POINCARÉ-SOBOLEV AND RELATED INEQUALITIES FOR SUBMANIFOLDS OF $\mathbb{R}^N$

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We prove Poincaré-Sobolev and related inequalities for rectifiable varifolds in $\mathbb{R}^N$. In particular, all our results apply to properly immersed submanifolds of $\mathbb{R}^N$.

Suppose $M \subset B_R = B_R(0) \subset \mathbb{R}^N = \mathbb{R}^{n+k}$ for some $R > 0$, and $V = v(M, \theta)$ is a countably $n$-rectifiable varifold in $B_R$ with generalised mean curvature vector $H$. $\mu$ is the weight measure defined by $\mu = \theta H^n | M$. $h: M \to R$ is a Lipschitz function.

In Theorem 1 we prove a Poincaré-Sobolev result for non-negative $h$ in case $\mu\{\xi: h(\xi) > 0\} < \omega_n R^n$ and $h \in W^{1,p}(\mu)$ for some $p < n$. This generalises a Poincaré result of Leon Simon; but in addition the relevant constant here does not depend on $\mu(B_R)$. Theorem 2 is an Orlicz space result in case $p = n$.

The proofs of Theorems 1 and 2 use a covering argument to obtain weak $L^p$ type estimates on $\mu\{\xi: h(\xi) > s\}$.

Theorems 3 and 4 are generalisations of Theorems 1 and 2 in case there is no restriction on $\mu\{\xi: h(\xi) \neq 0\}$ (again the constants in the estimates do not depend on $\mu(B_R)$). The conclusion of Theorem 4 is analogous to the conclusion of the John-Nirenberg theorem for functions of bounded mean oscillation.

We prove Poincaré-Sobolev and related inequalities for rectifiable varifolds in $\mathbb{R}^N$. In particular, all our results apply to properly immersed submanifolds of $\mathbb{R}^N$.

Theorem 1 is a refinement of a result due to Leon Simon. In [Sc; p. 70] and [S; Theorem 18.4, p. 91] one has a similar Poincaré inequality in case $p = 1$ and $|H|$ is bounded, but with a constant $c$ depending on $M(V[B_R])$. In Theorem 1, $c$ depends only on $p$ and the dimension of $V$. This is important in case we have no a priori density bound for $V$ at 0 (as in [H], which provided the motivation for the present paper).

We also remark that the Poincaré result in Theorem 1 for $p > 1$ does not seem to follow directly from the case $p = 1$—the usual trick of replacing $h$ by $h^r$ does not work since the integrals in the inequality occur over balls of different radius. Nonetheless, one can use the Sobolev inequality for functions with compact support and
a cut-off function argument to "bootstrap" up from the $p = 1$ case. However, the proof in Theorem 1 gives the Poincaré result directly for all $p$ and with the constant dependence as noted above. The Sobolev result then follows immediately (as pointed out by Leon Simon) by a simple cut-off function argument from the result in the compact support case (this latter was first established in [A; Theorem 7.3] and [MS]).

In Theorem 2 we prove an Orlicz space result in case $h \in \mathcal{W}^{1,n}(\mu)$, where $n$ is the dimension of $V$ and $\mu$ is the measure in $\mathbb{R}^N$ induced by $V$.

The proofs of Theorems 1 and 2 use a covering argument to obtain weak $L^p$ type estimates on $\mu(\xi : h(\xi) > s)$, and were motivated in part by the proof of the Sobolev inequality for functions with compact support in [S; Theorem 18.6, p. 93].

Theorems 3 and 4 are generalisations of Theorems 1 and 2 in case there is no restriction on $\mu(\xi : h(\xi) \neq 0)$ (again the constants in the estimates do not depend on $M(V|B_R)$). They follow directly from Theorems 1 and 2, as was also realised by Leon Simon in the context of his Poincaré inequality discussed previously [private communication]. The conclusion of Theorem 4 is analogous to the conclusion of the John-Nirenberg theorem for functions of bounded mean oscillation.

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**Notation.** Throughout this paper we use the notations and conventions of [S].

In each of the following theorems we take the following hypotheses:

\[(H): M \subset B_R = B_R(0) \subset \mathbb{R}^N = \mathbb{R}^{n+k} \text{ for some } R > 0, \text{ and } V = \nu(M, \theta) \text{ is a countably } n\text{-rectifiable varifold in } B_R \text{ with generalised mean curvature vector } H. \mu \text{ is the weight measure defined by } \mu = \theta H^n|M. \text{ } h: M \to \mathbb{R} \text{ is a Lipschitz function.}\]

*Convention.* All integrals are taken with respect to $\mu$, unless otherwise clear from context.

**Theorem 1.** Suppose (H). Suppose also that $h(\xi) \geq 0$ for all $\xi \in M$ and that $\mu(\xi : h(\xi) > 0) \leq \omega_n R^n(1 - \alpha)$ for some $\alpha > 0$. 

Then there are constants $c = c(n, p)$ and $\beta = \beta(n, \alpha) > 0$ such that
\[
\left[ \int_{B_{\beta R}} h^{np/(n-p)} \right]^{(n-p)/np} \leq \frac{c}{\alpha} \left[ \int_{B_{R}} h^p |H|^p + |\nabla^M h|^p \right]^{1/p}
\]
whenever $1 \leq p < n$.

**Remarks.** (1) The hypothesis $\mu \{ \zeta : h(\zeta) > 0 \} \leq \omega_n R^n (1 - \alpha)$ for some $\alpha > 0$ is clearly necessary, as one sees by letting $V = v(M, 1)$ where $M$ consists of two $n$-dimensional affine spaces passing through the origin, and setting $h = 1, 2$ respectively on the two spaces.

The necessity of taking the left integral in the theorem over $B_{\beta R}$, rather than over $B_R$, is clear if one considers a modification of the above example in which one of the affine spaces is displaced slightly from the origin.

(2) From Hölder's inequality one obtains under the same assumptions that
\[
\left[ \int_{B_{\beta R}} h^q \right]^{1/q} \leq c R^{1+q/n} \left[ \int_{B_{R}} h^p |H|^p + |\nabla^M h|^p \right]^{1/p}
\]
in case $1 \leq p < n$ and $1 \leq q \leq np/(n-p)$, or in case $p \geq n$ and $1 \leq q < \infty$. In the first case $c = c(n, p)$ and in the second case $c = c(n, q)$.

**Proof of Theorem.** Our main goal is to prove the estimate (11).

Without loss of generality assume $R = 1$.

Fix $s > 0$ and define
\[
f(\zeta) = \min \{ h(\zeta), s \}.
\]

In the following suppose
\[
0 < \beta < 1/2.
\]

We will later further restrict $\beta$.

Applying the monotonicity formula to $f^p$, we have for each $\xi \in B_{\beta}$ that
\[
\frac{\partial}{\partial \rho} \left[ \rho^{-n} \int_{B_{\rho}(\xi)} f^p \right] \geq -\rho^{-n} \int_{B_{\rho}(\xi)} [f^p |H| + |\nabla^M f^p|],
\]
(in the distributional sense in $r$) provided $0 < \rho < 1 - \beta$. (See [S;
18.1, p. 89], where this result is stated for \( C^1 \) functions. The extension to the Lipschitz case follows by first extending \( f \) to a Lipschitz function \( \tilde{f} \) on \( \mathbb{R}^{n+k} \), then mollifying in \( \mathbb{R}^{n+k} \), recalling that up to a set of \( H^n \) measure zero \( M \) is a disjoint union of sets \( M_i \), each of which is a subset of a \( C^1 \) manifold \( N_i \), and finally showing that for each \( i \) the integrals on each side of (3) (over \( M_i \cap B_\rho(\xi) \) instead of \( M \cap B_\rho(\xi) \)) are the limit of corresponding integrals with \( f \) replaced by the mollified function \( \tilde{f} \). This last step makes essential use of the fact that \( \nabla^M \) is a tangential derivative.

For \( \mu \) a.e. \( \xi \) with \( |\xi| < \beta \) and \( h(\xi) > s \), we see from (2) that

\[
(4) \quad s^p = f^p(\xi) \leq \sup_{0<\sigma<1-\beta} \omega_n^{-1}\sigma^{-n} \int_{B_\sigma(\xi)} f^p \\
\leq \omega_n^{-1}(1-\beta)^{-n} \int_{B_{1-\beta}\xi(\xi)} f^p \\
+ c \int_0^{1-\beta} \tau^{-n} \int_{B_{\tau}(\xi)} [f^p|H| + |\nabla^M f^p|] \\
\leq \omega_n^{-1}(1-\beta)^{-n} \omega_n(1-\alpha)s^p \\
+ c \int_0^{1-\beta} \tau^{-n} \int_{B_{\tau}(\xi)} [f^p|H| + |\nabla^M f^p|] \\
\leq (1-\alpha/2)s^p + c \int_0^{1-\beta} \tau^{-n} \int_{B_{\tau}(\xi)} [f^p|H| + |\nabla^M f^p|],
\]

for suitable \( \beta = \beta(n, \alpha) \), which we now fix.

It follows

\[
\sup_{0<\sigma<1-\beta} \omega_n^{-1}\sigma^{-n} \int_{B_\sigma(\xi)} f^p \\
\leq \frac{c}{\alpha} \int_0^{1-\beta} \tau^{-n} \int_{B_{\tau}(\xi)} [f^p|H| + |\nabla^M f^p|] \\
\leq \frac{c}{\alpha} \int_0^{1-\beta} \tau^{-n} \int_{B_{\tau}(\xi)} f^{p-1}[f|H| + |\nabla^M f|] \\
\leq \frac{c}{\alpha} \left[ \sup_{0<\sigma<1-\beta} \sigma^{-n} \int_{B_\sigma(\xi)} f^p \right]^{1-1/p} \\
\times \int_0^{1-\beta} \left[ \tau^{-n} \int_{B_{\tau}(\xi)} f^p|H|^p + |\nabla^M f|^p \right]^{1/p}.
\]
Thus for any $0 < \sigma < 1 - \beta$,

\begin{align*}
(5) \quad \left[ \sup_{0 < \sigma < 1 - \beta} \omega_n^{-1} \sigma^{-n} \int_{B_\sigma(\xi)} f^p \right]^{1/p} \\
& \leq \frac{c}{\alpha} \int_0^{1-\beta} \left[ \tau^{-n} \int_{B_{\tau}(\xi)} f^p |H|^p + |\nabla^M f|^p \right]^{1/p} \\
& \leq \frac{c}{\alpha} \int_0^{\rho_0} \left[ \tau^{-n} \int_{B_{\tau}(\xi)} f^p |H|^p + |\nabla^M f|^p \right]^{1/p} \\
& \quad + \frac{c}{\alpha} \int_{\rho_0}^{1-\beta} \left[ \tau^{-n} \int_{B_{\tau}(\xi)} f^p |H|^p + |\nabla^M f|^p \right]^{1/p} \\
& \leq \frac{c}{\alpha} \int_0^{\rho_0} \left[ \tau^{-n} \int_{B_{\tau}(\xi)} f^p |H|^p + |\nabla^M f|^p \right]^{1/p} + \frac{c_1 \Gamma}{\alpha} \rho_0^{1-n/p},
\end{align*}

where we set

\begin{align*}
(6) \quad \Gamma = \left[ \int_{B_1(0)} f^p |H|^p + |\nabla^M f|^p \right]^{1/p}.
\end{align*}

Now choose $s_0$ so that

\begin{align*}
(7) \quad \frac{c_1 \Gamma}{\alpha} \left( \frac{1}{10} \right)^{1-n/p} = \frac{1}{2} s_0.
\end{align*}

For each $s \geq s_0$ choose $\rho_0 = \rho_0(s)$ such that

\begin{align*}
(8) \quad \frac{c_1 \Gamma}{\alpha} (\rho_0^{1-n/p}) = \frac{1}{2^s},
\end{align*}

i.e.

\begin{align*}
(9) \quad \rho_0 = c_2 \left( \frac{\Gamma}{\alpha s} \right)^{p/(n-p)}.
\end{align*}

Note that

\begin{align*}
(10) \quad \rho_0 \leq \frac{1}{10}.
\end{align*}

From (5), (8), (10), (2), (4) we have for $s \geq s_0$ and $\rho_0$ as in (9), that

\begin{align*}
\left[ \sup_{0 < \sigma < 1 - \beta} \omega_n^{-1} \sigma^{-n} \int_{B_\sigma(\xi)} f^p \right]^{1/p} \\
& \leq \frac{c}{\alpha} \int_0^{\rho_0} \left[ \tau^{-n} \int_{B_{\tau}(\xi)} f^p |H|^p + |\nabla^M f|^p \right]^{1/p}.
\end{align*}
Hence

\[
\left[ \sup_{0 < \sigma < (1-\beta)/5} \sigma^{-n} \int_{B_{e}(\xi)} f^p \right]^{1/p} \leq \frac{c}{\alpha} \rho_0 \left[ \tau^{-n} \int_{B_{\frac{e}{5}}(\xi)} f^p |H|^p + \nabla^M f|^p \right]^{1/p}
\]

for some \(0 < \tau = \tau(\xi) < \rho_0\).

Since \(\rho_0 \leq 1/10 < (1-\beta)/5\) from (10) and (2), it follows from (9) that for this particular \(\tau = \tau(\xi) < \rho_0\) we have

\[
\int_{B_{\frac{e}{5}}(\xi)} f^p \leq \frac{c}{\alpha^p} \rho_0^p \int_{B_{\frac{e}{5}}(\xi)} f^p |H|^p + |\nabla^M f|^p,
\]

where \(\rho_0\) is as in (9).

Since this is true for \(\mu\) a.e. \(\xi \in B_{\beta} \cap \{h \geq s\}\), it follows from (10), (2) and a standard covering argument (see [S: Theorem 3.3, p. 11]) that

\[
\int_{B_{\frac{e}{5}} \cap \{h \geq s\}} f^p \leq \frac{c}{\alpha^p} \rho_0^p \int_{B_{\frac{e}{5}}} f^p |H|^p + |\nabla^M f|^p,
\]

and so for any \(s \geq s_0\) we have (using (9)) that

\[
(11) \quad \mu(B_{\beta} \cap \{h \geq s\}) \leq c \left( \frac{\Gamma \rho_0}{\alpha s} \right)^p \leq c \left( \frac{\Gamma}{\alpha s} \right)^{np/(n-p)}.
\]

(Since \(\mu(B_{\rho} \cap \{h > 0\}) < \omega_n\), this last inequality is true for all \(s > 0\).)

It follows from (11) and the fact \(\mu(B_{\beta} \cap \{h \geq 0\}) \leq \omega_n\) that

\[
(12) \quad \int_{B_{\frac{e}{5}}} h^p = p \int_0^{\infty} s^{p-1} \mu(B_{\beta} \cap \{h \geq s\})
\]

\[
= p \int_0^{\Gamma/\alpha} s^{p-1} \mu(B_{\beta} \cap \{h \geq s\})
\]

\[
+ p \int_{\Gamma/\alpha}^{\infty} s^{p-1} \mu(B_{\beta} \cap \{h \geq s\})
\]

\[
\leq c \left( \frac{\Gamma}{\alpha} \right)^p + c \int_{\Gamma/\alpha}^{\infty} s^{p-1} \left( \frac{\Gamma}{\alpha s} \right)^{np/(n-p)}
\]

\[
\leq c \left( \frac{\Gamma}{\alpha} \right)^p + c \int_1^{\infty} t^{p-1} t^{-np/(n-p)} dt \leq c \left( \frac{\Gamma}{\alpha} \right)^p.
\]

(Remarks. One can similarly estimate the integral of \(h^q\) for any \(1 \leq q < np/(n-p)\).)

Finally suppose \(\varphi \in C_c^{\infty}(B_1)\), \(0 \leq \varphi \leq 1\), \(\varphi \equiv 1\) on \(B_{\beta/2}\), \(\varphi \equiv 0\) on \(B_1 \sim B_{\beta}\), and \(|D\varphi| \leq c/\beta\). From the appropriate Sobolev inequality for functions with compact support (for example,
see [S; Theorem 18.6, p. 93], replace $h$ there with $h^r$ where $r = p(n - 1)/(n - p)$, and use Hölder's inequality it follows
\[
\left[ \int_{B_1} (\varphi h)^{np/(n-p)} \right]^{(n-p)/p n} \leq c \int_{B_1} \varphi h^n |H|^p + |\nabla^M \varphi h|^p \\
\leq \frac{c}{\alpha p} \left[ \int_{B_1} h^n |H|^p + |\nabla^M h|^p \right],
\]
using (12). Hence
\[
\left[ \int_{B_{h/2}} h^{np/(n-p)} \right]^{(n-p)/np} \leq \frac{c}{\alpha} \left[ \int_{B_1} h^n |H|^p + |\nabla^M h|^p \right]^{1/p}.
\]
This establishes the theorem.

**Theorem 2.** Under the same hypotheses as Theorem 1, there exist $\beta = \beta(n) > 0$, $\gamma_1 = \gamma_1(n) > 0$, and $\gamma_2 = \gamma_2(n)$, such that
\[
\int_{B_{\beta R}} \left( \frac{\alpha h}{\Gamma} \right)^n \exp \left( \frac{\gamma_1 \alpha h}{\Gamma} \right) \leq \gamma_2 R^n,
\]
where
\[
\Gamma = \left[ \int_{B_1} h^n |H|^n + |\nabla^M h|^n \right]^{1/n}.
\]

**Proof.** Choosing $R = 1$ and arguing exactly as in the proof of Theorem 1, with $p = n$, we obtain instead of (5) that
\[
(5)' \left[ \sup_{0<\sigma<-1-\beta} \omega_n^{-1} \sigma^{-n} \int_{B_{\rho_0}(\xi)} f^n \right]^{1/n} \leq \frac{c}{\alpha} \int_{B_1} \left[ \tau^{-n} \int_{B_1(\xi)} f^n |H|^n + |\nabla^M f|^n \right]^{1/n} \\
+ \frac{\varphi_1 \Gamma}{\alpha} \log(\rho_0^{-1}).
\]
Choose $s_0$ so that
\[
(7)' \frac{\varphi_1 \Gamma}{\alpha} \log \left( \frac{1}{10} \right)^{-1} = \frac{1}{2} s_0.
\]
For each $s \geq s_0$ choose $\rho_0 = \rho_0(s)$ such that
\[
(8)' \frac{\varphi_1 \Gamma}{\alpha} \log \rho_0^{-1} = \frac{1}{2} s,
\]
i.e.

\[(9)' \quad \rho_0 = \exp \left( -\frac{c_2 \alpha s}{\Gamma} \right) .\]

Arguing again exactly as before, we obtain for any \( s \geq s_0 \) that

\[(11)' \quad \mu(B_\beta \cap \{h \geq s\}) \leq c \left( \frac{\Gamma \rho_0}{\alpha s} \right)^n \leq c \left( \frac{\Gamma}{\alpha s} \right)^n \exp \left( -\frac{c_3 \alpha s}{\Gamma} \right) .\]

(This is then true for any \( s > 0 \) since \( \mu(B_\beta \cap \{h \geq 0\}) < \omega_n \).

By Fubini's theorem we see that if \( \phi(s) \) is a \( C^1 \) increasing function of \( s \) for \( s \geq 0 \), and \( \phi(0) = 0 \), then (since \( h > 0 \) on \( B_\beta \cap M \))

\[\int_{B_\beta} \phi(u) = \int_0^\infty \phi'(s) \mu(B_\beta \cap \{h \geq s\}) \, ds.\]

If we let

\[\phi(s) = \left( \frac{\alpha s}{\Gamma} \right)^n \exp \left( \frac{\gamma_1 \alpha s}{\Gamma} \right) ,\]

where \( \gamma_1 \) is yet to be chosen, it follows from (11)' and the fact \( \mu(B_\beta \cap \{h \geq s\}) < \omega_n \) that

\[\begin{align*}
\int_{B_\beta} \left( \frac{\alpha h}{\Gamma} \right)^n \exp \left( \frac{\gamma_1 \alpha h}{\Gamma} \right) \\
\leq \omega_n \int_0^{\Gamma/\alpha} \left[ \frac{\alpha}{\Gamma} \left( \frac{\alpha s}{\Gamma} \right)^{n-1} + \gamma_1 \left( \frac{\alpha s}{\Gamma} \right)^n \right] \exp \left( \frac{\gamma_1 \alpha s}{\Gamma} \right) \\
+ c \int_0^\infty \left[ \frac{\alpha}{\Gamma} \left( \frac{\alpha s}{\Gamma} \right)^{n-1} + \gamma_1 \frac{\alpha}{\Gamma} \left( \frac{\alpha s}{\Gamma} \right)^n \right] \\
\times \exp \left( \frac{\gamma_1 \alpha s}{\Gamma} \right) \left( \frac{\Gamma}{\alpha s} \right)^n \exp \left( -\frac{c_3 \alpha s}{\Gamma} \right)
\end{align*}\]

\[\leq \gamma_2, \quad \text{say},\]

where we choose \( \gamma_1 = c_3/2 \). \( \square \)

**Theorem 3.** Suppose (\( H \)). Suppose \( \alpha > 0 \) and choose \( N \) such that \( \mu(M) \leq N \omega_n(1 - \alpha) \).

Choose any \( \lambda_1 < \cdots < \lambda_M \) such that

\[\begin{align*}
\mu\{h < \lambda_1\} &\leq \omega_n - \alpha, \\
\mu\{\lambda_i < h < \lambda_{i+1}\} &\leq \omega_n - \alpha \quad \text{for } i = 1, \ldots, N, \\
\mu\{\lambda_M < h\} &\leq \omega_n - \alpha.
\end{align*}\]

This is clearly possible for some \( M \leq N - 1 \).
Then if \( 1 \leq p < n \) and \( p \leq q \leq np/(n - p) \), there exist constants \( c = c(n, p) \) and \( \beta = \beta(n, \alpha) \) such that

\[
\left[ \int_{B_{Br}} \left( \inf_i |h - \lambda_i| \right)^q \right]^{1/q} \leq \frac{c}{\alpha} R^{1+n/q-n/p} \left[ \int_{B_{Br}} \left( \inf_i |h - \lambda_i| \right)^p |H|^p + |\nabla^M h|^p \right]^{1/p}.
\]

The same result holds if \( p \geq n \) and \( p \leq q < \infty \), but with \( c = c(n, q) \).

**Remark.** The necessity of allowing distinct values for the \( \lambda_i \) is clear if one considers examples where \( V = v(M, 1) \), \( M \) consists of distinct affine spaces, and \( h \) takes a distinct constant value on each affine space.

**Proof of Theorem.** Let

\[
I_0 = (-\infty, \lambda_1],
I_1 = [\lambda_i, \lambda_{i+1}] \quad i = 1, \ldots, M - 1,
I_M = [\lambda_M, \infty).
\]

Define

\[
h_j(\xi) = \begin{cases} 
\inf_i |h(\xi) - \lambda_i|, & h(\xi) \in I_j, \\
0, & h(\xi) \notin I_j.
\end{cases}
\]

Let

\[
h(\xi) = \inf_i |h(\xi) - \lambda_i| = \sum_j h_j(\xi).
\]

Then for each \( \xi \in M \) there exists at most one \( j \) such that \( h_j(\xi) \neq 0 \). Moreover, each \( h_j(\xi) \) is Lipschitz. Finally, for \( H^n \) a.e. \( \xi \in M \cap \{h \in I_j\} \) we have \( \nabla^M h_j(\xi) = \nabla^M h(\xi) \), and so \( \nabla^M h(\xi) = \nabla^M h(\xi) \) for \( H^n \) a.e. \( \xi \in M \).

Taking \( \beta \) as in Theorem 1, it follows that

\[
\left[ \int_{B_{Br}} h^q \right]^{p/q} = \left[ \int_{B_{Br}} \left( \sum_j h_j^p \right)^{q/p} \right]^{p/q} \leq \sum_j \left[ \int_{B_{Br}} (h_j^p)^{q/p} \right]^{p/q}.
\]
(by Minkowski's inequality, using \( q \geq p \))

\[
\leq \sum_j \frac{c}{\alpha^p} R^{p+(np/q)-n} \left[ \int_{B_R} h_j^n |H|^p + |\nabla^M h_j|^p \right] 
\]

(by Theorem 1 and the remark following it)

\[
= \frac{c}{\alpha^p} R^{p+(np/q)-n} \left[ \int_{B_R} h_j^n |H|^p + |\nabla^M h_j|^p \right].
\]

**REMARK.** The restriction \( q > p \) is required in order that the constant \( c \) not depend on \( \mu(B_R) \).

**THEOREM 4.** Suppose the same hypotheses hold as in the previous theorem.

Then there exist \( \beta = \beta(n) > 0 \), \( \gamma_1 = \gamma_1(n) > 0 \), and \( \gamma_2 = \gamma_2(n) \), such that

\[
\int_{B_{\beta R}} \left( \frac{\alpha h}{\Gamma} \right)^n \exp \left( \frac{\gamma_1 \alpha h}{\Gamma} \right) d\mu \leq \gamma_2 R^n,
\]

where

\[
h(\xi) = \inf_i |h(\xi) - \lambda_i|,
\]

\[
\Gamma = \left[ \int_{B_R} h^n |H|^n + |\nabla^M h|^n \right]^{1/n}
\]

**Proof.** Define \( \lambda_i \) and \( h_j \) as in the proof of the previous theorem. Then

\[
\int_{B_{\beta R}} (\alpha h_j)^n \exp \left( \frac{\gamma_1 \alpha h_j}{\Gamma_j} \right) \leq \gamma_2 \Gamma_j^n,
\]

where \( \beta \), \( \gamma_1 \) and \( \gamma_2 \) are as in Theorem 2, and where

\[
\Gamma_j = \left[ \int_{B_R} h_j^n |H|^n + |\nabla^M h_j|^n \right]^{1/n}
\]

Replacing \( \Gamma_j \) by \( \Gamma \) on the left side (as \( \Gamma_j \leq \Gamma \)), and then summing the inequality over \( j \), we obtain the required result. \( \square \)
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