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DIAGONALIZING PROJECTIONS IN MULTIPLIER ALGEBRAS AND IN MATRICES OVER A C^*-ALGEBRA

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Assume that $\mathscr{A}$ is a C^*-algebra with the FS property ([3] and [16]). We prove that every projection in $M_n(\mathscr{A})$ $(n \geq 1)$ or in $L(\mathscr{H}_{\mathscr{A}})$ is homotopic to a projection whose diagonal entries are projections of $\mathscr{A}$ and off-diagonal entries are zeros. This yields partial answers for Questions 7 and 8 raised by M. A. Rieffel in [18]. If $\mathscr{A}$ is σ-unital but non-unital, then every projection in the multiplier algebra $M(\mathscr{A})$ is unitarily equivalent to a diagonal projection, and homotopic to a block-diagonal projection with respect to an approximate identity of $\mathscr{A}$ consisting of an increasing sequence of projections. The unitary orbits of self-adjoint elements of $\mathscr{A}$ and $M(\mathscr{A})$ are also considered.

0. Introduction. It is well known that a projection in $M_n(\mathbb{C})$ or in $L(\mathscr{H})$ is homotopic to a diagonal projection whose diagonal entries are either 1 or 0, where $M_n(\mathbb{C})$ is the algebra consisting of $n \times n$ scalar matrices and $L(\mathscr{H})$ is the algebra consisting of bounded operators on a separable Hilbert space $\mathscr{H}$. The following natural question comes up: if $\mathbb{C}$ is replaced by a C^*-algebra $\mathscr{A}$, is every projection in $M_n(\mathscr{A})$ or $L(\mathscr{H}_{\mathscr{A}})$ homotopic to a diagonal projection whose diagonal entries are projections of $\mathscr{A}$ and off-diagonal entries are zeros? Here $M_n(\mathscr{A})$ is the C^*-algebra of $n \times n$ matrices over $\mathscr{A}$ and $L(\mathscr{H}_{\mathscr{A}})$ can be regarded as bounded infinite matrices over $\mathscr{A}$ whose adjoints exist (see §1 for a more precise description). Certainly, diagonalizing projections of $M_n(\mathscr{A})$ for $n \geq 1$ would yield information about $K_0(\mathscr{A})$ (here diagonalizing projections in the sense of Murray-von Neumann is enough for this purpose).

Concerning the matrix algebra $M_n(\mathscr{A})$, R. V. Kadison proved ([**13**] and [**14**]) that if $\mathscr{A}$ is a von Neumann algebra, then every normal element in $M_n(\mathscr{A})$ is unitarily equivalent to a diagonal normal matrix over $\mathscr{A}$. Consequently, every projection in $M_n(\mathscr{A})$ is homotopic to a diagonal projection, since the unitary group of a von Neumann algebra is connected. In general, we certainly do not expect a positive answer for the question if $\mathscr{A}$ is an arbitrary C^*-algebra. K. Grove and

G. K. Pedersen have pointed out ([**11**, 1.3]) that if $\mathscr{A}$ is the algebra $C(S^2)$, the algebra of complex-valued continuous functions on S^2, then there exists a projection in $M_2(\mathscr{A})$ which is not unitarily equivalent to any diagonal projection. However, we do expect a positive answer for a large class of C^*-algebras.

The author has proved ([**22**]) that if $\mathscr{A}$ is a C^*-algebra with FS, then every projection in $M_n(\mathscr{A})$ or in $L(\mathscr{H}_{\mathscr{A}})$ is Murray-von Neumann equivalent to a diagonal projection. In this note, we will strengthen the previous results to unitary equivalence or homotopy. We prove that if $\mathscr{A}$ is a C^*-algebra with FS (not necessarily σ-unital), and if p is a projection of the multiplier algebra $M(\mathscr{A})$, then every projection q of $\mathscr{A}$ is homotopic to a projection $q' = p_1 + p_2$, where p_1 is a projection of $p\mathscr{A}p$ and p_2 is a projection of $(1-p)\mathscr{A}(1-p)$. As a special case, by induction we conclude that every projection in $M_n(\mathscr{A})$ is homotopic to a diagonal projection. This yields partial answers for Questions 7 and 8 raised by M. A. Rieffel in [**18**]. If $\mathscr{A}$ is σ-unital and $\{e_n\}$ is a fixed sequence of mutually orthogonal projections of $\mathscr{A}$ such that $\sum_{n=1}^{\infty} e_n = 1$, we prove that every projection in $M(\mathscr{A})$ is unitarily equivalent to a diagonal projection and homotopicto a block-diagonal projection with respect to the decomposition $\sum_{n=1}^{\infty} e_n = 1$. As a consequence, every projection in $L(\mathscr{H}_{\mathscr{A}})$ is unitarily equivalent (and hence homotopic) to a diagonal projection. In addition, the unitary orbits of self-adjoint elements of $\mathscr{A}$ or $M(\mathscr{A})$ are considered.

The class of C^*-algebras with FS includes many interesting subclasses of C^*-algebras. Obviously, AF algebras, the Calkin algebra, von Neumann algebras and AW^*-algebras have FS. The Bunce-Deddens algebras have FS ([**2**]). All purely infinite, simple C^*-algebras have FS ([**24**, Part I (1.3)] and [**25**]); in particular, the Cuntz algebras $\mathscr{O}_n$ and $\mathscr{O}_A$, where $2 \leq n \leq \infty$ and A is an irreducible scalar matrix, have FS. Certain irrational rotation C^*-algebras have FS ([**9**]). Many corona and multiplier algebras have FS ([**5**], [**24**, Part I] and [**24**, Part IV]). L. G. Brown and G. K. Pedersen have recently proved ([**5**]) that a C^*-algebra $\mathscr{A}$ has FS if and only if $M_n(\mathscr{A})$ has FS for all $n \geq 1$; and $\mathscr{A}$ has FS if and only if $\mathscr{A}$ has real rank zero. In [**21**], [**22**], [**23**] and [**24**] the author has investigated the multiplier and corona algebras of C^*-algebras with FS from various angles.

1. Notations. If $\mathscr{A}$ is a C^*-algebra, we denote the Banach space double dual of $\mathscr{A}$ by $\mathscr{A}^{**}$ and the multiplier algebra of $\mathscr{A}$ by $M(\mathscr{A})$; where $M(\mathscr{A}) = \{m \in \mathscr{A}^{**}: xm, mx \in \mathscr{A}\ \forall x \in \mathscr{A}\}$ ([**1**], [**7**], [**15**], among others).

Let $\mathcal{H}_{\mathcal{A}} = \{\{a_i\}: a_i \in \mathcal{A}$ and $\sum_{i=1}^{\infty} a_i^* a_i$ converges in norm$\}$. Then $\mathcal{H}_{\mathcal{A}}$ becomes a Hilbert $\mathcal{A}$-module with the $\mathcal{A}$-valued inner product

$$\langle \{a_i\}, \{b_i\} \rangle = \sum_{i=1}^{\infty} a_i^* b_i \quad \text{for all } \{a_i\}, \{b_i\} \in \mathcal{H}_{\mathcal{A}}.$$

We denote by $L(\mathcal{H}_{\mathcal{A}})$ the set of all bounded module maps with an adjoint and by $K(\mathcal{H}_{\mathcal{A}})$ a closed ideal of $L(\mathcal{H}_{\mathcal{A}})$ called the "compact maps"; more precisely, $K(\mathcal{H}_{\mathcal{A}})$ is the norm closure of the set of all "finite rank" module maps, $\{\sum_{i=1}^{n} \theta_{x_i, y_i} : x_i, y_i \in \mathcal{H}_{\mathcal{A}}$ and $n \in \mathbb{N}\}$. Here for any pair of elements x and y in $\mathcal{H}_{\mathcal{A}}$, $\theta_{x,y}$ is defined by $\theta_{x,y}(a) = x\langle y, a\rangle \in \mathcal{H}_{\mathcal{A}}$ for all $a \in \mathcal{H}_{\mathcal{A}}$ (**[15]**). It was proved (**[15]**) that

$$L(\mathcal{H}_{\mathcal{A}}) \cong M(\mathcal{A} \otimes \mathcal{K}) \quad \text{and} \quad K(\mathcal{H}_{\mathcal{A}}) \cong \mathcal{A} \otimes \mathcal{K}$$

as C^*-algebras, where $\mathcal{K}$ is the algebra consisting of compact operators on $\mathcal{H}$. The formulation of $L(\mathcal{H}_{\mathcal{A}})$ and $K(\mathcal{H}_{\mathcal{A}})$ are closely analogous to those of $\mathrm{L}(\mathcal{H})$ and $\mathcal{K}$.

If $\mathcal{A}$ is a unital C^*-algebra, we will denote the unitary group of $M_n(\mathcal{A})$ by $U_n(\mathcal{A})$ and the path component of $U_n(\mathcal{A})$ containing the identity by $U_n^0(\mathcal{A})$. In particular, we will denote $U_1^0(\mathcal{A})$ by $U_0(\mathcal{A})$.

If p and q are projections in $\mathcal{A}$, $p \sim q$ means that p and q are equivalent in the sense of Murray-von Neumann, and $p \approx q$ means that p and q are homotopic, i.e., in the same norm path component of projections in $\mathcal{A}$. It is well known that $p \approx q$ if and only if there exists a unitary element v in $U_0(\mathcal{A})$ such that $vpv^* = q$. We denote the matrix units of $\mathcal{K}$ by $\{e_{ij}\}$.

2. Key Lemmas. The following technical lemmas are the key of this paper:

2.1. LEMMA. *Suppose that $\mathcal{A}$ is a C^*-algebra with FS (not necessarily σ-unital) and p is a projection in $M(\mathcal{A})$. If q is a projection in $\mathcal{A}$, then for any $\varepsilon_0 > 0$ there exists a projection q' in $\mathcal{A}$ such that both $pq'p$ and $(1-p)q'(1-p)$ have finite spectra and $\|q - q'\| < \varepsilon_0$. More precisely, the projection q' has the following form:*

$$q' = \begin{pmatrix} f_0 & 0 & 0 \\ 0 & a_0 & b_0 \\ 0 & b_0^* & c_0 \end{pmatrix},$$

where f_0 and the range of a_0 are mutually orthogonal subprojections of p. Consequently $q' \approx q$ if $\varepsilon_0 < 1$.

Proof. Let $q = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ be the decomposition of q with respect to $p+(1-p) = 1$. It follows that $a-a^2 = bb^*$, $c-c^2 = b^*b$, $ab+bc = b$, $0 \leq a \leq p$ and $0 \leq c \leq 1-p$. (Actually these conditions are also sufficient for q to be a projection.) We will start with the idea in **[6]** and then go further to construct a projection $q' = \begin{pmatrix} a' & b' \\ b'^* & c' \end{pmatrix}$ such that both $\sigma(a')$ and $\sigma(c')$ are finite sets, and q' is close to q in norm.

Let $0 < \delta < 1$ be a fixed positive number and ε be another positive number such that $3\varepsilon < \delta$. Since $\mathscr{A}$ has FS, there exists a positive element c_1 in $(1-p)\mathscr{A}(1-p)$ with a finite spectrum such that

$$\|c - c_1\| < \varepsilon. \tag{1}$$

Set $e = \chi_{(\delta,\infty)}(c_1 - c_1^2)$. If δ_1 is the smaller root of $t^2 - t + \delta = 0$, then $e = \chi_{(\delta_1, 1-\delta_1)}(c_1)$ which is a projection in $(1-p)\mathscr{A}(1-p)$.

Set $c_0 = c_1 e + \chi_{(1-\delta_1, 1]}(c_1)$. Then $\sigma(c_0)$ is a finite set, $c_0 - c_0^2 = e(c_1 - c_1^2)e \in e\mathscr{A}e$ and $\|c_0 - c_1\| \leq \delta_1$. It follows that

$$\|c_0 - c\| \leq \varepsilon + \delta_1 < \varepsilon + \sqrt{\delta}. \tag{2}$$

Set $v = (eb^*be)^{-1/2}(eb^*)$, of course where $(eb^*be)^{-1}$ is taken in $e\mathscr{A}e$. Since $e(c_1 - c_1^2)e \geq \delta e$ and hence $eb^*be \geq (\delta - 3\varepsilon)e$, $(eb^*be)^{-1/2}$ exists. It is clear that $vv^* = e$.

Set $b_0 = v^*(c_0 - c_0^2)^{1/2}$. Then $b_0^* b_0 = c_0 - c_0^2$.

Set $a_0 = v^*(e - c_0)v$. Then $a_0 - a_0^2 = b_0 b_0^*$ and $a_0 b_0 + b_0 c_0 = b_0$.

If we first fix δ small enough, then we choose ε small enough and c_1 satisfying (1) such that $\|c - c_0\|$, $\|b - b_0\|$ and $\|(a-a^2)-(a_0-a_0^2)\|$ are all smaller than any preassigned positive number. However, $\|a - a_0\|$ can be equal to one. Here we give details for further reference.

It is obvious that

$$\|b^*b - (c_1 - c_1^2)\| \leq 3\|c - c_1\| < 3\varepsilon. \tag{3}$$

Since $\|(1-e)b^*b(1-e) - (1-e)(c_1 - c_1^2)(1-e)\| \leq 3\varepsilon$ and $\|(1-e)(c_1 - c_1^2)(1-e)\| \leq \delta$, it is easily seen that

$$\|b(1-e)\| \leq \sqrt{3\varepsilon + \delta}. \tag{4}$$

Since $eb^*be \geq (\delta - 3\varepsilon)e$, then

$$\|(eb^*be)^{-1}\| \leq (\delta - 3\varepsilon)^{-1}. \tag{5}$$

By **[12**, 126] and (3), we can choose ε small enough such that

$$\|(eb^*be)^{1/2} - [e(c_1 - c_2^2)e]^{1/2}\| < \delta. \tag{6}$$

By (4) and (6) we can choose ε small enough such that

$$(7)\qquad \begin{aligned}\|b_0 - b\| &\le \|v^*(c_0 - c_0^2)^{1/2} - v^*(eb^*be)^{1/2}\| + \|b(1-e)\| \\ &\le \|[e(c_1 - c_1^2)e]^{1/2} - (eb^*be)^{1/2}\| + \sqrt{3\varepsilon + \delta} \\ &< \delta + \sqrt{3\varepsilon + \delta}.\end{aligned}$$

Consequently,

$$(8)\qquad \begin{aligned}\|(a - a^2) - (a_0 - a_0^2)\| &= \|bb^* - b_0 b_0^*\| \\ &\le 2\|b_0 - b\| < 2\delta + 2\sqrt{3\varepsilon + \delta}.\end{aligned}$$

It is clear from construction that $q_0 = \left(\begin{smallmatrix} a_0 & b_0 \\ b_0^* & c_0 \end{smallmatrix}\right)$ is a projection. By Lemma (2.4) of **[21]**, $\sigma(a_0)\backslash\{0, 1\} = \sigma(1 - c_0)\backslash\{0, 1\}$, and hence $\sigma(a_0)$ is also a finite set. The idea of constructing the projection q_0 is due L. G. Brown (**[6]**) for different purpose.

We will go further to adjust q_0 to a projection $q' = \left(\begin{smallmatrix} a' & b' \\ b'^* & c' \end{smallmatrix}\right)$ so that $\|a - a'\|$ is small, too. Set $f = v^*v$. Then f is a subprojection of p and $fa_0 = a_0 f = a_0$. We claim that $\|faf - a_0\|$ can be arbitrarily small if δ, ε and c_1 are properly chosen. To prove this claim, we need the following estimates.

$$(9)\qquad \begin{aligned}\|e(b^*b)^{1/2}(1-e)\| &= \|e[(b^*b)^{1/2} - (c_1 - c_1^2)^{1/2}](1-e)\| \\ &\le \|(b^*b)^{1/2} - (c_1 - c_1^2)^{1/2}\|.\end{aligned}$$

Then by [**12**, 126] and

$$[(eb^*be)^{1/2}]^2 = eb^*be = [e(b^*b)^{1/2}e]^2 + e(b^*b)^{1/2}(1-e)(b^*b)^{1/2}e,$$

for a fixed $\delta > 0$ we can choose ε small enough (by (3)) such that

$$(10)\qquad \|(b^*b)^{1/2} - (c_1 - c_1^2)^{1/2}\| < \frac{\delta^2}{2} \quad \text{and}$$

$$(11)\qquad \|(eb^*be)^{1/2} - e(b^*b)^{1/2}e\| < \delta\sqrt{\frac{\delta}{2}}.$$

Since

$$\begin{aligned}f(a - a_0)f &= v^*ev(a - a_0)v^*ev \\ &= v^*e(vav^* - va_0v^*)ev \\ &= v^*[ec_0e - v(p-a)v^*]v,\end{aligned}$$

then

$$(12)\qquad \begin{aligned}\|f(a - a_0)f\| &\le \|ec_0e - ece\| + \|ece - v(p-a)v^*\| \\ &< \varepsilon + \|ece - v(p-a)v^*\|.\end{aligned}$$

Since $(1-a)b = bc$, $p(1-a)b = bp(c)$ for any polynomial $p(t)$. Approximating by polynomials, we obtain that $\sqrt{1-a}b = b\sqrt{c}$, and hence

$$b^*(1-a)b = c^2 - c^3 = (b^*b)^{1/2}c(b^*b)^{1/2}.$$

It follows that

$$\begin{aligned} v(p-a)v^* &= (eb^*be)^{-1/2}eb^*(p-a)be(eb^*be)^{-1/2} \\ &= (eb^*be)^{-1/2}e[b^*b - b^*ab]e(eb^*be)^{-1/2} \\ &= (eb^*be)^{-1/2}[e(b^*b)^{1/2}c(b^*b)^{1/2}e](eb^*be)^{-1/2} \\ &= (eb^*be)^{-1/2}[h_1 + h_2](eb^*be)^{-1/2}, \end{aligned}$$

where

$$\begin{aligned} h_1 &= e(b^*b)^{1/2}ece(b^*b)^{1/2}e \\ &= (eb^*be)^{1/2}c(eb^*be)^{1/2} + [e(b^*b)^{1/2}e - (eb^*be)^{1/2}]c(eb^*be)^{1/2} \\ &\quad + (eb^*be)^{1/2}c[e(b^*b)^{1/2}e - (eb^*be)^{1/2}] \\ &\quad + [e(b^*b)^{1/2}e - (eb^*be)^{1/2}]c[e(b^*b)^{1/2}e - (eb^*be)^{1/2}], \\ h_2 &= e(b^*b)^{1/2}(1-e)ce(b^*b)^{1/2}e \\ &\quad + e(b^*b)^{1/2}ec(1-e)(b^*b)^{1/2}e \\ &\quad + e(b^*b)^{1/2}(1-e)c(1-e)(b^*b)^{1/2}e. \end{aligned}$$

If δ is first fixed small enough, and ε and c_1 can be chosen such that $6\varepsilon < \delta$ and

$$\begin{aligned} (13) \qquad & \|(eb^*be)^{-1/2}h_1(eb^*be)^{-1/2} - ece\| \\ &\le 2\|(eb^*be)^{-1/2}\| \, \|e(b^*b)^{1/2}e - (eb^*be)^{1/2}\| \, \|c\| \\ &\quad + \|(eb^*be)^{-1/2}\|^2\|e(b^*b)^{1/2}e - (eb^*be)^{1/2}\|^2\|c\| \\ &\le 2\frac{\delta\sqrt{\delta/2}}{\sqrt{\delta-3\varepsilon}} + \left[\frac{\delta\sqrt{\delta/2}}{\sqrt{\delta-3\varepsilon}}\right]^2 < \delta^2 + 2\delta, \end{aligned}$$

(where using (5), (10) and (11)) and

$$\begin{aligned} (14) \qquad & \|(eb^*be)^{-1/2}h_2(eb^*be)^{-1/2}\| \\ &\le 2\|(eb^*be)^{-1/2}\|^2\|e(b^*b)^{1/2}(1-e)\| \, \|c\| \, \|(b^*b)^{1/2}\| \\ &\quad + \|(eb^*be)^{-1/2}\|^2\|e(b^*b)^{1/2}(1-e)\|^2\|c\| \\ &< (\delta - 3\varepsilon)^{-1}\left[\delta^2 + \frac{\delta^4}{4}\right] < 2\delta + \delta^2, \end{aligned}$$

where we used $\delta - 3\varepsilon > \delta/2$. Consequently,

$$\|v(p-a)v^* - ece\| \le 4\delta + 2\delta^2, \quad \text{and so}$$
$$\|f(a-a_0)f\| < \varepsilon + 4\delta + 2\delta^2 \quad \text{by (12).}$$

If δ is fixed small enough and ε is chosen small enough, then $\|faf-a_0\|$ can be arbitrarily small if c_1 satisfies (1).

Moreover, by properly choosing $\delta > 0$, ε and c_1 in a similar way we can require that $\|(p-f)af\|$ is less than any preassigned positive number. This can be done as follows.

Since $a - a^2 = bb^*$ and the spectral mapping theorem, it is clear $\|b\| \le 1/2$. Since $(1-a)b = bc$, we have

$$\begin{aligned}
-(1-f)av^* &= (1-f)(1-a)be(eb^*be)^{-1/2} \\
&= bce(eb^*be)^{-1/2} - be(eb^*be)^{-1}eb^*bce(eb^*be)^{-1/2} \\
&= bce(eb^*be)^{-1/2} - be(eb^*be)^{-1}eb^*bece(eb^*be)^{-1/2} \\
&\quad - be(eb^*be)^{-1}eb^*b(1-e)ce(eb^*be)^{-1/2} \\
&= b(1-e)ce(eb^*be)^{-1/2} \\
&\quad - be(eb^*be)^{-1}eb^*b(1-e)ce(eb^*be)^{-1/2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
(15)\quad \|(1-f)af\| &\le \|(1-f)av^*\| \\
&\le \|b\|\,\|(1-e)ce\|\,\|(eb^*be)^{-1/2}\| \\
&\quad + \|b\|\,\|(eb^*be)^{-1}\|\,\|e(b^*b)(1-e)\|\,\|c\|\,\|(eb^*be)^{-1/2}\| \\
&< \frac{\varepsilon}{2}\left[\frac{1}{\sqrt{\delta-3\varepsilon}}\right] + \frac{1}{2}\left[\frac{1}{\delta-3\varepsilon}\right](3\varepsilon)\left[\frac{1}{\sqrt{\delta-3\varepsilon}}\right] \\
&< \left[\frac{\varepsilon}{2}\right]\sqrt{\frac{2}{\delta}} + \left[\frac{3\varepsilon}{2}\right]\left[\frac{2}{\delta}\right]^{3/2},
\end{aligned}$$

where we use (1), (3), (5) and the facts:

$$\|(1-e)ce\| = \|(1-e)(c-c_1)e\| \le \|c-c_1\|, \quad \text{and}$$
$$\|eb^*b(1-e)\| = \|e[b^*b-(c_1-c_1^2)](1-e)\| \le \|b^*b-(c_1-c_1^2)\|.$$

As a consequence of the last estimate and (8), for any $0 < \lambda < 1/2$, we can fix δ small enough and then choose ε small enough such that $\sigma((p-f)a(p-f)) \subset [0, \lambda] \cup [1-\lambda, 1]$. This is because of the following estimates:

$$\begin{aligned}
(p-f)[(a-a^2)-(a_0-a_0^2)](p-f) &= (p-f)(a-a^2)(p-f) \\
&= (p-f)a(p-f) - [(p-f)a(p-f)]^2 - (p-f)afa(p-f),
\end{aligned}$$

$$\begin{aligned}
&\|(p-f)a(p-f) - [(p-f)a(p-f)]^2\| \\
&\quad \le \|(p-f)[(a-a^2)-(a_0-a_0^2)](p-f)\| + \|(1-f)af\|^2 \\
&\quad \le \|(a-a^2)-(a_0-a_0^2)\| + \|(p-f)af\|^2.
\end{aligned}$$

Set $f_0 = \chi_{[1/2,1]}((p-f)a(p-f))$. Then f_0 is a projection in $(p-f)\mathscr{A}(p-f)$ such that $f_0a_0 = a_0f_0 = 0$ and $\|f_0-(p-f)a(p-f)\| \le \lambda$. Set $a' = a_0 + f_0$, $b' = b_0$ and $c' = c_0$. Then $q' = \left(\begin{smallmatrix} a' & b' \\ b'^* & c' \end{smallmatrix}\right)$ is a projection in $\mathscr{A}$ such that

$$
\begin{aligned}
(16)\quad \|q'-q\| &\le \|(f_0+a_0)-a\| + 2\|b_0-b\| + \|c_0-c\| \\
&\le \|f(a-a_0)f\| + 2\|fa(p-f)\| \\
&\quad + \|f_0-(p-f)a(p-f)\| + 2\|b_0-b\| + \|c_0-c\|.
\end{aligned}
$$

Combining all above estimates, we first fix λ small enough, then fix δ small enough, and then choose ε small enough and c_1 satisfying (1) so that each term on the right-hand side of (16) is small. Then $\|q-q'\|$ is small. It is clear that $\sigma(pq'p) = \sigma(f_0+a_0)$ is a finite set. The last sentence in the statement of this lemma is well known. □

2.2. LEMMA. *Suppose that $\mathscr{A}$ is a C^*-algebra (not necessarily σ-unital) and p is a projection in $M(\mathscr{A})$. If q is a projection in $\mathscr{A}$ such that $\sigma(pqp) \ne [0, 1]$, then there exist two projections q_1 and q_2 in $\mathscr{A}$ such that $q_1 \le p$, $q_2 \le 1-p$ and $q \approx q_1+q_2$.*

Proof. Let $q = \left(\begin{smallmatrix} a & b \\ b^* & c \end{smallmatrix}\right)$ be the composition of q with respect to $p+(1-p) = 1$. Then $a = pqp$, $c = (1-p)q(1-p)$ and $b = pq(1-p)$. By [**21**, 2.4], $\sigma(a)\backslash\{0, 1\} = \sigma(1-c)\backslash\{0, 1\}$.

If $b = 0$, then $q_1 = a$ and $q_2 = c$ are as desired. Assume that $b \ne 0$. If $1 \notin \sigma(c)$, then $\|c\| < 1$. By the argument of [**8**, 1], q is path connected to a subprojection q_1 of p. We can assume that $1 \in \sigma(c)$. Since $\sigma(c) \ne [0, 1]$ and 0 is always in $\sigma(c)$, there is a λ in $(0, 1)\backslash\sigma(c)$. Then there exists a positive number ε such that $(\lambda-\varepsilon, \lambda+\varepsilon) \cap \sigma(c) = \varnothing$. Since $b \ne 0$, we can assume that $\sigma(c) \cap (\lambda+\varepsilon, 1) \ne \varnothing$ (Otherwise, $\sigma(a) \cap (\lambda+\varepsilon, 1) \ne \varnothing$, we consider a instead.) We will use a variation of [**8**, 1] to construct a path of projections for our purpose.

Define a family of continuous positive functions $\{f_t\}_{t\in[0,1]}$ from $[0, 1]$ to $[0, 1]$ with the following properties:

(1) $\lim_{t\to t_0} \|f_t - f_{t_0}\|_\infty = 0$ for any t_0 in $[0, 1]$;

(2) $f_1(s) = s$ for all s in $[0, 1]$;

(3)

$$
f_0(s) = \begin{cases} 1, & \text{if } \lambda \le s \le 1, \\ \text{linear}, & \text{if } \lambda-\varepsilon < s < \lambda, \\ 0, & \text{if } 0 \le s \le \lambda-\varepsilon; \end{cases}
$$

(4) For all t in $(0, 1)$, $f_t(s) \leq s$ if $s \in [0, \lambda - \varepsilon]$ and $f_t(s) \geq s$ if $s \in [\lambda, 1]$.

Since q is a projection, $bc = (1 - a)b$. Approximating by polynomials, we obtain that $bg(c) = g(1 - a)b$ for any continuous function g on $[0, 1]$. Set

$$c_t = f_t(c),$$
$$b_t = b\left[\frac{f_t(c) - f_t(c)^2}{c - c^2}\right]^{1/2},$$
$$a_t = p - f_t(p - a).$$

Then b_t and c_t are well defined elements in $\mathscr{A}$ by the properties of f_t. Although $p - a$ is not in $p\mathscr{A}p$ if p is in $M(\mathscr{A})\backslash\mathscr{A}$, $p - f_t(p - a)$ is in $p\mathscr{A}p$ for $t \in [0, 1]$. To see this, first, $f_t(p - a)$ is well defined for each $t \in [0, 1]$ since $\sigma(p - a)\backslash\{0, 1\} = \sigma(c)\backslash\{0, 1\}$. Second, if we denote by π the canonical map from $(p\mathscr{A}p)^+$ to $(p\mathscr{A}p)^+/p\mathscr{A}p$, where $(p\mathscr{A}p)^+$ is the C^*-algebra obtained by joining an identity to $p\mathscr{A}p$, then $p - f_t(p - a) \in p\mathscr{A}p$, since $\pi(p - f_t(p - a)) = \pi(p) - f_t(\pi(p)) = 0$. It is easily verified that

$$a_t - a_t^2 = b_t b_t^*,$$
$$a_t b_t = b_t(1 - c_t),$$
$$c_t - c_t^2 = b_t^* b_t.$$

Thus $q(t) = \left(\begin{smallmatrix} a_t & b_t \\ b_t^* & c_t \end{smallmatrix}\right)$ is a projection in $\mathscr{A}$ for each t in $[0, 1]$. By the property (1) of $\{f_t\}$, $\{q(t)\}_{t\in[0,1]}$ is contained in the same path component of projections in $\mathscr{A}$. Then $q(0) \approx q(1) = q$. Since $(\lambda - \varepsilon, \lambda) \cap \sigma(c) = \varnothing$, $c_0 = f_0(c) = \chi_{[\lambda, 1]}(c)$ is a projection of $(1 - p)\mathscr{A}(1 - p)$. It is obvious that

$$q(0) = \left(\begin{smallmatrix} a_0 & b_0 \\ b_0^* & c_0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_0 & 0 \\ 0 & c_0 \end{smallmatrix}\right).$$

Consequently, a_0 is a projection of $p\mathscr{A}p$. Set $q_1 = a_0$ and $q_2 = c_0$, as desired. □

Roughly speaking, with respect to a fixed sequential increasing approximate identity of $\mathscr{A}$ a block-diagonal projection of $M(\mathscr{A})$ whose blocks are with the same size is homotopic to a diagonal projection. More precisely, we have the following lemma:

2.3. LEMMA. *Suppose that $\mathscr{A}$ is a σ-unital, non-unital C^*-algebra with FS and $\sum_{i=1}^{\infty}(s_{i1} + s_{i2} + \cdots + s_{in}) = 1$, where $\{s_{ij}: i \geq 1, 1 \leq j \leq n\}$ are mutually orthogonal projections in $\mathscr{A}$ and the sum converges* *in the strict topology. If p is a projection in $M(\mathscr{A})$ with the form $\sum_{i=1}^{\infty} p_i$, where p_i is a projection in $(s_{i1}+s_{i2}+\cdots+s_{in})\mathscr{A}(s_{i1}+s_{i2}+\cdots+s_{in})$ for $i\geq 1$, then $p\approx\sum_{i=1}^{\infty}(p_{i1}+p_{i2}+\cdots+p_{in})$, where p_{ij} is a projection in $s_{ij}\mathscr{A}s_{ij}$ for $i\geq 1$ and $1\leq j\leq n$.*

Proof. It suffices to prove the case if $n=2$. If $n>2$, we simply employ the same proof recursively $n-1$ times by induction to reach the conclusion.

We write

$$p_i=\begin{pmatrix} a_i^* & b_i \\ b_i^* & c_i \end{pmatrix}$$

with respect to $s_{i1}+s_{i2}$. By Lemma (2.1), for each $i\geq 1$ we can find a projection

$$p'_i=\begin{pmatrix} f_i & 0 & 0 \\ 0 & a'_i & b'_i \\ 0 & b'^*_i & c'_i \end{pmatrix}$$

in $(s_{i1}+s_{i2})\mathscr{A}(s_{i1}+s_{i2})$ such that $\|p'_i-p_i\|<1/4$, and both a'_i and c'_i have finite spectra. Here we use the proof of Lemma (2.1) to properly choose a positive number δ_i and a positive element c'_{1i} in $s_{i2}\mathscr{A}s_{i2}$ with a finite spectrum, then we have that

$$e_i=\chi_{(\delta_i,1-\delta_i)}(c'_{1i}),\qquad c'_i=c'_{1i}e_i+\chi_{(1-\delta_i,1)}(c'_{1i}),$$
$$v_i=(e_ib_i^*b_ie_i)^{-1/2}(e_ib_i^*),\qquad b'_i=v_i^*(c'_i-c'^2_{ii})^{1/2},$$
$$a'_i=v_i^*(e_i-c'_{1i})v_i$$

and f_i is a projection of $s_{i1}\mathscr{A}s_{i1}$ orthogonal to the range projection of a'_i.

Let $p'=\sum_{i=1}^{\infty}p'_i$. Then $\|p'-p\|<1/4$, and hence $p\approx p'$.

Let $\sigma(c'_i)=\{\lambda_{i1},\lambda_{i2},\cdots,\lambda_{il_i}\}$ for each $i\geq 1$. It follows from the construction or [21, 2.4] that $\sigma(a'_i)=\{1-\lambda_{i1},1-\lambda_{i2},\dots,1-\lambda_{il_i}\}$. We can write $c'_i=\sum_{j=1}^{l_i}\lambda_{ij}r_{ij}$, where $\{r_{ij}\colon 1\leq j\leq l_i\}$ is a set of mutually orthogonal projections in $s_{i2}\mathscr{A}s_{i2}$. Let λ be any number in the open interval $(\frac{1}{2},\frac{3}{4})$ but not in $\bigcup_{i=1}^{\infty}\sigma(c'_i)$. Let $\varepsilon=\min\{\lambda-\frac{1}{2},\frac{3}{4}-\lambda\}$. For $i\geq 1$, if λ_{ij} is in the open interval $(\lambda-\varepsilon,\lambda)$, we replace λ_{ij} by $\lambda'_{ij}=\lambda-\varepsilon$, and if λ_{ij} is in $(\lambda,\lambda+\varepsilon)$, we replace λ_{ij} by $\lambda'_{ij}=\lambda+\varepsilon$. If λ_{ij} is not in $(\lambda-\varepsilon,\lambda+\varepsilon)$, then we let $\lambda'_{ij}=\lambda_{ij}$. Set $c''_i=\sum_{j=1}^{l_i}\lambda'_{ij}r_{ij}$ for $i\geq 1$, and correspondingly set $b''_i=v_i^*(c''_i-c''^2_i)^{1/2}$ and $a''_i=v_i^*(e_i-c''_i)v_i$. Then

$$\|a'_i-a''_i\|\leq\|c'_i-c''_i\|<\varepsilon\quad\text{and}$$
$$\|b'_i-b''_i\|\leq\|(c'_i-c'^2_i)^{1/2}-(c''_i-c''^2_i)^{1/2}\|<\tfrac{1}{8}.$$

It follows that

$$p_i'' = \begin{pmatrix} f_i & 0 & 0 \\ 0 & a_i'' & b_i'' \\ 0 & b_i''^* & c_i'' \end{pmatrix}$$

is a projection in $(s_{i1}+s_{i2})\mathscr{A}(s_{i1}+s_{i2})$ such that $\|p_i'-p_i''\| \le 2\varepsilon+\frac{1}{4} < 1$. Define $p'' = \sum_{i=1}^{\infty} p_i''$. Then $\|p' - p''\| < 1$, and hence $p' \approx p''$. The remaining job is to prove that p'' is homotopic to a desired diagonal projection.

Let $\{f_t\}_{t\in[0,1]}$ be the family of continuous functions defined in the proof of Lemma (2.2). Since $\sigma(c_i'')$ does not intersect with the open interval $(\lambda - \varepsilon, \lambda + \varepsilon)$ for $i \ge 1$, we can define

$$c_i(t) = f_t(c_i''),$$

$$b_i(t) = b_i'' \left[\frac{f_t(c_i'') - f_t(c_i'')^2}{c_i'' - c_i''^2} \right]^{1/2},$$

$$a_i(t) = p - f_t(p - a_i'' - f_i).$$

Then $a_i(t)$, $b_i(t)$ and $c_i(t)$ are well defined elements in $(s_{i1} + s_{i2})\mathscr{A}(s_{i1} + s_{i2})$ for each t in $[0, 1]$ and $i \ge 1$ by the properties of f_t. Thus for each t in $[0, 1]$

$$p_i(t) = \begin{pmatrix} a_i(t) & b_i(t) \\ b_i(t)^* & c_i(t) \end{pmatrix}$$

is a projection in $(s_{i1} + s_{i2})\mathscr{A}(s_{i1} + s_{i2})$. It is easily seen that

$$p_i(1) = p_i'' \quad \text{and} \quad p_i(0) = \begin{pmatrix} a_i(0) & 0 \\ 0 & c_i(0) \end{pmatrix},$$

where $a_i(0)$ is a projection of $s_{i1}\mathscr{A}s_{i1}$ and $c_i(0)$ is a projection of $s_{i2}\mathscr{A}s_{i2}$. Define $p(t) = \sum_{i=1}^{\infty} p_i(t)$ for each t in $[0, 1]$. Then $\{p(t)\}_{t\in[0,1]}$ is a path of projection in $M(\mathscr{A})$. It is obvious that

$$p(1) = p'' \quad \text{and} \quad p(0) = \sum_{i=1}^{\infty} \begin{pmatrix} a_i(0) & 0 \\ 0 & c_i(0) \end{pmatrix}.$$

Since the choice of $\{f_t\}_{t\in[0,1]}$ does not depend on i, the path $\{p(t): t \in [0, 1]\}$ is continuous in the norm topology.

Set $p_{i1} = a_1(0)$, $p_{i2} = c_i(0)$ for $i \ge 1$. Then

$$p \approx p' \approx p'' \approx p(0) = \sum_{i=1}^{\infty}(p_{i1} + p_{i2}), \quad \text{as desired.} \qquad \square$$

3. Diagonalizing projections in $\mathscr{A}$ and in $M_n(\mathscr{A})$. Since we will frequently employ the following well-known fact in this paper, we state it as a lemma.

3.1. LEMMA. *If $\mathscr{A}$ is a C^*-algebra, and if p and q are two mutually orthogonal projections in $\mathscr{A}$, then $p \sim q$ if and only if $p \approx q$.*

Proof. Let v be a partial isometry in $\mathscr{A}$ such that $vv^* = p$ and $v^*v = q$. Define $w = v + v^* + (1 - p - q)$. Then w is a self-adjoint unitary in $M(\mathscr{A})$ such that $w^*pw = q$. It is well known that $w \in U_0(\mathscr{A})$. It follows that $p \approx q$. □

3.2. THEOREM. *Suppose that $\mathscr{A}$ is a C^*-algebra with FS and p_1, $p_2, \dots, p_n$ $(n \geq 1)$ are mutually orthogonal projections in $M(\mathscr{A})$ such that $\sum_{i=1}^n p_i = 1$. If p is a projection in $\mathscr{A}$, then $p \approx \sum_{i=1}^n q_i$, where q_i is a projection in $\mathscr{A}$ such that $q_i \leq p_i$ for $1 \leq i \leq n$.*

Proof. Recursively using Lemma (2.1) and Lemma (2.2), we reach the conclusion. □

The following theorem can be regarded as an analogue of the well-known fact: Every projection in $M_n(\mathbb{C})$ is homotopic to a diagonal projection whose entries are either 1 or 0.

3.3. THEOREM. *Assume that $\mathscr{A}$ is a C^*-algebra with FS and $n \geq 1$. If p is a projection in $M_n(\mathscr{A})$, then $p \approx \sum_{i=1}^n p_i \otimes e_{ii}$, where $\{p_i\}$ is a set of projections in $\mathscr{A}$ such that*

$$p_1 \leq p_2 \leq \cdots \leq p_{n-1} \leq p_n.$$

Proof. It has been recently proved (**[5]**) that $\mathscr{A} \otimes \mathscr{K}$ has FS if and only if $\mathscr{A}$ has FS. By Theorem (3.2) we have $p \approx \sum_{i=1}^n p_i' \otimes e_{ii}$, where $\{p_i'\}$ is a set of projections in $\mathscr{A}$. The remaining work is to adjust $\{p_i'\}$. We use induction on n.

If $n = 2$, $p \approx p_1' \otimes e_{11} + p_2' \otimes e_{22}$, where p_1' and p_2' are projections in $\mathscr{A}$. Combining Lemma (2.1) and Lemma (2.2), we obtain that $p_1' \approx q_1 + q_2$ in $\mathscr{A}$, where q_1 and q_2 are two projections in $\mathscr{A}$ such that $q_1 \leq p_2'$ and $q_2 \leq 1 - p_2'$. It follows that $p \approx (q_1 + q_2) \otimes e_{11} + p_2' \otimes e_{22}$. Working in the hereditary C^*-subalgebra of $M_n(\mathscr{A})$ generated by $(1 - q_1) \otimes e_{11} + 1 \otimes e_{22}$, we have $q_2 \otimes e_{11} + p_2' \otimes e_{22} \approx (p_2' + q_2) \otimes e_{22}$ by Lemma (3.1). It follows that $p \approx q_1 \otimes e_{11} + (p_2' + q_2) \otimes e_{22}$. Let $p_1 = q_1$ and $p_2 = q_2 + p_2'$.

Assume that $p \approx \sum_{i=1}^n p_i' \otimes e_{ii}$ such that $p_2' \leq p_3' \leq \cdots \leq p_n'$. Applying Lemma (2.1) and Lemma (2.2) to p_1', and p_n', we have $p_1' \approx q_n + q_n'$, where q_n and q_n' are projections in $\mathscr{A}$ such that $q_n \leq 1 - p_n'$ and $q_n' \leq p_n'$. By the same argument as in the last paragraph we have that $p \approx q_n' \otimes e_{11} + \sum_{i=2}^{n-1} p_i' \otimes e_{ii} + (p_n' + q_n) \otimes e_{nn}$. Repeating this argument to q_n' and p_{n-1}', we have that $q_n' \approx q_{n-1}' + q_{n-1}$, where q_{n-1}' and q_{n-1} are two projections in $\mathscr{A}$ such that $q_{n-1} \leq p_n' - p_{n-1}'$ and $q_{n-1}' \leq p_{n-1}'$. It follows that $p \approx q_{n-1}' \otimes e_{11} + \sum_{i=2}^{n-2} p_i' \otimes e_{ii} + (p_{n-1}' + q_{n-1}) \otimes e_{n-1,n-1} + (p_n' + q_n) \otimes e_{nn}$.

Proceeding in this way, we write $p_1' = \sum_{i=1}^{n} q_i$, where $\{q_i\}$ is a set of mutually orthogonal projections in $\mathscr{A}$ such that $q_i \leq p_{i+1}' - p_i'$ for $2 \leq i \leq n$ (where $p_{n+1}' = 1$), $q_1 \leq p_2'$, and $p \approx q_1 \otimes e_{11} + \sum_{i=2}^{n} (p_i' + q_i) \otimes e_{ii}$. Let $p_1 = q_1$ and $p_i = p_i' + q_i$ for $2 \leq i \leq n$. Then $p_1 \leq p_2 \leq \cdots \leq p_n$ and $p \approx \sum_{i=1}^{n} p_i \otimes e_{ii}$. □

M. A. Rieffel raised a question in **[18**, 7]: If $\mathscr{A}$ is a unital C^*-algebra with cancellation, and if two projections p and q in $M_n(\mathscr{A})$ represent the same class in $K_0(\mathscr{A})$, are p and q in the same path component of projections in $M_n(\mathscr{A})$? Since $\mathscr{A}$ has cancellation, $[p] = [q]$ in $K_0(\mathscr{A})$ if and only if $p \sim q$ ([**3**] or [**4**]). Hence, Rieffel's question is equivalent to whether two Murray-von Neumann equivalent projections in $M_n(\mathscr{A})$ are in the same path component of projections in $M_n(\mathscr{A})$. The following corollary provides a partial answer for his question in the case that $\mathscr{A}$ has FS:

3.4. Corollary. *If $\mathscr{A}$ is a unital C^*-algebra with FS and cancellation, and if p and q are two projections in $M_n(\mathscr{A})$, then $p \sim q$ if and only if $p \approx q$.*

Proof. Of course we need only to show that $p \sim q$ implies $p \approx q$. Since $M_n(\mathscr{A})$ has FS, by Theorem (3.2) we have $p \approx q_1 + q_2$, where q_1 is a subprojection of q and q_2 is a subprojection of $1 - q$. Since $\mathscr{A}$ has cancellation and $p \sim q$, $q_2 \sim q - q_1$. Working in $(1 - q_1)M_n(\mathscr{A})(1 - q_1)$, by Lemma (3.1) we can find a unitary v in $U_0((1 - q_1)M_n(\mathscr{A})(1 - q_1))$ such that $vq_2v^* = q - q_1$. Set $u = q_1 + v$. Then u is a unitary in $U_0(M_n(\mathscr{A}))$ such that $uq_1 = q_1u$. Thus $p \approx q_1 + q_2 \approx q$. □

Concerning the unitary orbit of elements in $M_n(\mathscr{A})$, we have the following corollary:

3.5. Corollary. *If $\mathscr{A}$ is a C^*-algebra with FS and x is a normal element in $M_n(\mathscr{A})$ with finite spectrum, then there is a unitary element u in $U_n^0(\mathscr{A})$ such that $uxu^* = \sum_{j=1}^{n} [\sum_{i=1}^{n} \lambda_i p_{ij}] \otimes e_{jj}$, where $\{p_{ij}\}$ is a set of projections in $\mathscr{A}$ such that $p_{i_1 j} \perp p_{i_2 j}$ in $\mathscr{A} \otimes e_{jj}$ if $i_1 \neq i_2$.*

Proof. By operator calculus we write $x = \sum_{i=1}^{m} \lambda_i p_i$, where $\{\lambda_i\}$ is a set of complex numbers and $\{p_i\}$ is a set of mutually orthogonal projections in $M_n(\mathscr{A})$. By Theorem (3.2) we can find a unitary element u_1 in $U_n^0(\mathscr{A})$ such that $u_1 p_1 u_1^* = \sum_{j=1}^{n} p_{1j} \otimes e_{jj}$ $(= q_1)$ for some projections $\{p_{1j}\}$ in $\mathscr{A}$. Working in $(I_n - q_1)M_n(\mathscr{A})(I_n - q_1)$ and repeating the same argument, we can find a unitary u_2' in $U_0[(I_n - q_1)M_n(\mathscr{A})(I_n - q_1)]$ such that $u_2'(u_1 p_2 u_1^*)u_2'^2 = \sum_{j=1}^{n} p_{2j} \otimes e_{jj}$ for some projections $\{p_{2j}\}$ in $\mathscr{A}$. It follows from $p_1 p_2 = 0$ that $p_{1j} p_{2l} = 0$ for $1 \leq j < l \leq n$. Set $u_2 = q_1 + u_2'$. Then u_2 is a unitary in $U_n^0(\mathscr{A})$ and $u_2 u_1 (p_1 + p_2) u_1^* u_2^* = \sum_{i=1}^{2} \sum_{j=1}^{n} p_{ij} \otimes e_{jj} = \sum_{j=1}^{n} (\sum_{i=1}^{2} p_{ij}) \otimes e_{jj}$.

Proceeding in this way we can find unitary elements $\{u_i\colon 1 \leq i \leq m\}$ in $U_n^0(\mathscr{A})$ such that

$$\begin{aligned} & u_m u_{m-1} \cdots u_1 (p_1 + p_2 + \cdots + p_m) u_1^* \cdots u_{m-1}^* u_m^* \\ &= \sum_{i=1}^{m} \left[\sum_{j=1}^{n} p_{ij} \otimes e_{jj} \right] = \sum_{j=1}^{n} \left[\sum_{i=1}^{m} p_{ij} \right] \otimes e_{jj}. \end{aligned}$$

Let $u = u_m \cdots u_2 u_1$. It is obvious that u is in $U_n^0(\mathscr{A})$ and

$$uxu^* = \sum_{j=1}^{n} \left[\sum_{i=1}^{m} \lambda_i p_{ij} \right] \otimes e_{jj}. \qquad \square$$

It is well known that the unitary orbit of a self-adjoint matrix in $M_n(\mathbb{C})$ contains a diagonal self-adjoint matrix. If $\mathbb{C}$ is replaced by a unital C^*-algebra with FS, we have the following weaker analogue:

3.6. COROLLARY. *If $\mathscr{A}$ is a C^*-algebra with FS and x is a self-adjoint element in $M_n(\mathscr{A})$ $(n \geq 1)$, then for any $\varepsilon > 0$ there exist a unitary element u in $U_n^0(\mathscr{A})$ and elements a_i in $\mathscr{A}$ with finite spectra such that*

$$\left\| uxu^* - \sum_{i=1}^{n} a_i \otimes e_{ii} \right\| < \varepsilon.$$

Proof. Since $M_n(\mathscr{A})$ has FS, there is a self-adjoint element h in $M_n(\mathscr{A})$ with finite spectrum such that $\|x - h\| < \varepsilon$. By the same argument as in the proof of Corollary (3.5) we can find a unitary element u in $U_n^0(\mathscr{A})$ such that $uhu^* = \sum_{i=1}^{n} a_i \otimes e_{ii}$, where $\{a_i\}$ is a set of self-adjoint elements in $\mathscr{A}$ with finite spectra. Therefore,

$$\left\| uxu^* - \sum_{i=1}^{n} a_i \otimes e_{ii} \right\| = \|x - h\| < \varepsilon. \qquad \square$$

3.7. REMARK. Concerning the computation of K_0-groups of a C^*-algebra, M. A. Rieffel raised a question in **[18**, 8]: What is the smallest n such that the projections in $M_n(\mathscr{A})$ generate $K_0(\mathscr{A})$? Theorem (3.3) provides a partial answer for his question for the class of C^*-algebras with FS (actually it has been given in **[22]** although it was not mentioned there). In fact, if $\mathscr{A}$ is a C^*-algebra with FS, then the smallest such an integer is $n = 1$; in other words, $K_0(\mathscr{A})$ is generated by the set of Murray-von Neumann equivalence classes of projections in $\mathscr{A}$.

4. Diagonalizing projections in $M(\mathscr{A})$.

4.1. THEOREM. *Assume that $\mathscr{A}$ is a σ-unital C^*-algebra with FS and $\{e_n\}$ is a fixed increasing sequential approximate identity consisting of projections. If p is a projection in $M(\mathscr{A})$, then the following hold*:

(i) *There is a unitary u in $M(\mathscr{A})$ connected to the identity by a path of unitaries, where the path is continuous in the strict topology, such that $upu^* = \sum_{i=1}^{\infty} p_i$, where $p_i \leq e_i$ for $i \geq 1$; in other words, each strict path component of projections in $M(\mathscr{A})$ contains a diagonal projection with respect to $\{e_n\}$.*

(ii) *There exist a unitary v in $U_0(M(\mathscr{A}))$ and a subsequence $\{e_{m_i}\}$ of $\{e_n\}$ such that $vpv^* = \sum_{i=1}^{\infty} p'_i$, where p'_i is a projection of $(e_{m_i} - e_{m_{i-1}})\mathscr{A}(e_{m_i} - e_{m_{i-1}})$ for $i \geq 1$; in other words, each norm path component of projections in $M(\mathscr{A})$ contains a block-diagonal projection with respect to $\{e_n\}$.*

Before proving this theorem, we state the following corollary, which can be regarded as an analogue of the well known fact that a projection on a separable Hilbert space is unitarily equivalent to a diagonal projection whose diagonal entries are either 1 or 0.

4.2. COROLLARY. *If $\mathscr{A}$ is a σ-unital C^*-algebra with FS, and if p is a projection in $L(\mathscr{H}_{\mathscr{A}})$, then there is a unitary u in $L(\mathscr{H}_{\mathscr{A}})$ such that $upu^* = \sum_{i=1}^{\infty} p_i \otimes e_{ii}$, where $\{p_i\}$ is a sequence of projections in $\mathscr{A}$. Consequently, $p \approx \sum_{i=1}^{\infty} p_i \otimes e_{ii}$ (by* **[8]**).

Proof of Theorem (4.1).

Case 1. If p is a projection of $\mathscr{A}$.

Choose $n \geq 1$ large enough such that $\|p(1 - e_n)p\|$ is small. Then Lemma (2.1) of **[10]** applies. We find a unitary u in $U_0(M(\mathscr{A}))$ such that $upu^* \le e_n$. By Theorem (3.2), $p \approx upu^* \approx \sum_{i=1}^n p_i$, where $p_i \le e_i - e_{i-1}$ for $1 \le i \le n$. Hence both (i) and (ii) hold.

Case 2. If p is a projection in $M(\mathscr{A})\backslash\mathscr{A}$.

Let $\{q_n\}$ and $\{q'_n\}$ be two increasing sequences of projections in $\mathscr{A}$ such that $q_n \nearrow p$ and $q'_n \nearrow 1-p$ in the strict topology. Set $f_n = q_n + q'_n$. Then $\{f_n\}$ is an increasing sequential approximate identity of $\mathscr{A}$ consisting of projections. By the argument of **[10**, 2.4] we find a unitary element v in $U_0(M(\mathscr{A}))$ such that

$$e_{m_1} \le vf_{n_1}v^* \le e_{m_2} \le vf_{n_2}v^* \le e_{m_3} \le \cdots,$$

where $\{n_i\}$ and $\{m_i\}$ are increasing sequences. It is clear that

$$vpv^* = \sum_{i=1}^{\infty} vp(f_{n_i} - f_{n_{i-1}})v^* = \sum_{i=1}^{\infty} v(q_{n_i} - q_{n_{i-1}})v^*$$

and $v(q_{n_i} - q_{n_{i-1}})v^* \le v(f_{n_i} - f_{n_{i-1}})v^* = (vf_{n_i}v^* - e_{m_i}) + (e_{m_i} - vf_{n_{i-1}}v^*)$ (where $q_{n_0} = 0$ and $f_{n_0} = 0$).

We first prove (i). By Theorem (3.2) we find a unitary w_i in $U_0(\mathscr{A}_i)$, where $\mathscr{A}_i = [v(f_{n_i} - f_{n_{i-1}})v^*]\mathscr{A}[v(f_{n_i} - f_{n_{i-1}})v^*]$, such that $w_i v(q_{n_i} - q_{n_{i-1}})v^* w_i^* = r_i + r'_i$, where $r_i \le vf_{n_i}v^* - e_{m_i}$ and $r'_i \le e_{m_i} - vf_{n_{i-1}}v^*$. Set $w = \sum_{i=1}^{\infty} w_i$. Then w is a unitary in $M(\mathscr{A})$ such that w is path connected (in the strict topology) to the identity and

$$wvpv^*w^* = \sum_{i=1}^{\infty}(r_i + r'_i) \le \sum_{i=1}^{\infty}[(vf_{n_i}v^* - e_{m_i}) + (e_{m_i} - vf_{n_{i-1}}v^*)].$$

Since $r_i + r'_{i+1} \le e_{m_{i+1}} - e_{m_i}$, we can apply Theorem (3.2) again to get a unitary w'_i in $U_0(\mathscr{B}_i)$, where $\mathscr{B}_i = (e_{m_{i+1}} - e_{m_i})M(\mathscr{A})(e_{m_{i+1}} - e_{m_i})$ such that

$$w'_i(r_i + r'_{i+1})w'^*_i = \sum_{j=m_i+1}^{m_{i+1}} p_j,$$

where p_j is in $(e_j - e_{j-1})\mathscr{A}(e_j - e_{j-1})$ for $m_i < j \le m_{i+1}$.

Define $w' = \sum_{i=1}^{\infty} w'_i$. Then w' is a unitary in $M(\mathscr{A})$ such that w' is path connected in the strict topology to the identity and $w'wvpv^*w^*w'^* = \sum_{i=1}^{\infty} p_i$. Set $u = w'wv$, as (i) desired.

To prove (ii), we start with $p \approx vpv^* = \sum_{i=1}^{\infty} v(q_{n_i} - q_{n_{i-1}})v^*$, where $s_i = v(q_{n_i} - q_{n_{i-1}})v^* \le v(f_{n_i} - f_{n_{i-1}})v^* = (vf_{n_i}v^* - e_{m_i}) + (e_{m_i} - vf_{n_{i-1}}v^*)$ for each $1 \ge 1$ and $q_{n_0} = 0$ and $f_{n_0} = 0$. With respect to

$$v(f_{n_i} - f_{n_{i-1}})v^* = (vf_{n_i}v^* - e_{m_i}) + (e_{m_i} - vf_{n_{i-1}}v^*),$$

we can write

$$s_i = \begin{pmatrix} a_i & b_i \\ b_i^* & c_i \end{pmatrix} \quad \text{for } i \geq 1.$$

By Lemma (2.3),

$$vpv^* \approx \sum_{i=1}^{\infty}(s_i + s_i'),$$

where s_i is a projection in $(vf_{n_i}v^* - e_{m_i})\mathscr{A}(vf_{n_i}v^* - e_{m_i})$ and s_i' is a projection in $(e_{m_i} - vf_{n_{i-1}}v^*)\mathscr{A}(e_{m_i} - vf_{n_{i-1}}v^*)$. Let $p_i' = s_i' + s_{i-1}$ for $i \geq 1$, where $s_0 = 0$, as desired. □

The following theorem asserts that the unitary orbit of each self-adjoint element of $M(\mathscr{A})$ contains an "almost" diagonal form, which is a natural analogue of the classical Weyl-von Neumann theorem.

4.3. THEOREM. *Assume that $\mathscr{A}$ is a σ-unital C^*-algebra with FS and also $M(\mathscr{A})$ has FS. If $\{e_n\}$ is a fixed increasing approximate identity of $\mathscr{A}$ consisting of projections and h is a self-adjoint element in $M(\mathscr{A})$, then there exist a unitary u in $M(\mathscr{A})$, an element a in $\mathscr{A}$, some mutually orthogonal subprojection p_{ij} $(1 \leq j \leq n_i)$ of $e_i - e_{i-1}$ for each $i \geq 1$ and a real bounded scalar sequence $\{\lambda_{ij}\}$ such that*

$$\sum_{ij} p_{ij} = 1, \quad \text{and} \quad uhu^* = \sum_{i=1}^{\infty}\left[\sum_{j=1}^{l_i}\lambda_{ij}p_{ij}\right] + a,$$

where a can be chosen such that $\|a\|$ is arbitrarily small. Moreover, u is connected to the identity by a path of unitaries in $M(\mathscr{A})$, where the path is continuous in the strict topology.

4.4. COROLLARY. *If $\mathscr{A}$ is a unital C^*-algebra with FS and $L(\mathscr{H}_{\mathscr{A}})$ has FS also, then for any self-adjoint element h in $L(\mathscr{H}_{\mathscr{A}})$ there are a unitary u in $L(\mathscr{H}_{\mathscr{A}})$, an element a in $K(\mathscr{H}_{\mathscr{A}})$, a sequence of projections $\{p_{ij}\}$ in $\mathscr{A}$ and a real bounded scalar sequence $\{\lambda_{ij}\}$ such that*

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{l_i}p_{ij}\right)\otimes e_{ii} = 1 \quad \text{and} \quad uhu^* = \sum_{i=1}^{\infty}\left[\sum_{j=1}^{l_i}\lambda_{ij}p_{ij}\right]\otimes e_{ii} + a,$$

where p_{ij} $(i \leq j \leq l_i)$ are mutually orthogonal for each fixed i, and a can be chosen with an arbitrarily small norm.

Proof of Theorem (4.3).. Since $\mathscr{A}$ is σ-unital and both $\mathscr{A}$ and $M(\mathscr{A})$ have FS, by **[21**, 3.1] we can find mutually orthogonal projections p_i in $\mathscr{A}$ with $\sum_{i=1}^{\infty} p_i = 1$, a real bounded scalar sequence

$\{\lambda_i\}$ and an element b in $\mathscr{A}$ with arbitrarily small norm such that $h = \sum_{i=1}^{\infty} \lambda_i p_i + b$. Let $f_n = \sum_{i=1}^{n} p_i$. Then $\{f_n\}$ is an increasing approximate identity consisting of projections. By the same argument as in [**10**, 2.4] we can find a unitary v in $M(\mathscr{A})$ such that $v \sim 1$, and

$$e_{m_1} \leq v f_{n_1} v^* \leq e_{m_2} \leq v f_{n_2} v^* \leq e_{m_3} \leq \cdots ,$$

where $\{n_i\}$ and $\{m_i\}$ are increasing sequences. Since

$$v \left(\sum_{j=n_{i-1}+1}^{n_i} p_i \right) v^* = (v f_{n_i} V^* - e_{m_i}) + (e_{m_i} - V f_{n_{i-1}} v^*)$$

(where $f_{n_0} = 0$), by the same arguments in the proof of Theorem (4.1) we can find a unitary w_i of $[v(f_{n_i} - f_{n_{i-1}})v^*]M(\mathscr{A})[v(f_{n_i} - f_{n_{i-1}})v^*]$ path connected to the identity $v(f_{n_i} - f_{n_{i-1}})v^*$ such that

$$w_i v \left(\sum_{j=n_{i-1}+1}^{n_i} p_i \right) v^* w_i^* = \sum_{j=n_{i-1}+1}^{n_i} w_i v p_i' v^* w_i^* + \sum_{j=n_{i-1}+1}^{n_i} w_i v p_i'' v^* w_i^* ,$$

where

$$p_i' + p_i'' = p_i, \quad r_i = \sum_{j=n_{i-1}+1}^{n_i} w_i v p_i' v^* w_i^* = v f_{n_i} v^* - e_{m_i} \quad \text{and}$$

$$r_i' = \sum_{j=n_{i-1}+1}^{n_i} w_i v p_i'' v^* w_i^* = e_{m_i} - v f_{n_{i-1}} v^*$$

Let $w = \sum_{i=1}^{\infty} w_i$. Then w is a unitary in $M(\mathscr{A})$ such that w is connected to the identity by a path of unitaries, where the path is continuous in the strict topology. Since $r_j + r_{j+1}' \leq e_{m_{j+1}} - e_{m_j}$, by the same arguments in the proof of Theorem (4.1), we obtain a unitary w_j' of $(e_{m_{j+1}} - e_{m_j})M(\mathscr{A})(e_{m_{j+1}} - e_{m_j})$ path connected to the identity $e_{m_{j+1}} - e_{m_j}$ such that

$$w_j'(r_j + r_{j+1}')w_j'^* = \sum_{i=m_j+1}^{m_{j+1}} \sum_{j=1}^{l_i} p_{ij} ,$$

where $\{p_{ij} : 1 \leq j \leq l_i\}$ is a set of mutually orthogonal subprojections in $(e_i - e_{i-1})\mathscr{A}(e_i - e_{i-1})$.

Define $w' = \sum_{i=1}^{\infty} w_i'$. Then w' is a unitary in $M(\mathscr{A})$ such that w' is path connected to the identity, where the path is continuous in the strict topology. Set $u = w'wv$. Then u is path connected to the identity, where the path is continuous in the strict topology. It is easily verified that uhu^* has a desired form. (Notice that $\{\lambda_i\}$ is equal to $\{\lambda_{ij}\}$ as sets.) □

4.5. REMARKS. (i) The condition " $M(\mathscr{A})$ has FS" in the hypotheses of Theorem (4.3) and Corollary (4.4) has been studied in **[5]**, **[21]** and **[24]**. Actually many multiplier algebras have the FS property.

(ii) Several applications of the results in this note have been given in the author's subsequent papers **[24**, Part II, III, IV].

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