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## **RELATIONS AMONG GENERALIZED CHARACTERISTIC CLASSES**

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In this paper, we extend Brown and Peterson's algebraic calculations, using methods of homotopy theory, to the consideration of manifolds with structure and to characteristic classes arising from generalized cohomology theories.

**0. Introduction.** In [BP], E. Brown and F. Peterson made the first calculation of relations among the Stiefel-Whitney classes of the stable normal bundles of manifolds. Specifically, they computed

$$I_n = \bigcap_{M^n} \text{Ker } \nu_M^* \subset H^*(BO; \mathbb{Z}/2),$$

where  $\nu_M: M^n \rightarrow BO$  classifies the stable normal bundle of  $M^n$ , and the intersection is taken over all compact differentiable manifolds of dimension  $n$ . These calculations have, via the Brown-Gitler spectra [BG], proven to be of considerable value. Although they arose in the context of the Immersion Conjecture for compact differentiable manifolds and were instrumental in its solution [C1], these spectra were also used by M. Mahowald [Ma], and subsequently at odd primes by R. Cohen [C2], to produce infinite families in the homotopy groups of spheres. G. Carlsson used the Spanier-Whitehead duals of these spectra to prove the Segal Conjecture for elementary abelian 2-groups [Ca], and H. Miller then used the algebra thus developed by Carlsson in his proof of the Sullivan Conjecture [Mi].

These theories should be related to the bordism theories coming from our chosen class of manifolds. We wish to calculate

$$I_n = \bigcap_{(M^n, \tilde{\nu})} \text{Ker } \tilde{\nu}^* \subset E^*(B),$$

where  $B$  is the classifying space associated to a certain class of manifolds, denoted by pairs  $(M^n, \tilde{\nu})$ ;  $\tilde{\nu}: M^n \rightarrow B$  is a lifting of  $\nu_M$  ( $B$  comes equipped with a map to  $BO$ ); and  $E^*$  is the cohomology theory. We will place the following conditions on  $E^*$ , where  $TB$  is the Thom-spectrum associated to  $B$ .

2.1. (a)  $TB$  has an  $E$ -orientation, and

(b) Given a class  $u \in E^q(M^n)$ , there exists a class  $v \in E^{n-q}(M^n)$  such that  $\langle u \cdot v, [M^n] \rangle \neq 0$ .

In Section 2 we define a map

$$\psi_q: E^q(TB) \rightarrow \pi_n(TB \wedge E_{n-q+})^*,$$

where  $E_{n-q}$  is the  $(n-q)$ th space in the  $\Omega$ -spectrum  $E$  representing  $E^*$ , and for a group  $G$ ,  $G^* = \text{Hom}(G, \pi_0 E)$ . We show that if  $J_n = \Phi(I_n) \subset E^*(TB)$ , where  $\Phi$  is the Thom isomorphism, then we have the following:

2.6. THEOREM. *If  $E^*$  satisfies 2.1, then  $J_n \cap E^q(TB) = \text{Ker } \psi_q$ .*

Dualizing to homology, we obtain our main result:

2.7. THEOREM. *Under assumptions 2.1, the following diagram commutes:*

$$\begin{array}{ccc} \pi_n(TB \wedge E_{n-q+}) & \xrightarrow{\psi_q^*} & (E^q(TB))^* \\ \downarrow (i_{n-q})_* & & \uparrow \eta_q \\ TB_q(E) & \xrightarrow{\chi} & E_q(TB) \end{array}$$

Here,  $i_{n-q}$  is the stabilization map,  $\chi$  is induced by the switch-map  $TB \wedge E \rightarrow E \wedge TB$ , and  $\eta_q$  is evaluation. In those cases where  $\eta_q$  is an isomorphism, therefore, we have reduced our original calculation to that of the stabilization map and  $\chi$ . At the end of Section 2, we show that these results reduce to those of Brown and Peterson, by setting  $B = BO$  and  $E^* = H^*(-\mathbb{Z}/2)$ .

In the final sections of this paper, we apply this program to the case  $B = BU$ , where  $U$  is the infinite unitary group (thus, the manifolds under consideration are stably almost-complex). We use the Morava  $K$ -theories as our generalized (co)homology theories, since they are complex-oriented and satisfy the strong duality conditions which we need. The paper ends with a calculation of the image of the stabilization map, which we now summarize briefly. The Thom-spectrum  $MU$ , localized at the odd prime  $p$ , is made up of similar spectra  $BP$ . Let  $m \geq 1$ , and let  $K(m)$  be the corresponding Morava  $K$ -theory at the prime  $p$ . We show that  $BP_*K(m)$  is generated as a  $\pi_*$   $BP$ -module by elements  $v_m^r \tilde{\xi}^J$ , where  $J = (j_1, j_2, \dots)$  is a nonnegative finite sequence with each  $j_k < p^m$ . The image of the suspension map may be described as follows: given  $\alpha \in BP_*K(m)$ , let  $d(\alpha)$  be

the minimum  $q$  such that there is an  $\alpha_q \in \pi_*(BP \wedge K(m)_{q+})$  with  $(\iota_q)_*(\alpha_q) = \alpha$ . Finally, let  $|J| = \sum j_i (< \infty)$ .

4.9. THEOREM.  $d(v_m^r \tilde{\xi}^J) = 2|J| - 2r(p^m - 1)$ .

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**1. Preliminaries.** Throughout this paper we shall be concerned with manifolds with structure in the sense of Stong [St]. In this section we recall briefly the definition and basic properties of such objects. Henceforth all manifolds are assumed to be compact and differentiable.

Suppose one is given a sequence of spaces  $B_k$ , fibrations  $f_k: B_k \rightarrow BO(k)$ , and maps  $g_k: B_k \rightarrow B_{k+1}$  such that the diagram

$$\begin{array}{ccc} B_k & \xrightarrow{g_k} & B_{k+1} \\ \downarrow f_k & & \downarrow f_{k+1} \\ BO(k) & \xrightarrow{j_k} & BO(k+1) \end{array}$$

commutes,  $j_k$  being induced by the usual inclusion  $O(k) \hookrightarrow O(k+1)$ . Let  $b = \varinjlim B_k$  and as usual  $BO = \varinjlim BO(k)$ .

1.1. DEFINITION. A  $(B, f)$ -manifold is a pair  $(M^n, \tilde{\nu})$ , where  $M^n$  is an  $n$ -dimensional manifold and  $\tilde{\nu}: M^n \rightarrow B$  is a lifting of the stable normal bundle classifying map of  $M^n$ .

For example, if we let  $B_{2k} = B_{2k+1} = BU(k)$ , the classifying space for the unitary group, let  $f_{2k}: BU(k) \rightarrow BO(2k)$  be induced by the standard map  $U(k) \rightarrow O(2k)$ , and define  $f_{2k+1} = j_{2k} \circ f_{2k}$ , then we are considering stably almost-complex manifolds.

For a space  $X$ , we define as usual  $\Omega_n(B, f; X)$  to be the set of equivalence classes of triples  $(M^n, \tilde{\nu}, h)$ , where  $(M^n, \tilde{\nu})$  is a  $(B, f)$ -manifold and  $h: M \rightarrow X$  is a map, under the relation of cobordism. Note that for our above example,  $\Omega_n(B; f; \text{pt.}) = \Omega_n^U$ , the complex cobordism groups.

Now let  $TB_k$  be the Thom space of the bundle  $f_k^*(\gamma_k)$ , for  $\gamma_k$  the universal bundle over  $BO(k)$ . The spaces  $TB_k$  form a spectrum, which we denote  $TB$ . Then one has the Thom-Pontrjagin theorem for  $(B, f)$ -manifolds.

1.2. THEOREM.  $\Omega_n(B; X) \cong \pi_n(TB \wedge X_+)$ . □

The map  $T: \Omega_n(B, f; X) \rightarrow \pi_n(TB \wedge X_+)$  is constructed as follows: there is a stable map  $\Delta^*: T\nu_M \rightarrow T\nu_M \wedge M_+$ , induced by the diagonal map  $\Delta: M \rightarrow M \times M$ . Let  $r: S^n \rightarrow T\nu_M$  be the stable map given by the Thom-Pontrjagin construction. Then  $T([M^n, \tilde{\nu}, h])$  is the composition

$$S_n \xrightarrow{r} T\nu_M \xrightarrow{\Delta^*} T\nu_M \wedge M_+ \xrightarrow{T\tilde{\nu} \wedge h} TB \wedge X_+.$$

In the remainder of this section we recall some facts about generalized cohomology theories and duality in manifolds that will be useful to us. The reference here is [A2].

For what follows we shall let  $E = \{E_q\}$  be the  $\Omega$ -spectrum representing our homology theory  $E_*$  and cohomology theory  $E^*$ . We shall assume unless stated otherwise that  $E$  is a ring spectrum with multiplication  $\mu$ , so that our theories  $E^*$  and  $E_*$  come equipped with the cup, cap and Kronecker products.

We recall that an  $E$ -orientation (or Thom class) for a  $k$ -plane bundle  $\xi$  over a space  $X$  with Thom space  $T_\xi$  is an element  $u_E \in \tilde{E}^k(T\xi)$  which restricts to a generator  $\tilde{E}_*(S^k)$  (as an  $\tilde{E}^*(S^0)$ -module) on each fiber. If  $(M^n, \tilde{\nu})$  is a  $(B, f)$ -manifold, then for  $k$  sufficiently large the normal bundle  $\nu_M^k$  to any immersion of  $M^n$  in  $\mathbb{R}^{n+k}$  has a  $TB$ -orientation. If each  $\nu_M^k$  has an  $E$ -orientation for  $k$  sufficiently large, then one obtains a stable Thom class  $u_E \in E^0(T\nu_M)$ , which we call an  $E$ -orientation for  $M^n$ . In particular, every  $(B, f)$ -manifold has a  $TB$ -orientation. If  $M^n$  has an  $E$ -orientation, then we have a Thom isomorphism  $\Phi_u: E^q(M^n) \rightarrow E^q(T\nu_M)$ , given by cup-product with  $u_E: E^{(M^n)} \rightarrow E^p(T\nu_M)$ .

Poincaré duality holds for manifolds  $M^n$  with an  $E$ -orientation in the usual way: there exists a class  $[M^n] \in E_n(M^n)$  such that the map  $\frown [M^n]: E^{(M^n)} \rightarrow E_{n-p}(M^n)$  is an isomorphism for all  $p$ . As usual, we call such a class a **fundamental class** for  $M^n$ .

We recall that  $T\nu_M$  and  $M_+^n$  are  $S$ -duals, with the duality isomorphism  $s: E^q(T\nu_M) \rightarrow E_{n-q}(M_+^n)$  given by:  $s(v)$  is the composition

$$S^n \xrightarrow{r} T\nu_M \xrightarrow{\Delta^*} T\nu_M \wedge M_+^n \xrightarrow{v \wedge 1} \Sigma^q E \wedge M_+^n.$$

Then Poincaré duality, the Thom isomorphism and  $S$ -duality are all related by the following result.

1.3. LEMMA [A2]. *Suppose  $T\nu_M$  has a Thom class  $u_E$ . Then the following diagram commutes up to sign:*

$$\begin{array}{ccc}
 E^q(M^n) & \xrightarrow{\Phi_u} & E^q(T\nu_M) \\
 \sim(M^n) \searrow & & \downarrow s \\
 & & E_{n-q}(M^n)
 \end{array} \quad \square$$

1.4. COROLLARY.  $s(u_E) \in E_n(\mathfrak{M}^n)$  is a fundamental class for  $M^n$ . □

In other words, we may take as a representative of  $[M^n]$  the following composition:

$$S^n \xrightarrow{r} T\nu_M \xrightarrow{\Delta^*} T\nu_M \wedge M_+^n \xrightarrow{u_E \wedge 1} E \wedge M_+^n.$$

2. The generalized Brown-Peterson process. As we stated earlier, our goal is to compute the ideal

$$I_n = \bigcap_{(M^n, \tilde{\nu})} \text{Ker } \tilde{\nu}^* \subset E^*(B),$$

where the intersection is over all  $(B, f)$ -manifolds  $(M^n, \tilde{\nu})$ , for a judicious choice of  $E^*$ .

We fix our choice  $(B, f)$  of structure for our manifolds, and demand the following two conditions of our cohomology theory  $E^*$ :

2.1. (a)  $TB$  has an  $E$ -orientation, and

(b) Given a class  $u \in E^q(M^n)$ , there exists a class  $v \in E^{n-q}(M^n)$  such that  $\langle u \cdot v, [M^n] \rangle \neq 0$ .

Condition 2.1(a) means that for each  $k$ , the bundle  $f_k^*(\gamma_k)$  over  $B_k$  has an  $E$ -orientation  $U_k$  such that the composition  $E^k(TB_k) \rightarrow E^k(\Sigma TB_{k-1}) \rightarrow E^{k-1}(TB_{k-1})$  carries  $U_k$  to  $U_{k-1}$ .

Two further remarks are in order here. First, 2.1(a) implies that each  $(B, f)$ -manifold  $(M^n, \tilde{\nu})$  has an  $E$ -orientation: from the preceding comment we see that there is a stable class  $U \in E^0(TB)$ , called the **stable Thom class** for  $TB$ ; cup-product with  $U$  yields a stable Thom isomorphism

$$\Phi: E^q(B) \rightarrow E^q(TB).$$

The composition  $T\nu_M \xrightarrow{T\tilde{\nu}} TB \xrightarrow{U} E$  then yields a stable Thom class for  $T\nu_M$  over the  $(B, f)$ -manifold  $(M^n, \tilde{\nu})$ . Second, 2.1(b) is stronger than Poincaré duality as described in Section 1; indeed, if, for example  $E^* = H^*(-; \mathbb{Z})$ , then 2.1(b) needn't hold if  $u$  is a torsion

class. Poincaré duality alone is insufficient to prove Lemma 2.2 below, which is a key step in our reduction of the calculation. One needs other methods, for example, to find the ideal of relations for  $B = BSO$  and  $E^* = H^*(-; \mathbb{Z})$  (see [Sh]).

Let  $G$  be an abelian group. In what follows we shall let  $G^* = \text{Hom}(G, \pi_0 E)$ , where  $E$  is understood from context.

Let  $\{E_q\}$  be the  $\Omega$ -spectrum representing  $E^*$ , and define a map  $\varphi_q: E^q(B) \rightarrow \Omega_n(B, f; E_{n-q})^*$  by the rule

$$\varphi_q(v)([M^n, \tilde{\nu}, h]) = \langle \tilde{\nu}^*(v) \cdot h^*(\iota_{n-q}), [M^n] \rangle,$$

where  $\iota_{n-q} \in E^{n-q}(E_{n-q})$  is the fundamental class. We shall see in a moment that  $\varphi_q$  is well-defined.

**2.2. LEMMA.** *If  $E^*$  satisfies conditions 2.1, then  $I_n \cap E^q(B) = \text{Ker } \varphi_q$ .*

*Proof.* Let  $v \in E^q(B)$ . Then

$$\begin{aligned} v \in I_n & \text{ iff } \tilde{\nu}^*(v) = 0 \text{ for all } (M^n, \tilde{\nu}) \\ & \text{ iff } \langle \tilde{\nu}^*(v) \cdot y, [M^n] \rangle = 0 \text{ for all } (M^n, \tilde{\nu}) \\ & \text{ and all } y \in E^{n-q}(M^n) \\ & \text{ iff } \langle \tilde{\nu}^*(v) \cdot h^*(\iota_{n-q}), [M^n] \rangle \text{ for all } (M^n, \tilde{\nu}) \\ & \text{ and all } h: M^n \rightarrow E_{n-q} \\ & \text{ iff } v \in \text{Ker } \varphi_q. \quad \square \end{aligned}$$

We now define a second map,  $\psi_q: E^q(B) \rightarrow \pi_n(TB \wedge E_{n-q^+})^*$ , by the following: for  $v \in E^q(B)$ ,  $\alpha \in \pi_n(TB \wedge E_{n-q^+})$ ,  $\psi_q(v)(\alpha)$  is the composition

$$S^n \xrightarrow{\alpha} TB \wedge E_{n-q^+} \xrightarrow{\Phi(v) \wedge \iota_{n-q}} \Sigma^q E \wedge \sigma^{n-q} E \xrightarrow{\quad} \Sigma^n E.$$

**2.3. PROPOSITION.** *The following diagram commutes:*

$$\begin{array}{ccc} E^q(B) & \xrightarrow{\varphi_q} & \Omega_n(B, f; E_{n-q})^* \\ & \searrow \psi_q & \uparrow T^* \\ & & \pi_n(TB \wedge E_{n-q^+})^* \end{array}$$

where  $T: \Omega_n(B, f; E_{n-q}) \rightarrow \pi_n(TB \wedge E_{n-q^+})$  is the Thom-Pontrjagin map defined in Section 1.

Note as a corollary that  $\varphi_q$  is well-defined.

First we need the following

2.4. LEMMA. Let  $u_M$  be the Thom class  $T\nu_M \rightarrow TB \rightarrow E$  as above. Let  $v \in E^q(M^n)$ , and let  $j_n \in E_n(S^n)$  be induced by the unit  $1 \in \pi_0 E$ . Then

$$\langle v, [M^n] \rangle = \langle r^*(u_M \cdot v), j_n \rangle,$$

where  $r: S^n \rightarrow T\nu_M$  is as before.

*Proof.* By definition  $\langle v, [M^n] \rangle$  is the composition

$$S^n \xrightarrow{[M^n]} E \wedge M_+^n \xrightarrow{1 \wedge v} E \wedge \Sigma^q E \xrightarrow{\mu} \Sigma^q E.$$

By Corollary 1.4, this is the same as

$$S^n \xrightarrow{r} T\nu_M \xrightarrow{\Delta^*} T\nu_M \wedge M_+^n \xrightarrow{u_M \wedge 1} E \wedge M_+^n \xrightarrow{1 \wedge v} E \wedge \Sigma^q E \xrightarrow{\mu} \Sigma^q E.$$

But the above composition  $T\nu_M \rightarrow E$  is equal to  $u_E \cdot v$ , by definition. The result follows immediately.  $\square$

*Proof of Proposition 2.3.* Let

$$v \in E^q(B), \quad [M^n, \tilde{\nu}, h] \in \Omega_n(B, f; E_{n-q+}).$$

Then if we let  $\alpha \in \pi_n(TB \wedge E_{n-q+})$  be the composition

$$S^n \xrightarrow{r} T\nu_M \xrightarrow{\Delta^*} T\nu_M \wedge M_+^n \xrightarrow{T\tilde{\nu} \wedge h} TB \wedge E_{n-q+},$$

we have that  $((T^* \circ \varphi_q(v))([M^n, \tilde{\nu}, h]) = \langle \alpha^*(\Phi(v) \cdot \iota_{n-q}), j_n \rangle$ , where  $\Phi(v) \cdot \iota_{n-q}$  is given by the composition

$$TB \wedge E_{n-q+} \xrightarrow{\Phi(v) \wedge \iota_{n-q}} \Sigma^q E \wedge \Sigma^{n-q} E \xrightarrow{\mu} \Sigma^n E.$$

By Lemma 2.4,

$$\begin{aligned} \varphi_q([M^n, \tilde{\nu}, h]) &= \langle \tilde{\nu}^*(v) \cdot h^*(\iota_{n-q}), [M^n] \rangle \\ &= \langle r^*((u_M \cdot \tilde{\nu}^*(v)) \cdot h^*(\iota_{n-q})), j_n \rangle. \end{aligned}$$

Now  $\Phi(v)$  is the composition

$$TB \xrightarrow{\Delta^*} TB \wedge B_+ \xrightarrow{U \wedge v} E \wedge \Sigma^q E \xrightarrow{\mu} \Sigma^q E.$$

Hence  $\alpha^*(\Phi(v) \cdot \iota_{n-q}) = r^*(T\tilde{\nu}^*(U \cdot v) \cdot h^*(\iota_{n-q}))$ . Thus in order to complete the proof of Proposition 2.3 it remains to show that

$$T\tilde{\nu}^*(U \cdot v) = u_M \cdot \tilde{\nu}^*(v) \in E^q(T\nu_M).$$



Since  $u_M$  is the composition  $T\nu_M \xrightarrow{T\tilde{v}} TB \xrightarrow{U} E$ ,  $u_M \cdot \tilde{v}^*(v)$  is given by

$$T\nu_M \xrightarrow{\Delta^*} T\nu_M \wedge M_+^n \xrightarrow{T\tilde{v} \wedge \tilde{v}} TB \wedge B_+ \xrightarrow{U \wedge v} E \wedge \Sigma^q E \xrightarrow{\mu} \Sigma^q E$$

and  $\tilde{v}^*(U \cdot v)$  is given by

$$T\nu_M \xrightarrow{T\tilde{v}} TB \xrightarrow{\Delta^*} TB \wedge B_+ \xrightarrow{U \wedge v} E \wedge \Sigma^q E \xrightarrow{\mu} \wedge \Sigma^q E.$$

So we need only show that the following commutes:

$$\begin{array}{ccccccc} T\nu_M & & \xrightarrow{T\tilde{v}} & & TB & & \xrightarrow{\Delta^*} & & TB \wedge B_+ \\ \downarrow \parallel & & & & & & & & \downarrow \parallel \\ T\nu_m & & \xrightarrow{\Delta^*} & & T\nu_M \wedge M_+^n & & \xrightarrow{i\tilde{v} \wedge \tilde{v}} & & TB \wedge B_+ \end{array}$$

But the first line is induced by

$$M^n \rightarrow B \rightarrow B \times B,$$

and the second line by

$$M^n \xrightarrow{\Delta} M^n \times M^n \xrightarrow{\tilde{v} \times \tilde{v}} B \times B.$$

Since these two compositions are equal, the proposition follows.  $\square$

2.5. COROLLARY.  $I_n \cap E^q(B) = \text{Ker } \psi_q$ .  $\square$

Making use of the Thom isomorphism, we now study instead  $J_n = \Phi(I_n) \subset E^*(TB)$ . Define a map, which by abuse of notation we still call  $\psi_q$ , from  $E_q(TB)$  to  $\pi_n(TB \wedge E_{n-q^+})^*$  by the rule:  $\psi_q(v)(\alpha)$  is the composition

$$S^n \xrightarrow{\alpha} TB \wedge E_{n-q^+} \xrightarrow{v \wedge I_{n-q}} \Sigma^q E \wedge \Sigma^{n-q} E \xrightarrow{\mu} \Sigma^n E.$$

We have the following immediate consequence:

2.6. COROLLARY.  $J_n \cap E^q(TB) = \text{Ker } \psi_q$ .  $\square$

Using the fact that  $\pi_n(TB \wedge E_{n-q^+}) = TB_n(E_{n-q^+})$ , we have a map

$$\psi_q^*: TB_n(E_{n-q^+}) \rightarrow E^q(TB)^*.$$

We last have a map  $\eta_q: E_q(TB) \rightarrow E^q(TB)^*$  given by

$$\eta_q(x)(v) = \langle v, x \rangle$$

for  $x \in E_q(TB)$ ,  $v \in E^q(TB)$ . Then the main result of this generalized Brown-Peterson process is the following, which is proven simply by checking the two composites on a homotopy level:

2.7. THEOREM. *The following diagram commutes:*

$$\begin{array}{ccc}
 TB_n(E_{n-q^+}) & \xrightarrow{\psi_q} & E^q(TB)^* \\
 \downarrow (i_{n-q})_* & & \uparrow \eta_q \\
 TB_n(\Sigma^{n-q}E) = TB_q(E) & \xrightarrow{\chi} & E_q(TB)
 \end{array}$$

where  $\chi$  is induced by the switch-map  $TB \wedge E \rightarrow E \wedge TB$ . □

2.8. COROLLARY. *If  $\eta_q$  is an isomorphism and  $E = TB$ , then*

$$J_n \cap E^q E = \text{Ker}((i_{n-q})^* \circ \chi^*),$$

where  $\chi: E_*E \rightarrow E_*E$  is the canonical anti-automorphism associated to the Hopf algebra  $E_*E$ . □

In particular, for  $TB = MO$ , i.e., for unoriented cobordism, the above calculation reduces to that of Brown and Peterson. In fact, since  $MO$  splits as a wedge of Eilenberg-Mac Lane spectra  $K\mathbb{Z}/2$ , if we restrict our attention to the  $K\mathbb{Z}/2$  summand containing the Thom class, one may easily verify that  $(T\nu_M)^*: H^*(K\mathbb{Z}/2) \rightarrow H^*(T\nu_M)$  (with  $\mathbb{Z}/2$  coefficients) is given by  $(T\nu_M)^*(a) = a \cdot u_M$ , where  $a \in A$ , the Steenrod algebra, and  $u_M \in H^0(T\nu_M)$  is the Thom class. Then if

$$J_n(0) = \bigcap_{M^n} \text{Ker}(T\nu_M)^*,$$

using the fact that  $a \in \text{Ker}(i_p)^*: A \rightarrow H^*(K(\mathbb{Z}/2, p))$  if and only if the element  $a$  has excess  $e(a) > p$ , we obtain the following result of Brown and Peterson's [BP]:

2.9. COROLLARY.  $J_n(0) = \{a \in A \mid \dim(\chi(a)) + e(\chi(a)) > n\}$ . □

**3.  $MU$ ,  $BP$ , and the Morava  $K$ -Theories.** In the remainder of this paper we restrict ourselves to the study of stably almost-complex manifolds, where  $B_{2k} = B_{2k+1} = BU(k)$ , and the resulting Thom spectrum is  $MU$ . Now  $MU$  localized at a prime  $p$  splits into a wedge of suspensions of  $BP$  summands. Unfortunately, neither  $MU$  nor  $BP$  satisfies condition 2.1(b) in general. Thus we are led to use  $E = K(m)$ , the Morava  $K$ -theories, as our generalized (co)homology

theories. In this section we collect some facts about  $K(m)$  and related spectra.

Fix a prime  $p$ . For the  $BP$  spectrum associated to  $p$  we have

$$\pi_*BP \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots], \quad \dim(v_i) = 2(p^i - 1),$$

where  $\mathbb{Z}_{(p)}$  represents the integers localized at  $p$ . The Morava  $K$ -theories are  $BP$ -module spectra related to  $BP$  by maps  $K_m: BP \rightarrow K(m)$ . We collect their basic properties in the following (see, for example, [RW2]):

**3.1. PROPOSITION.** (a) For  $p \neq 2$ ,  $K(m)$  is a commutative ring spectrum.

- (b)  $\pi_*K(m) \cong (\mathbb{Z}/p)[v_m, v_m^{-1}]$ .
- (c)  $K(m)_*(X \times Y) \cong K(m)_*(X) \otimes_{\pi_*K(m)} K(m)_*(Y)$  for spaces  $X, Y$ .
- (d) As a map of coefficient rings,

$$(K_m)_*(v_m) = v_m, \quad \text{and} \quad (K_m)_*(v_q) = 0, \quad q \neq m.$$

- (e)  $K(m)_*(X) \cong (K(m)^*(X))^*$  for  $X$  a space or a spectrum.
- (f) Let  $\underline{K(m)}_q$  be the  $q$ th space in the  $\Omega$ -spectrum for  $K(m)$ . Then there are homotopy equivalences for each  $q$ ,  $\underline{K(m)}_{q+2(p^m-1)} \rightarrow \underline{K(m)}_q$ .  $\square$

Note that 3.1(e) follows from the Universal Coefficient Theorem spectral sequence (see [A1]), since  $\pi_*K(m)$  is a “graded field” and hence  $K(m)_*(X)$  is free over  $\pi_*K(m)$ .

Next we introduce some intermediate theories lying between  $BP_*$  and  $K(m)_*$  which will be of use to us. Let  $E$  be a ring spectrum, and let  $x \in \pi_n E$ . Then multiplication on the left by  $x$  induces a map  $x: \Sigma^n E \rightarrow E$ . Let  $I(m) \subset \pi_*BP$  be the ideal defined by

$$I(0) = 0, \quad I(1) = (p), \quad I(m) = (p, v_1, \dots, v_{m-1}) \quad \text{for } m > 1.$$

**3.2. PROPOSITION [JW1].** There exist spectra  $P(m)$ ,  $m = 0, 1, 2, \dots$ , such that

- (a)  $P(0) = BP$ ;
- (b)  $\pi_*P(m) \cong BP/I(m) \cong \mathbb{Z}_{(p)}[v_m, v_{m+1}, \dots]$  for  $m \geq 1$ ;
- (c)  $P(m)$  is a left  $BP$ -module spectrum;
- (d)  $P(m+1)$  is related to  $P(m)$  by a stable cofibration

$$\Sigma^{2(p^m-1)}P(m) \xrightarrow{v_m} P(m) \xrightarrow{g_m} P(m+1);$$

(e)  $(g_m)_*: \pi_*P(m) \rightarrow \pi_*P(m+1)$  is given on generators by

$$\begin{aligned} (g_m)_*(v_i) &= 0 && \text{if } i \leq m, \\ &= v_i && \text{if } i > m. \end{aligned}$$

(f) for  $p > 2$ ,  $P(m)$  is a commutative ring spectrum. □

Thus  $P(m+1)$  may be obtained from  $P(m)$  by “killing” the element  $v_m$  via the cofibration of 3.2(d). Proceeding in this manner, we may start from  $P(m)$  and kill the generators  $v_{m+1}, v_{m+2}, \dots$  of  $\pi_*P(m)$  to obtain in the limit the  $BP$ -module spectrum  $k(m)$ . We have then that  $\pi_*k(m) \cong (\mathbb{Z}/p)[v_m]$ . If we let  $T_m = \{1, v_m, v_m^2, \dots\}$  be the multiplicative set of nonnegative powers of the element  $v_m \in \pi_*k(m)$ , then we may obtain  $K(m)_*$  by localizing the homology theory  $k(m)_*$  with respect to  $T_m$  via the techniques described in [JW2].

Finally, we note that the maps

$$MU \rightarrow BP \rightarrow P(1) \rightarrow \dots \rightarrow P(m) \rightarrow k(m) \rightarrow K(m)$$

give  $MU$  an orientation with respect to the cohomology theories  $BP^*, P(m)^*, k(m)^*$ , and  $K(m)^*$ .

**4. Calculation of relations for stably almost-complex manifolds.** We now return to the generalized Brown-Peterson process and apply it to the case  $B_{2k} = B_{2k+1} = BU$  as before. By 3.1(e) and the remark at the end of the last section, the cohomology theory  $K(m)^*$  satisfies conditions 2.1(a') and 2.1(b) for stably almost-complex manifolds. By Corollary 2.6, we need to determine the kernel of the map  $\psi_q: K(m)^q(MU) \rightarrow MU_n(K(m)_{n-q^+})^*$ . Dually, we need to determine the cokernel (and hence the image) of the map

$$\psi_q^*: MU_n \underline{K(m)}_{n-q^+} \rightarrow K(m)_q MU.$$

Here we are making use of 3.1(e). By Theorem 2.7, then, since  $\eta_q$  is an isomorphism, we need to calculate the image of

$$MU_n \underline{K(m)}_{n-q^+} \xrightarrow{(i_{n-q})^*} MU_n \Sigma^{n-q} K(m) = MU_q K(m) \xrightarrow{\chi} K(m)_q MU.$$

Since  $MU$  localized at  $p$  is made up of  $BP$ -summands, it suffices, modulo  $\chi$ , to calculate the image of the stabilization map

$$BP_* \underline{K(m)}_s \xrightarrow{(i_s)^*} BP_* K(m).$$

We make use of the following, where  $E(x_1, \dots, x_t)$  is the exterior algebra on the generators  $x_1, \dots, x_t$ .

4.1. LEMMA.  $K(m)_*P(m) \cong K(m)_*BP \otimes E(\tau_0, \dots, \tau_{m-1})$  as modules over  $\pi_*K(m)$ , where  $\dim(\tau_j) = 2p^j - 1$ .

*Proof.* The Atiyah-Hirzebruch spectral sequence for  $k(m)_*BP$  collapses, yielding

$$k(m)_*BP \cong H_*BP \otimes \pi_*k(m) \cong (\mathbb{Z}/p)[v_m; c_1, c_2, \dots]$$

as  $\mathbb{Z}/p$ -algebras, where  $\dim(c_j) = 2(p^j - 1)$ .

If we apply  $k(m)_*( )$  to the cofibration of 3.2(d), we obtain an exact sequence for  $q < m$ :

$$\begin{aligned} \dots \rightarrow k(m)_sP(q) \xrightarrow{v_q} k(m)_{s+r}P(q) \rightarrow k(m)_{s+r}P(q+1) \\ \rightarrow k(m)_{s-1}P(q) \rightarrow \dots \end{aligned}$$

where  $r = 2(p^q - 1)$ .

But multiplication by  $v_q$  is zero in  $k(m)_*( )$ . Hence  $k(m)_*P(q)$  injects in  $k(m)_*P(q+1)$ . Furthermore, when  $s = 1$  we obtain a new element  $\tau_q \in k(m)_{2q^a-1}P(q+1)$  which is external, as one easily checks inductively by using our knowledge of  $k(m)_*BP$  (recall that  $P(0) = BP$ ). Thus for  $q < m$ ,  $k(m)_*P(q+1) \cong k(m)_*P(q) \otimes E(\tau_q) \cong k(m)_*BP \otimes E(\tau_0, \dots, \tau_q)$ . Localizing now with respect to  $\{1, v_m, v_m^2, \dots\}$  gives the desired result.  $\square$

4.2. COROLLARY. *The map  $BP_*K(m) \rightarrow P(m)_*K(m)$  is injective.*  $\square$

With 4.2 in mind, we shall make use of the following commutative diagram:

$$(4.3) \quad \begin{array}{ccc} BP_*\underline{K(m)}_q & \xrightarrow{(i_q)_*} & BP_*K(m) \\ \downarrow & & \downarrow \\ P(m)_*\underline{K(m)}_q & \xrightarrow{(i_q)_*} & P(m)_*K(m), \end{array}$$

and calculate the image of  $(i_q)_*$  on  $P(m)_*$ -homology.

4.4. REMARK. Because of problems with the multiplication in the spectra  $P(m)$ ,  $k(m)$ , and  $K(m)$  at the prime 2 [R], we restrict our attention to  $p$  odd from now on.

Wilson has calculated  $P(m)_*\underline{K(m)}_q$  for each  $q$ , by considering  $P(m)_*\underline{K(m)}_* = \{P(m)_*\underline{K(m)}_q\}$  as a Hopf ring. The general reference for Hopf rings is [RW1]; here we recall only that there are structure

maps

$$* : P(m)_* \underline{K(m)}_k \otimes P(m)_* \underline{K(m)}_k \rightarrow P(m)_* \underline{K(m)}_k \quad (\text{for each } k), \text{ and}$$

$$\circ : P(m)_* \underline{K(m)}_k \otimes P(m)_* \underline{K(m)}_n \rightarrow P(m)_* \underline{K(m)}_{k+n} \quad (\text{for all } n, k)$$

satisfying certain properties (associativity, distributivity, having a unit, etc.) The map  $*$  is induced by the loop-space multiplication on  $\underline{K(m)}_k$ , and  $\circ$  is induced by the multiplication

$$\mu : P(m) \wedge P(m) \rightarrow P(m) \quad \text{and} \quad m_{k,n} : \underline{K(m)}_k \wedge \underline{K(m)}_n \rightarrow \underline{K(m)}_{k+n}.$$

Using these two maps, the Hopf ring  $P(m)_* \underline{K(m)}_*$  is generated by elements  $e_1 \in P(m)_1 \underline{K(m)}_1$ ,  $a_{(i)} \in P(m)_{2^i} \underline{K(m)}_1^*$  for  $i < m$ , and  $b_{(i)} \in P(m)_{2^i} \underline{K(m)}_2$ , which we now describe. For  $q < 2p^m - 1$ ,  $\tilde{P}(m)_q \underline{K(m)}_1 \cong \tilde{H}_q(K(\mathbb{Z}/p, 1); \mathbb{Z}/p)$  since  $\underline{K(m)}_1 \simeq K(\mathbb{Z}/p, 1)$  through dimension  $2(p^m - 1)$ , and  $P(m) \simeq K\mathbb{Z}/p$  in stable dimensions less than  $2(p^m - 1)$ .  $H_1(K(\mathbb{Z}/p, 1); \mathbb{Z}/p)$  and  $H_{2p^1}(K(\mathbb{Z}/p, 1); \mathbb{Z}/p)$  are isomorphic to  $\mathbb{Z}/p$ ; use this isomorphism on the canonical generators to define  $e_1$  and  $a_{(i)}$ .  $P(m)_* CP^\infty$  is free over  $\pi_* P(m)$  on generators  $\beta_i \in P(m)_{2^i} CP^\infty$ . Using these elements and the  $K(M)$ -orientation for  $CP^\infty$ , represented by a map  $CP^\infty \rightarrow \underline{K(m)}_2$ , one defines  $b_{(i)} \in P(m)_{2^i} \underline{K(m)}_2$ .

For  $I = (i_0, i_1, \dots, i_{m-1})$  and  $J = (j_0, j_2, \dots)$  nonnegative finite sequences with  $i_k = 0$  or 1 and  $j_k < p^m$ , define

$$\begin{matrix} I & J & \circ & i_0 & \circ & i_{m-1} & \circ & j_0 & \circ & j_1 \\ ab & = & a_{(0)} & \circ \cdots \circ & a_{(m-1)} & \circ & b_{(0)} & \circ & b_{(0)} & \circ \cdots \end{matrix}$$

Then Wilson's theorem states that, as a  $\pi_* P(m)$ -algebra,  $P(m)_* \underline{K(m)}_*$  is described in terms of the above elements as follows. For  $j_0 < p^m - 1$ , each  $a^I b^J \circ e_1$  is an exterior generator; and depending on  $I$  and  $J$  each  $a^I b^J$  is either a polynomial or a truncated polynomial generator, all using the  $*$  product. Here,  $P(m)_* \underline{K(m)}_*$  is considered as graded over  $\mathbb{Z}/2(p^m - 1)$  instead of over  $\mathbb{Z}$ , by use of 3.1(f). The homotopy equivalence of 3.1(f) is given by the "periodicity operator"  $[v_m] \in \pi_0 \underline{K(m)}_{-2(p^m-1)}$  as:

$$\underline{K(m)}_{q+r} \approx S^0 \wedge \underline{K(m)}_{q+r} \xrightarrow{[v_m]^{\wedge 1}} \underline{K(m)}_{-r} \wedge \underline{K(m)}_{q+2r} \xrightarrow{\mu} \underline{K(m)}_q,$$

where  $r = 2(p^m - 1)$ .

4.5. PROPOSITION (Wilson [W]). *The following relations hold in  $P(m)_* \underline{K(m)}_*$ , where  $\lambda : BP \rightarrow P(m)$  is the induced map from 3.2:*

- (a)  $e_1 \circ$ —is the homology suspension map.
- (b)  $e_1 \circ e_1 = b_{(0)}$ .

- (c)  $a_{(i)} \circ a_{(j)} = -a_{(j)} \circ a_{(i)}$ .
- (d)  $\lambda_*(v_m)e_1 = [v_m] \circ b_{(0)}^{\circ p^m - 1} \circ e_1$ .
- (e)  $[v_m] \circ b_{(k)}^{\circ p^m} = \sum_{i=0}^k \lambda_*(v_{m+i}^{p^{k-i}})b_{(k-i)} \pmod *, k > 0$ . □

Our goal is to determine the image of the stabilization map  $P(m)_* \underline{K(m)}_q \rightarrow P(m)_* \underline{K(m)}$ . First we calculate the stable object  $P(m)_* \underline{K(m)}$ . Let  $R_m = \pi_* P(m)[v_m^{-1}] \cong (\mathbb{Z}/p)[v_m, v_m^{-1}, v_{m+1}, \dots]$ , and let  $E(x)$  and  $P(x)$  denote, respectively, the exterior and polynomial algebras on the generator  $x$ .

4.6. THEOREM. *As  $\pi_* P(m)$ -modules,*

$$P(m)_* \underline{K(m)} \cong E_{R_m}(\tilde{\tau}_0, \tilde{\tau}_1, \dots, \tilde{\tau}_{m-1}) \otimes P_{R_m}(\xi_1, \xi_2, \dots)$$

*modulo the relations*

$$\xi_k^{P^k} = v_m^{-1} \sum_{i=0}^k v_{m+i}^{p^{k-i}} \xi_{k+i},$$

where  $\dim(\tilde{\tau}_i) = 2p^i - 1$  and  $\dim(\xi_i) = 2(p^i - 1)$ .

*Proof.* The stabilization map  $P(m)_* \underline{K(m)}_q \rightarrow P(m)_* \underline{K(m)}$  is given, from 4.5(a), by  $\circ$ -multiplication with  $e_1$  infinitely often. Stabilization kills  $*$ -products and  $e_1$  stabilizes to  $1 \in P(m)_0 \underline{K(m)}$ , so we need only concern ourselves with elements of the form  $[v_n]^r \circ a^I b^J$ , where  $r \in \mathbb{Z}$  and  $I$  and  $J$  are as before, with the additional property that  $j_0 = 0$  (by 4.5(b)). By 4.5(d), all of these elements survive to  $P(m)_* \underline{K(m)}$ .

In particular, let  $\tilde{\tau}_i$  and  $\xi_j$  be the stable images of  $a_{(i)}$  and  $b_{(j)}$  respectively, for  $0 \leq i \leq m - 1$  and  $j > 0$ . One may easily verify that for  $\alpha \in P(m)_* \underline{K(m)}_r$ ,  $\beta \in P(m)_* \underline{K(m)}_s$ ,

$$\iota_{r+s}(\alpha \circ \beta) = \iota_r(\alpha) \iota_s(\beta)$$

in  $P(m)_* \underline{K(m)}$ . Using this result we have that  $\tilde{\tau}_i \tilde{\tau}_j = -\tilde{\tau}_j \tilde{\tau}_i$  (from (4.5(c)), and  $a^I b^J$  stabilizes to  $\tilde{\tau}^I \xi^J$ , defined analogously.

By 4.5(d) we have that  $[v_m] \in P(m)_0 \underline{K(m)}_{2-2p^m}$  stabilizes to the same element as the image

$$v_m \in \pi_{2(p^m-1)} P(m) \rightarrow P(m)_{2(p^m-1)} \underline{K(m)}.$$

(That is to say, multiplication by  $v_m$  is the same on the left and on the right in  $P(m)_* \underline{K(m)}$ .) Hence the coefficient ring for  $P(m)_* \underline{K(m)}$  becomes  $\pi_* P(m)[v_m^{-1}] = R_m$ .

Finally, since stabilization is a  $\pi_*P(m)$ -module map, 4.5(e) stabilizes to the relation

$$v_m \xi_k^{p^k} = \sum_{i=0}^k v_{m+i}^{p^{k-i}} \xi_{k-i}, \quad \text{or} \quad \xi_k^{p^k} = v_m^{-1} \sum_{i=0}^k v_{m+i}^{p^{k-i}} \xi_{k+i}.$$

This finishes the proof of 4.6. □

From 4.6 we can tell how far each element of  $P(m)_*K(m)$  desuspends. Given  $\alpha \in P(m)_*K(m)$ , let  $d(\alpha)$  be the minimum  $q$  such that there is an  $\alpha_q \in P(m)_*K(m)_q$  with  $(i_q)_*(\alpha_q) = \alpha$ . Define  $\tilde{\tau}^I \xi^J$  in analogy with  $a^I b^J$  (except that there is no  $\xi_0$ ).

4.7. COROLLARY.  $d(v_m^r \tilde{\tau}^I \xi^J) = |I| + 2|J| - 2r(p^m - 1)$ , where

$$|I| = \sum_{s=0}^{m-1} i_s \quad \text{and} \quad |J| = \sum_{t=1}^{\infty} j_t (< \infty).$$

*Proof.* Since  $a^I \in P(M)_*K(m)_{|I|}$  and  $b^J \in P(m)_*K(m)_{|J|}$ , we need note only that  $d(v_m^r) = -2r(p^m - 1)$ . □

We now return our attention to  $BP_*K(m)_*$  and  $BP_*K(m)$ , by making use of 4.3. First we prove the following:

- 4.8. LEMMA. (a)  $\tilde{\tau}_1 \notin \text{Im } \lambda_* \subset P(m)_*K(m)$ .  
 (b)  $b_{(i)} \in \text{Im } \lambda_* \subset P(m)_*K(m)_2$ .

*Proof.* (a) Since  $\lambda_*(xy) = \lambda_*(x)\lambda_*(y)$  (see, for example, [Wü]), we have that if  $\lambda_*(\alpha) = \tilde{\tau}_1$ , then  $\alpha^2 = 0$ . But by the proof of 4.1,  $BP_*k(m)$  has no exterior elements, and the same holds, after localization, for  $BP_*K(m)$ . Hence no such  $\alpha$  exists.

(b) We have that  $BP_*CP^\infty$  is free over  $\pi_*BP$  on generators  $\tilde{\beta}_i \in BP_{2i}CP^\infty$ . By the commutativity of

$$\begin{array}{ccc} BP_*CP^\infty & \xrightarrow{\theta_*} & BP_*K(m)_2 \\ \downarrow \lambda_* & & \downarrow \lambda_* \\ P(m)_*CP^\infty & \xrightarrow{\theta_*} & P(m)_*K(m)_2 \end{array}$$

where  $\theta: CP^\infty \rightarrow K(m)_2$  is the orientation, since  $\lambda_*(\tilde{\beta}_i) = \beta_i$  and  $(\theta \circ \lambda)_*(\tilde{\beta}_i) = b_{(i)}$ , there is an element  $\tilde{b}_i \in BP_{2i}K(m)_2$  with  $\lambda_*(\tilde{b}_i) = b_{(i)}$ . □

By the commutativity of  $\lambda_*$  with the stabilization map, we conclude that there is an element  $\tilde{\xi}_i \in BP_{2(p^i-1)}K(m)$  with  $\lambda_*(\tilde{\xi}_i) = \xi_i$ .



By defining the function  $d$  in analogy with 4.7, the following is a consequence of 4.7 and 4.8:

4.9. COROLLARY.  $d(v_m^r \tilde{\xi}^J) = 2|J| - 2r(p^m - 1)$ . □

Using the fact that stabilization about is a  $\pi_*BP$ -module map, by 4.8 this completes the description of  $\text{Im } \iota_*: BP_*K(m)_* \rightarrow BP_*K(m)_*$ .

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