\Delta(X, L) = \Delta(V, L_V)$,
(b) $V$ is a regular member.

**Proof.** As $0 \to \mathcal{O}_X \to \mathcal{L} \to \mathcal{L}_V \to 0$ is exact, therefore

$$ h^0(V, \mathcal{L}_V) \geq h^0(X, \mathcal{L}) - 1. $$

Hence $\Delta(X, L) - \Delta(V, L_V) = h^0(V, \mathcal{L}_V) - h^0(X, \mathcal{L}) + 1 \geq 0$, because $(D^n) = (D|_{V^n-1})$ where $\mathcal{L} = \mathcal{O}(D)$. By the above equation, the last part of this theorem is clear.

**Theorem B.** If $X$ is a variety and $L$ is a very ample line bundle, then $\Delta(X, L) \geq 0$.

**Proof.** It is a well-known fact (see Fujita [1]).

**Theorem C.** Let $(X, L)$ be a polarized non-singular variety. If $\Delta(X, L) = 0$, then $(X, L)$ is isomorphic to $(\mathbb{P}(E), H_E)$ or $(\mathbb{P}^2, H_{p^2}(2))$ where $E$ is a vector bundle on $\mathbb{P}^1$, $H_E$ is a tautological bundle on $\mathbb{P}(E)$ and $H_{p^2}(i) = \mathcal{O}(i)$ on $\mathbb{P}^2$ ($i \in \mathbb{Z}$).

**Proof.** This is a well-known classical theorem (see Fujita [1]).

**Theorem D ([2]).** Let $(X, L)$ be a polarized non-singular variety. If $g(X, L) = 0$ and $L$ is very ample, then $\Delta(X, L) = 0$.

**Proof.** We prove this theorem by the induction on $n = \dim X$. If $n = 1$, then this theorem is trivial. We may assume that $n \geq 2$. Let $V$ be a reduced irreducible non-singular member of $|D|$ where $\mathcal{L} = \mathcal{O}(D)$. By the induction hypothesis, we assume $\Delta(V, L_V) = 0$ because $g(V, L_V) = g(X, L) = 0$. Hence $H^1(V, \mathcal{L}_V^{(t+1)}) = 0$ for every $t \geq 0$ by Theorem C. Therefore the long exact sequence

$$ \cdots \to H^1(X, \mathcal{L}^{(t+1)}) \to H^1(X, \mathcal{L}^{(t)}) \to H^1(V, \mathcal{L}_V^{(t)}) $$

says that $h^1(X, \mathcal{L}^{(t+1)}) \geq h^1(X, \mathcal{L}^{(t)})$ for any $t \geq 0$. As $H^1(X, \mathcal{L}^{(-s)}) = 0$
for sufficiently large $s$, we obtain $H^1(X, \mathcal{O}_X) = 0$. Therefore $V$ is a regular member. Hence we obtain this theorem.

**Theorem E.** Let $(X, L)$ be a polarized non-singular variety and let $d = (D^n)$ where $\mathcal{L} = \mathcal{O}(D)$ and $n = \dim X$. Moreover we assume that $\Delta(X, L) \leq g(X, L)$ and $L$ is very ample. In this case, the following are true:

(a) if $d \geq 2\Delta(X, L) - 2$, then every reduced irreducible non-singular member $V \in |D|$ is a regular member;

(b) if $d \geq 2\Delta(X, L) + 1$, then $(X, L)$ is projectively normal and $\Delta(X, L) = g(X, L)$;

(c) if $d \geq 2\Delta(X, L) + 2$, then the ideal of $(X, L)$ is generated by degree 2.

**Proof.** See Fujita [2]. As $L$ is very ample, the proof is the same in the case of characteristic $p > 0$.

**Theorem F.** Let $X \subset \mathbb{P}^N$ be a closed non-singular subvariety which is not contained in any hyperplane. If the degree of $X$ is 4, then $X$ is of the following type:

(a) hypersurface,

(b) $(2, 2)$ complete intersection,

(c) Segre variety $\mathbb{P}^1 \times \mathbb{P}^3$ in $\mathbb{P}^7$,

(d) Veronese surface $\mathbb{P}^2$ in $\mathbb{P}^5$,

(e) the variety obtained by hyperplane section or projection of (a), (b), (c), (d), (e).

**Proof.** See Swinnerton-Dyer [10].

By the above theorems, we obtain that $(X, L)$ is projectively normal for $(D^n) = 3, 4$ where $\mathcal{L} = \mathcal{O}(D)$ and $n = \dim X$. Moreover $(X, L)$ is also projectively normal if $(D^n) = 5$ and the codimension of $\phi_{|D|}(X)$ is 1, 3, 4. So we consider the case that $(D^n) = 5$ and the codimension of $\phi_{|D|}(X)$ is 2.

### 2. Codimension 2 case.

Throughout §2, we assume that $h^0(X, \mathcal{L}) = n + 3$ where $n = \dim X$, $\mathcal{L} = \mathcal{O}(D)$, $(D^n) = 5$ and $L$ is very ample. In this case, $g(X, L) = 1$ or 2 because $g(X, L) = 0$ implies that $\Delta(X, L) = 0$ by the Theorem D. This contradicts $(D^n) = 5$ and $h^0(X, L) = n + 3$. If $g(X, L) \geq 2$, then $g(X, L) = 2$ by Theorem E in §1.
**Theorem 1.** If $g(X, L) = 2$, then $(X, L)$ is projectively normal and the defining ideal of $(X, L)$ is generated by degree 2 and 3.

To prove this theorem, we prepare two lemmas.

**Lemma 1.** Let $(X, L)$ be as above. Let $V$ be a reduced irreducible non-singular member of $|D|$. If the homogeneous ideal of $(V, L_V)$ is generated by degree 2 and 3, then the homogeneous ideal of $(X, L)$ is generated by degree 2 and 3.

**Proof.** Let $I(k)$ be the polynomials defined by

$$I(k) = \ker[S^k H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}^\otimes k)]$$

where $S^k$ is a $k$th symmetric product and let $I_V(k)$ be the polynomials defined by

$$I_V(k) = \ker[S^k H^0(V, \mathcal{L}_V) \to H^0(V, \mathcal{L}_V^\otimes k)].$$

We prove this lemma by induction on $k$. In the case of $k = 2, 3$, this lemma is trivial. We assume that $I(k)$ is generated by $I(2)$ and $I(3)$. By Theorem E (a) in §1, $V$ is a regular member. Moreover $(X, L)$ and $(V, L_V)$ are projectively normal by Theorem E(b) in §1. Therefore we obtain the following diagram:

$$
\begin{array}{cccc}
0 & \to & I(k) & \to & I(k+1) & \to & I_V(k+1) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & S^k H^0(X, \mathcal{L}) & \to & S^{k+1} H^0(X, \mathcal{L}) & \to & S^{k+1} H^0(V, \mathcal{L}_V) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^0(X, \mathcal{L}^\otimes k) & \to & H^0(X, \mathcal{L}_V^\otimes(k+1)) & \to & H^0(V, \mathcal{L}_V^\otimes(k+1)) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
$$

By the snake lemma, $\pi$ is a surjective map. By the assumption, $I_V(k+1)$ is generated by degree 2 and 3. Therefore $I(k+1)$ is generated by degree 2 and 3.

**Lemma 2.** If $C$ is a non-singular curve and $L$ is a very ample line bundle on $C$ and $\Delta(C, L) = 2$, then $(C, L)$ is projectively normal and its ideal is generated by degree 2 and 3.

**Proof.** See Saint-Donat [9].

**Proof of Theorem 1.** It is clear by Lemma 1 and Lemma 2.
Next we prepare the following notation.

**DEFINITION.** Let \((X, L)\) be a polarized non-singular variety and let \(L\) be a very ample line bundle. We define \(c(X, L)\) by

\[
c(X, L) = \min \{ \sum_{i} X_i \mid X = X_n \supset X_{n-1} \supset \cdots \supset X_1 \supset \cdots \supset X_1 \text{ with } X_i \text{ being a reduced irreducible non-singular member of } |D_{i+1}| \text{ where } L_{X_i} = \mathcal{O}(D_i) \text{ and } \Delta(X_n, L_{X_n}) = \cdots = \Delta(X_i+1, L_{X_{i+1}}) > \Delta(X_i, L_{X_i}) \}
\]

where \(n = \dim X\). In the case of \(\Delta(X_1, L_{X_1}) = \Delta(X, L)\), we put \(c(X, L) = 0\).

If \(\Delta(X, L) = 2\) and \(g(X, L) = 2\), then \(c(X, L) = 0\). If \(\Delta(X, L) = 2\) and \(g(X, L) = 1\), then \(1 \leq c(X, L) \leq \dim X - 1\). Therefore, Theorem 1 is in the case of \(c(X, L) = 0\).

**THEOREM 2.** If \(c(X, L) = 1\), then \((X, L)\) is projectively normal and the ideal defining \((X, L)\) is generated by degree 3.

We prepare the following two lemmas.

**LEMMA 3.** If \(C \subset \mathbb{P}^3\) is a non-singular elliptic curve of degree 5 which is not contained in any hyperplane, then

\[
H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k)) \rightarrow H^0(C, \mathcal{O}_C(k))
\]

is surjective for every \(k \geq 2\).

**Proof.** Let \(\mathcal{O}_C(1) = \mathcal{O}(D)\). We obtain the following diagram:

\[
\begin{array}{ccc}
\mathbb{P}^4 & \rightarrow & \mathbb{P}^3 \\
\phi|_{\mathcal{O}_D} \downarrow & & \downarrow \text{projection} \\
C \leftarrow & &
\end{array}
\]

As \((C, \mathcal{O}(D))\) is projectively normal, hence

\[
H^0(C, \mathcal{O}_C(k)) \otimes H^0(C, \mathcal{O}_C(m)) \rightarrow H^0(C, \mathcal{O}_C(k + m))
\]

is surjective for every \(k, m \geq 1\). By the assumption, the canonical map

\[
H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \rightarrow H^0(C, \mathcal{O}_C(1))
\]

is injective. Now we show that

\[
H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \rightarrow H^0(C, \mathcal{O}_C(2))
\]
is an isomorphism. As $h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = h^0(C, \mathcal{O}_C(2)) = 10$, therefore we may show that

$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \rightarrow H^0(C, \mathcal{O}_C(2))$

is injective. If this is not true, then there exists some quadratic surface $Q$ in $\mathbb{P}^3$ with $Q \supset C$. If $Q$ is non-singular, then the degree of $C = a + b$ and the genus of $C = ab - a - b + 1$ for some integers $a$, $b$. This cannot occur because the degree of $C = 5$ and the genus of $c = 1$. If $Q$ is singular, then the genus of $C = a^2 - a$ for odd degree $2a + 1$ of $C$. Hence degree of $C = 5$ and genus of $C = 1$ does not occur. Therefore the above map is injective, hence is an isomorphism.

Next we show that $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) \rightarrow H^0(C, \mathcal{O}_C(3))$ is surjective. We take the basis of $H^0(C, \mathcal{O}_C(1))$ with

$$
H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) = [x_0, x_1, x_2, x_3], \\
H^0(C, \mathcal{O}_C(1)) = [x_0, x_1, x_2, x_3, x_4]
$$

where $[x_0, \ldots, x_N]$ means that $x_1, \ldots, x_N$ are bases of a vector space. As

$$
H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = [x_0^2, x_1^2, x_2^2, x_3^2, x_0x_1, x_0x_2, x_0x_3, x_1x_2, x_1x_3, x_2x_3]
$$

and $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \cong H^0(C, \mathcal{O}_C(2))$, therefore $H^0(C, \mathcal{O}_C(2))$ has the above basis. But $x_ix_4$ $(i = 0, \ldots, 4)$ are contained in $H^0(C, \mathcal{O}_C(2))$, and therefore we obtain the following relations:

$$
(*) \quad x_ix_4 = f_i(x_0, x_1, x_2, x_3)
$$

where $i = 0, 1, 2, 3, 4$ and $f_i$ $(i = 1, 2, 3, 4)$ are homogeneous polynomials of degree 2. As $(C, \mathcal{O}_C(1))$ is projectively normal, hence

$$
H^0(C, \mathcal{O}_C(1)) \otimes \mathbb{P}^3 \rightarrow H^0(C, \mathcal{O}_C(3))
$$

is surjective. Therefore we obtain the generators of $H^0(C, \mathcal{O}_C(3))$ as follows,

$$
(1) \quad \left\{ \begin{array}{l}
x_0^3, x_1^3, x_2^3, x_3^3 \\
x_0^2x_1, x_0^2x_2, x_0^2x_3, x_1^2x_0, x_1^2x_2, x_1^2x_3 \\
x_2^2x_0, x_2^2x_1, x_2^2x_3, x_3^2x_0, x_3^2x_1, x_3^2x_2 \\
x_0x_1x_2, x_0x_1x_3, x_0x_2x_3, x_1x_2x_3
\end{array} \right.
$$

$$
(2) \quad \left\{ \begin{array}{l}
x_4^3, x_4^2x_0, x_4^2x_1, x_4^2x_2, x_4^2x_3 \\
x_4x_0^2, x_4x_1^2, x_4x_2^2, x_4x_3^2 \\
x_4x_0x_1, x_4x_0x_2, x_4x_0x_3, x_4x_1x_2, x_4x_1x_3, x_4x_2x_3.
\end{array} \right.
$$
The part (1) is clearly the image of $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$. And the relation (*) says that the part (2) is also in the image of $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$. Because

$$x_4 x_i x_j = f_i(x_0, x_1, x_2, x_3) x_j \quad (i, j \neq 4),$$
$$x_4^2 x_i = f_4(x_0, x_1, x_2, x_3) x_i \quad (i = 0, 1, 2, 3),$$
$$x_4^3 = f_4(x_0, x_1, x_2, x_3) x_4$$

by the relation (*); moreover the relation (*) says $f_4 x_4$ is in the image of $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$. Hence

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(C, \mathcal{O}_C(3))$$

is surjective. Finally we prove this lemma. If $k = 2, 3$, then this lemma is true by the above argument. We consider the case in which $k \geq 4$. First, we show this lemma in the case that $k$ is even. Let $k = 2m$. We show in this case by the induction on $m$. In this, we give the following diagram:

$$
\begin{array}{ccc}
H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2m)) & \to & H^0(C, \mathcal{O}_C(2m)) \\
\uparrow & & \uparrow \\
H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2(m-1))) \otimes H^0(\mathbb{P}^3, \mathcal{O}(2)) & \to & H^0(C, \mathcal{O}_{\mathbb{P}^3}(2(m-1))) \otimes H^0(C, \mathcal{O}(2))
\end{array}
$$

By the hypothesis of induction and projective normality of $(C, \mathcal{O}_C(1))$, we obtain

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2m)) \to H^0(C, \mathcal{O}_C(2m))$$

is surjective. Next we consider the case in which $k$ is odd and $k \geq 5$. But this case is clear by the same argument. Therefore we obtain this lemma.

**Lemma 4.** If $C \subset \mathbb{P}^3$ is as in Lemma 3, then the homogeneous ideal of $C \subset \mathbb{P}^3$ is generated by degree 3.

**Proof.** Let $I_k$ be the kernel of $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k)) \to H^0(C, \mathcal{O}_C(k))$. We show that

$$I_k \otimes H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \to I_{k+1}$$

is surjective for every $k \geq 3$. We take a divisor $D$ with $\mathcal{O}_C(1) \cong \mathcal{O}(D)$ and support of $D$ consists of 5 distinct points. As $D \subset \mathbb{P}^2$, we define $I'_k$ ($k = 1, 2, \ldots$) by

$$0 \to I'_k \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k)) \to H^0(D, \mathcal{O}_D(k)) \cong H^0(D, \mathcal{O}_D).$$
If \( k \geq 2 \), then we give the following diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & I_k \\
\downarrow & & \downarrow \\
I_{k+1} & \rightarrow & I'_{k+1} \\
\downarrow & & \downarrow \\
H^0(\mathbb{P}^3, \mathcal{O}(k)) & \rightarrow & H^0(\mathbb{P}^3, \mathcal{O}(k+1)) \\
\downarrow & & \downarrow \\
H^0(\mathbb{P}^2, \mathcal{O}(k+1)) & \rightarrow & 0 \\
\end{array}
\]

By the snake lemma,

\[
0 \rightarrow I_k \rightarrow I_{k+1} \rightarrow I'_{k+1} \rightarrow 0
\]

is exact for every \( k \geq 2 \). Moreover we define \( \lambda \) so the following diagram commutes:

\[
\begin{array}{ccccc}
I_k \otimes H^0(\mathbb{P}^3, \mathcal{O}_p^3(1)) & \rightarrow & I'_k \otimes H^0(\mathbb{P}^3, \mathcal{O}_p^3(1)) \\
\downarrow & & \downarrow \\
I'_k \otimes H^0(\mathbb{P}^2, \mathcal{O}_p^2(1)) & \leftarrow & \lambda \\
\end{array}
\]

As \( I_k \rightarrow I'_k \) is surjective if \( k \geq 3 \) and \( H^0(\mathbb{P}^3, \mathcal{O}_p^3(1)) \rightarrow H^0(\mathbb{P}^2, \mathcal{O}_p^2(1)) \) is surjective, therefore \( \lambda \) is surjective for \( k \geq 3 \). Next we define \( \psi: I_k \rightarrow I_k \otimes H^0(\mathbb{P}^3, \mathcal{O}_p^3(1)) \) with \( \psi(s) = s \otimes \delta \) where \( \delta \) is a section of \( H^0(\mathbb{P}^3, \mathcal{O}_p^3(1)) \) which is defining \( \mathbb{P}^2 \). This shows that the following diagram

\[
\begin{array}{ccc}
0 & \rightarrow & I_k \\
\downarrow & & \downarrow \\
I_{k+1} & \rightarrow & I'_{k+1} \\
\downarrow & & \downarrow \\
H^0(\mathbb{P}^3, \mathcal{O}_p^3(1)) & \rightarrow & I'_k \otimes H^0(\mathbb{P}^2, \mathcal{O}_p^2(1)) \\
\end{array}
\]

is commutative for \( k \geq 2 \). Therefore if \( I'_k \otimes H^0(\mathbb{P}^2, \mathcal{O}_p^2(1)) \rightarrow I'_{k+1} \) is surjective for every \( k \geq 3 \), then this lemma is proved. So we show that

\[
I'_k \otimes H^0(\mathbb{P}^2, \mathcal{O}_p^2(1)) \rightarrow I'_{k+1}
\]

is surjective for \( k \geq 3 \). Let \( V = H^0(\mathbb{P}^2, \mathcal{O}_p^2(1)) \) and let \( V^k = \) the image of \( H^0(\mathbb{P}^2, \mathcal{O}_p^2(k)) \rightarrow H^0(D, \mathcal{O}_D(K)) \). As the support of \( D \) is not collinear, \( V \rightarrow H^0(D, \mathcal{O}_D(1)) \) is injective. We show that \( V^k = H^0(D, \mathcal{O}_D(k)) \) for \( k \geq 2 \). If \( V \neq H^0(D, \mathcal{O}_D(2)) \), then the dimension of \( \ker[H^0(\mathbb{P}^2, \mathcal{O}_p^2(2)) \rightarrow H^0(D, \mathcal{O}_D(2))] \) is at least 2. Therefore there exist distinct quadratics \( Q_1 \) and \( Q_2 \) with \( Q_i \supset D \ (i = 1, 2) \). \( Q_1 \) and \( Q_2 \) satisfy \( Q_1 \cap Q_2 = \) finite points. Because if \( Q_1 \cap Q_2 \) has component, then there exist distinct lines \( l_1, l_2, l_3 \) with

\[
l_1 \cap D = 4 \text{ points}
\]
and

\[ Q_1 = l_1 + l_2, \quad Q_2 = l_1 + l_3. \]

Hence \( \mathbb{P}^3 - l_1 \rightarrow \mathbb{P}^1 \) be a projection with center \( l_1 \), and let \( C \rightarrow \mathbb{P}^1 \) be a restriction map to \( C \). Let \( f: C \rightarrow \mathbb{P}^1 \) be an associated morphism defined by the above map \( C \rightarrow \mathbb{P}^1 \). As \( l_1 \cap D = 4 \) points, therefore \( f \) is a bijective morphism. Hence the genus of \( C \) = the genus of \( \mathbb{P}^1 = 0 \). This is a contradiction. So \( Q_1 \cap Q_2 = \) finite points. As \( Q_1 \) and \( Q_2 \) are conics, \( Q_1 \cap Q_2 \) contains at most 4 points by Bezout’s theorem. But \( Q_1 \cap Q_2 \) contains \( D \) with degree 5; this is a contradiction. Hence \( V^2 = H^0(D, \mathcal{O}_D(2)) \). We take \( s \in V \) with

\[
\begin{array}{ccc}
H^0(D, \mathcal{O}_D(k)) & \simeq & H^0(D, \mathcal{O}_D(k + 1)). \\
\downarrow & & \downarrow \\
t & \mapsto & ts
\end{array}
\]

In this, we obtain the following commutative diagram:

\[
\begin{array}{ccc}
H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k)) & \rightarrow & H^0(D, \mathcal{O}_D(k)) \\
\sigma \downarrow & \Leftrightarrow & \downarrow \zeta \\
H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k + 1)) & \rightarrow & H^0(D, \mathcal{O}_D(k + 1))
\end{array}
\]

where \( \sigma, \zeta \), are defined by \( f \mapsto fs \). Therefore we obtain

\[
H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k)) \rightarrow H^0(D, \mathcal{O}_D(k))
\]

is surjective if \( k \geq 2 \). Hence

\[
V^k = H^0(D, \mathcal{O}_D(k))
\]

for every \( k \geq 2 \). Let \( K(V^k, V) \) be ker\( [V^k \otimes V \rightarrow V^{k+1}] \) and \( K(V, s) \) be ker\( [V^s \rightarrow V^s] \) where \( k \) and \( s \) are positive integers. We consider the following commutative diagram:

\[
\begin{array}{ccc}
& 0 & 0 & 0 \\
\downarrow & & & \downarrow \\
K(V^{k-1}, V) \otimes V & \xrightarrow{\alpha} & K(V^k, V) & \xrightarrow{\xi} & K(V^{k-1}, V) \\
\downarrow & \Leftrightarrow & \downarrow & \Leftrightarrow & \downarrow \\
V^{k-1} \otimes V \otimes V & \xrightarrow{\beta} & V^k \otimes V & \xrightarrow{\rho} & V^{k-1} \otimes V \\
\downarrow & \Leftrightarrow & \downarrow & \Leftrightarrow & \downarrow \\
V^k \otimes V & \rightarrow & V^{k+1} & \xrightarrow{\zeta} & V^k \\
\downarrow & & \downarrow & & \downarrow \\
0 & 0 & 0
\end{array}
\]
where \( \beta(a \otimes b \otimes c) = ab \otimes c \), \( \alpha \) is induced by \( \beta \), \( \zeta(f) = fs \), \( \rho(f \otimes g) = fs \otimes g \), \( \xi \) is induced by \( \rho \) and \( s \) is an element of \( V \) defined as above. If \( k \geq 3 \), \( \rho \) and \( \zeta \) are isomorphisms. Hence we obtain that \( \alpha \) is a surjective map. Next we consider the following commutative diagram:

\[
\begin{array}{cccc}
0 & \to & K(V, k) \otimes V & \overset{u}{\to} & K(V, k+1) & \overset{\nu}{\to} & K(V^k, V) & \to 0 \\
& & w \uparrow & & \alpha \uparrow & & \\
& & K(V, k) \otimes V & \overset{\nu'}{\to} & K(V^{k-1}, V) \otimes V & \to 0
\end{array}
\]

where \( u, \nu, \nu' \) and \( w \) are canonical maps and the surjectivity of \( \nu \) and \( \nu' \) is induced by the following commutative diagram and the snake lemma:

\[
\begin{array}{cccc}
0 & \to & K(V, k) \otimes V & \overset{u}{\to} & K(V, k+1) & \overset{\nu}{\to} & K(V^k, V) & \to 0 \\
& & \text{id} \downarrow & & \downarrow & & \downarrow & \\
0 & \to & K(V, k) \otimes V & \to & V^{\otimes (k+1)} & \to & V^k \otimes V & \to 0 \\
& & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & 0 & \to & V^{k+1} & \overset{\text{id}}{\to} & V^{k+1} & \to 0 \\
& & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & 0 & \to & 0 & & 0 & \\
\end{array}
\]

Therefore \( K(V, k+1) = \text{im}(w) + \text{im}(u) \) if \( k \geq 3 \). Hence we obtain that \( I'_k \otimes H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \to I'_{k+1} \) is surjective for \( k \geq 3 \). Hence we prove this lemma.

**Proof of Theorem 2.** First we show that

\[
H^0(X, \mathcal{L}^{\otimes k}) \to H^0(X, \mathcal{L}^{\otimes k})
\]

is surjective for \( k \geq 1 \). If \( k = 1 \), then this is clear. Now we can take

\[
X = X_n \supset X_{n-1} \supset \cdots \supset X_2 \supset X_1
\]

such that \( X_i \) is a reduced irreducible non-singular member of \( |D_{i+1}| \)

where \( \mathcal{L}_{X_i} = \mathcal{O}(D_i) \) \( (i = 1, 2, \ldots, n = \dim X) \) and

\[
2 = \Delta(X_n, L_{X_n}) = \cdots = \Delta(X_2, L_{X_2}) > \Delta(X_1, L_{X_1}) = 1
\]

because \( c(X, L) = 1 \). As \( X_1 \) is an elliptic curve of degree 5 in \( \mathbb{P}^3 \), therefore \( H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k)) \to H^0(X_1, L^{\otimes k}_{X_1}) \) is surjective for \( k \geq 2 \) by Lemma 3. We consider the following diagram:

\[
\begin{array}{cccc}
0 & \to & H^0(\mathbb{P}^4, \mathcal{O}(k-1)) & \to & H^0(\mathbb{P}^4, \mathcal{O}(k)) & \to & H^0(\mathbb{P}^3, \mathcal{O}(k)) & \to 0 \\
& & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & H^0(X_2, \mathcal{L}^{\otimes (k-1)}_{X_2}) & \to & H^0(X_2, \mathcal{L}^{\otimes k}_{X_2}) & \to & H^0(X_1, \mathcal{L}^{\otimes k}_{X_1}) & \\
\end{array}
\]
By induction on \( k \), \( L_{X_2} \) is projective normal. So it is clear that \( L \) is projectively normal because \( \Delta(X_n, L_{X_n}) = \cdots = \Delta(X_2, L_{X_2}) \). The last part of this theorem is obtained by Lemma 4 and the same argument.

**Corollary.** If \((X, L)\) is a polarized non-singular surface, \((D^2) = 5\) where \( \mathcal{L} = \mathcal{O}(D) \) and \( L \) is very ample, then \((X, L)\) is projectively normal.

To conclude this section, we give two examples of varieties of degree 5 and codimension 2.

**Example 1.** Let \( f: S \to \mathbb{P}^2 \) be a blowing up with center \( p_1, \ldots, p_8 \in \mathbb{P}^2 \) where \( p_1, \ldots, p_8 \) are in general position. We put \( f^{-1}(p_i) = E_i \) (\( i = 1, \ldots, 8 \)) and \( D = f^*(4I) - 2E_1 - E_2 - \cdots - E_8 \) where \( I \subset \mathbb{P}^2 \) is a line. This \( D \) is very ample, \((D^2) = 5\) and \( \mathcal{g}(S, \mathcal{O}(D)) = 2 \) (see Hartshorne [5]). Therefore \( c(S, \mathcal{O}(D)) = 0 \).

**Example 2.** Let \( f: S = \mathbb{P}(\mathcal{E}) \to C \) be a ruled surface over an elliptic curve \( C \) where \( \mathcal{E} \) is an indecomposable locally free sheaf of rank 2 on \( C \). Let \( \text{deg} (\mathcal{E}) = 1 \). Let \( C_0 \) be a section of \( f \) with \( \text{Pic}(S) = \mathbb{Z}C_0 \oplus f^* \text{Pic}(C) \). Let \( D \) be a divisor in \( \text{Pic}(S) \) with \( D = C_0 + f^*(T) \) and \( \text{deg}(T) = 2 \). This \( D \) is very ample (see Hartshorne [5]). Let \( l \) be a fiber of \( f \). As \( D \) is numerically equivalent to \( C_0 + 2l \), therefore \((D^2) = 5\) and \( (D.(D + K_S)) = 0 \). Therefore \( \mathcal{g}(S, \mathcal{O}(D)) = 1 \). This is an example of \( c(S, \mathcal{O}(D)) = 1 \).

**References**


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DEPARTMENT OF MATHEMATICS
FACULTY OF EDUCATION
YAMAGUCHI UNIVERSITY
YAMAGUCHI, JAPAN
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