

# Pacific Journal of Mathematics

## **RELATIONS AMONG GENERALIZED CHARACTERISTIC CLASSES**

ALEXANDER EBEN KOONCE

## RELATIONS AMONG GENERALIZED CHARACTERISTIC CLASSES

ALEXANDER E. KOONCE

In this paper, we extend Brown and Peterson's algebraic calculations, using methods of homotopy theory, to the consideration of manifolds with structure and to characteristic classes arising from generalized cohomology theories.

**0. Introduction.** In [BP], E. Brown and F. Peterson made the first calculation of relations among the Stiefel-Whitney classes of the stable normal bundles of manifolds. Specifically, they computed

$$I_n = \bigcap_{M^n} \text{Ker } \nu_M^* \subset H^*(BO; \mathbb{Z}/2),$$

where  $\nu_M: M^n \rightarrow BO$  classifies the stable normal bundle of  $M^n$ , and the intersection is taken over all compact differentiable manifolds of dimension  $n$ . These calculations have, via the Brown-Gitler spectra [BG], proven to be of considerable value. Although they arose in the context of the Immersion Conjecture for compact differentiable manifolds and were instrumental in its solution [C1], these spectra were also used by M. Mahowald [Ma], and subsequently at odd primes by R. Cohen [C2], to produce infinite families in the homotopy groups of spheres. G. Carlsson used the Spanier-Whitehead duals of these spectra to prove the Segal Conjecture for elementary abelian 2-groups [Ca], and H. Miller then used the algebra thus developed by Carlsson in his proof of the Sullivan Conjecture [Mi].

These theories should be related to the bordism theories coming from our chosen class of manifolds. We wish to calculate

$$I_n = \bigcap_{(M^n, \tilde{\nu})} \text{Ker } \tilde{\nu}^* \subset E^*(B),$$

where  $B$  is the classifying space associated to a certain class of manifolds, denoted by pairs  $(M^n, \tilde{\nu})$ ;  $\tilde{\nu}: M^n \rightarrow B$  is a lifting of  $\nu_M$  ( $B$  comes equipped with a map to  $BO$ ); and  $E^*$  is the cohomology theory. We will place the following conditions on  $E^*$ , where  $TB$  is the Thom-spectrum associated to  $B$ .

2.1. (a)  $TB$  has an  $E$ -orientation, and

(b) Given a class  $u \in E^q(M^n)$ , there exists a class  $v \in E^{n-q}(M^n)$  such that  $\langle u \cdot v, [M^n] \rangle \neq 0$ .

In Section 2 we define a map

$$\psi_q: E^q(TB) \rightarrow \pi_n(TB \wedge E_{n-q+})^*,$$

where  $E_{n-q}$  is the  $(n-q)$ th space in the  $\Omega$ -spectrum  $E$  representing  $E^*$ , and for a group  $G$ ,  $G^* = \text{Hom}(G, \pi_0 E)$ . We show that if  $J_n = \Phi(I_n) \subset E^*(TB)$ , where  $\Phi$  is the Thom isomorphism, then we have the following:

2.6. THEOREM. *If  $E^*$  satisfies 2.1, then  $J_n \cap E^q(TB) = \text{Ker } \psi_q$ .*

Dualizing to homology, we obtain our main result:

2.7. THEOREM. *Under assumptions 2.1, the following diagram commutes:*

$$\begin{array}{ccc} \pi_n(TB \wedge E_{n-q+}) & \xrightarrow{\psi_q^*} & (E^q(TB))^* \\ \downarrow (\iota_{n-q})_* & & \uparrow \eta_q \\ TB_q(E) & \xrightarrow{\chi} & E_q(TB) \end{array}$$

Here,  $\iota_{n-q}$  is the stabilization map,  $\chi$  is induced by the switch-map  $TB \wedge E \rightarrow E \wedge TB$ , and  $\eta_q$  is evaluation. In those cases where  $\eta_q$  is an isomorphism, therefore, we have reduced our original calculation to that of the stabilization map and  $\chi$ . At the end of Section 2, we show that these results reduce to those of Brown and Peterson, by setting  $B = BO$  and  $E^* = H^*(-\mathbb{Z}/2)$ .

In the final sections of this paper, we apply this program to the case  $B = BU$ , where  $U$  is the infinite unitary group (thus, the manifolds under consideration are stably almost-complex). We use the Morava  $K$ -theories as our generalized (co)homology theories, since they are complex-oriented and satisfy the strong duality conditions which we need. The paper ends with a calculation of the image of the stabilization map, which we now summarize briefly. The Thom-spectrum  $MU$ , localized at the odd prime  $p$ , is made up of similar spectra  $BP$ . Let  $m \geq 1$ , and let  $K(m)$  be the corresponding Morava  $K$ -theory at the prime  $p$ . We show that  $BP_*K(m)$  is generated as a  $\pi_*$   $BP$ -module by elements  $v_m^r \xi^J$ , where  $J = (j_1, j_2, \dots)$  is a nonnegative finite sequence with each  $j_k < p^m$ . The image of the suspension map may be described as follows: given  $\alpha \in BP_*K(m)$ , let  $d(\alpha)$  be

the minimum  $q$  such that there is an  $\alpha_q \in \pi_*(BP \wedge K(m)_{q+})$  with  $(\iota_q)_*(\alpha_q) = \alpha$ . Finally, let  $|J| = \sum j_t$  ( $< \infty$ ).

4.9. THEOREM.  $d(v_m^r \xi^J) = 2|J| - 2r(p^m - 1)$ .

This paper consists primarily of work done on my Ph.D. thesis at Stanford University under the supervision of Ralph L. Cohen. It is with great pleasure that I thank Professor Cohen for his guidance, support and inspiration. In addition, I gratefully acknowledge the assistance of W. Stephen Wilson in realizing the results of Section 4 from his paper [W].

**1. Preliminaries.** Throughout this paper we shall be concerned with manifolds with structure in the sense of Stong [St]. In this section we recall briefly the definition and basic properties of such objects. Henceforth all manifolds are assumed to be compact and differentiable.

Suppose one is given a sequence of spaces  $B_k$ , fibrations  $f_k: B_k \rightarrow BO(k)$ , and maps  $g_k: B_k \rightarrow B_{k+1}$  such that the diagram

$$\begin{array}{ccc} B_k & \xrightarrow{g_k} & B_{k+1} \\ \downarrow f_k & & \downarrow f_{k+1} \\ BO(k) & \xrightarrow{j_k} & BO(k+1) \end{array}$$

commutes,  $j_k$  being induced by the usual inclusion  $O(k) \hookrightarrow O(k+1)$ . Let  $b = \varinjlim B_k$  and as usual  $BO = \varinjlim BO(k)$ .

1.1. DEFINITION. A  $(B, f)$ -manifold is a pair  $(M^n, \tilde{\nu})$ , where  $M^n$  is an  $n$ -dimensional manifold and  $\tilde{\nu}: M^n \rightarrow B$  is a lifting of the stable normal bundle classifying map of  $M^n$ .

For example, if we let  $B_{2k} = B_{2k+1} = BU(k)$ , the classifying space for the unitary group, let  $f_{2k}: BU(k) \rightarrow BO(2k)$  be induced by the standard map  $U(k) \rightarrow O(2k)$ , and define  $f_{2k+1} = j_{2k} \circ f_{2k}$ , then we are considering stably almost-complex manifolds.

For a space  $X$ , we define as usual  $\Omega_n(B, f; X)$  to be the set of equivalence classes of triples  $(M^n, \tilde{\nu}, h)$ , where  $(M^n, \tilde{\nu})$  is a  $(B, f)$ -manifold and  $h: M \rightarrow X$  is a map, under the relation of cobordism. Note that for our above example,  $\Omega_n(B; f; \text{pt.}) = \Omega_n^U$ , the complex cobordism groups.

Now let  $TB_k$  be the Thom space of the bundle  $f_k^*(\gamma_k)$ , for  $\gamma_k$  the universal bundle over  $BO(k)$ . The spaces  $TB_k$  form a spectrum, which we denote  $TB$ . Then one has the Thom-Pontrjagin theorem for  $(B, f)$ -manifolds.

1.2. THEOREM.  $\Omega_n(B; X) \cong \pi_n(TB \wedge X_+)$ . □

The map  $T: \Omega_n(B, f; X) \rightarrow \pi_n(TB \wedge X_+)$  is constructed as follows: there is a stable map  $\Delta^*: T\nu_M \rightarrow T\nu_M \wedge M_+$ , induced by the diagonal map  $\Delta: M \rightarrow M \times M$ . Let  $r: S^n \rightarrow T\nu_M$  be the stable map given by the Thom-Pontrjagin construction. Then  $T([M^n, \tilde{\nu}, h])$  is the composition

$$S_n \xrightarrow{r} T\nu_M \xrightarrow{\Delta^*} T\nu_M \wedge M_+ \xrightarrow{T\tilde{\nu} \wedge h} TB \wedge X_+.$$

In the remainder of this section we recall some facts about generalized cohomology theories and duality in manifolds that will be useful to us. The reference here is [A2].

For what follows we shall let  $E = \{E_q\}$  be the  $\Omega$ -spectrum representing our homology theory  $E_*$  and cohomology theory  $E^*$ . We shall assume unless stated otherwise that  $E$  is a ring spectrum with multiplication  $\mu$ , so that our theories  $E^*$  and  $E_*$  come equipped with the cup, cap and Kronecker products.

We recall that an  $E$ -orientation (or Thom class) for a  $k$ -plane bundle  $\xi$  over a space  $X$  with Thom space  $T_\xi$  is an element  $u_E \in \tilde{E}^k(T\xi)$  which restricts to a generator  $\tilde{E}_*(S^k)$  (as an  $\tilde{E}^*(S^0)$ -module) on each fiber. If  $(M^n, \tilde{\nu})$  is a  $(B, f)$ -manifold, then for  $k$  sufficiently large the normal bundle  $\nu_M^k$  to any immersion of  $M^n$  in  $\mathbb{R}^{n+k}$  has a  $TB$ -orientation. If each  $\nu_M^k$  has an  $E$ -orientation for  $k$  sufficiently large, then one obtains a stable Thom class  $u_E \in E^0(T\nu_M)$ , which we call an  $E$ -orientation for  $M^n$ . In particular, every  $(B, f)$ -manifold has a  $TB$ -orientation. If  $M^n$  has an  $E$ -orientation, then we have a Thom isomorphism  $\Phi_u: E^q(M^n) \rightarrow E^q(T\nu_M)$ , given by cup-product with  $u_E: E^{(M^n)} \rightarrow E^p(T\nu_M)$ .

Poincaré duality holds for manifolds  $M^n$  with an  $E$ -orientation in the usual way: there exists a class  $[M^n] \in E_n(M^n)$  such that the map  $\cap [M^n]: E^{(M^n)} \rightarrow E_{n-p}(M^n)$  is an isomorphism for all  $p$ . As usual, we call such a class a **fundamental class** for  $M^n$ .

We recall that  $T\nu_M$  and  $M_+^n$  are  $S$ -duals, with the duality isomorphism  $s: E^q(T\nu_M) \rightarrow E_{n-q}(M_+^n)$  given by:  $s(v)$  is the composition

$$S^n \xrightarrow{r} T\nu_M \xrightarrow{\Delta^*} T\nu_M \wedge M_+^n \xrightarrow{v \wedge 1} \Sigma^q E \wedge M_+^n.$$

Then Poincaré duality, the Thom isomorphism and  $S$ -duality are all related by the following result.

1.3. LEMMA [A2]. Suppose  $T\nu_M$  has a Thom class  $u_E$ . Then the following diagram commutes up to sign:

$$\begin{array}{ccc}
 E^q(M^n) & \xrightarrow{\Phi_u} & E^q(T\nu_M) \\
 \sim(M^n) \searrow & & \downarrow s \\
 & & E_{n-q}(M^n)
 \end{array} \quad \square$$

1.4. COROLLARY.  $s(u_E) \in E_n(\mathfrak{M}^n)$  is a fundamental class for  $M^n$ . □

In other words, we may take as a representative of  $[M^n]$  the following composition:

$$S^n \xrightarrow{r} T\nu_M \xrightarrow{\Delta^*} T\nu_M \wedge M_+^n \xrightarrow{u_E \wedge 1} E \wedge M_+^n.$$

2. The generalized Brown-Peterson process. As we stated earlier, our goal is to compute the ideal

$$I_n = \bigcap_{(M^n, \tilde{\nu})} \text{Ker } \tilde{\nu}^* \subset E^*(B),$$

where the intersection is over all  $(B, f)$ -manifolds  $(M^n, \tilde{\nu})$ , for a judicious choice of  $E^*$ .

We fix our choice  $(B, f)$  of structure for our manifolds, and demand the following two conditions of our cohomology theory  $E^*$ :

2.1. (a)  $TB$  has an  $E$ -orientation, and

(b) Given a class  $u \in E^q(M^n)$ , there exists a class  $v \in E^{n-q}(M^n)$  such that  $\langle u \cdot v, [M^n] \rangle \neq 0$ .

Condition 2.1(a) means that for each  $k$ , the bundle  $f_k^*(\gamma_k)$  over  $B_k$  has an  $E$ -orientation  $U_k$  such that the composition  $E^k(TB_k) \rightarrow E^k(\Sigma TB_{k-1}) \rightarrow E^{k-1}(TB_{k-1})$  carries  $U_k$  to  $U_{k-1}$ .

Two further remarks are in order here. First, 2.1(a) implies that each  $(B, f)$ -manifold  $(M^n, \tilde{\nu})$  has an  $E$ -orientation: from the preceding comment we see that there is a stable class  $U \in E^0(TB)$ , called the **stable Thom class** for  $TB$ ; cup-product with  $U$  yields a stable Thom isomorphism

$$\Phi: E^q(B) \rightarrow E^q(TB).$$

The composition  $T\nu_M \xrightarrow{T\tilde{\nu}} TB \xrightarrow{U} E$  then yields a stable Thom class for  $T\nu_M$  over the  $(B, f)$ -manifold  $(M^n, \tilde{\nu})$ . Second, 2.1(b) is stronger than Poincaré duality as described in Section 1; indeed, if, for example  $E^* = H^*(-; \mathbb{Z})$ , then 2.1(b) needn't hold if  $u$  is a torsion

class. Poincaré duality alone is insufficient to prove Lemma 2.2 below, which is a key step in our reduction of the calculation. One needs other methods, for example, to find the ideal of relations for  $B = BSO$  and  $E^* = H^*(-; \mathbb{Z})$  (see [Sh]).

Let  $G$  be an abelian group. In what follows we shall let  $G^* = \text{Hom}(G, \pi_0 E)$ , where  $E$  is understood from context.

Let  $\{E_q\}$  be the  $\Omega$ -spectrum representing  $E^*$ , and define a map  $\varphi_q: E^q(B) \rightarrow \Omega_n(B, f; E_{n-q})^*$  by the rule

$$\varphi_q(v)([M^n, \tilde{\nu}, h]) = \langle \tilde{\nu}^*(v) \cdot h^*(i_{n-q}), [M^n] \rangle,$$

where  $i_{n-q} \in E^{n-q}(E_{n-q})$  is the fundamental class. We shall see in a moment that  $\varphi_q$  is well-defined.

**2.2. LEMMA.** *If  $E^*$  satisfies conditions 2.1, then  $I_n \cap E^q(B) = \text{Ker } \varphi_q$ .*

*Proof.* Let  $v \in E^q(B)$ . Then

$$\begin{aligned} v \in I_n & \text{ iff } \tilde{\nu}^*(v) = 0 \text{ for all } (M^n, \tilde{\nu}) \\ & \text{ iff } \langle \tilde{\nu}^*(v) \cdot y, [M^n] \rangle = 0 \text{ for all } (M^n, \tilde{\nu}) \\ & \text{ and all } y \in E^{n-q}(M^n) \\ & \text{ iff } \langle \tilde{\nu}^*(v) \cdot h^*(i_{n-q}), [M^n] \rangle \text{ for all } (M^n, \tilde{\nu}) \\ & \text{ and all } h: M^n \rightarrow E_{n-q} \\ & \text{ iff } v \in \text{Ker } \varphi_q. \quad \square \end{aligned}$$

We now define a second map,  $\psi_q: E^q(B) \rightarrow \pi_n(TB \wedge E_{n-q^+})^*$ , by the following: for  $v \in E^q(B)$ ,  $\alpha \in \pi_n(TB \wedge E_{n-q^+})$ ,  $\psi_q(v)(\alpha)$  is the composition

$$S^n \xrightarrow{\alpha} TB \wedge E_{n-q^+} \xrightarrow{\Phi(v) \wedge i_{n-q}} \Sigma^q E \wedge \sigma^{n-q} E \xrightarrow{\quad} \Sigma^n E.$$

**2.3. PROPOSITION.** *The following diagram commutes:*

$$\begin{array}{ccc} E^q(B) & \xrightarrow{\varphi_q} & \Omega_n(B, f; E_{n-q})^* \\ & \searrow \psi_q & \uparrow T^* \\ & & \pi_n(TB \wedge E_{n-q^+})^* \end{array}$$

where  $T: \Omega_n(B, f; E_{n-q}) \rightarrow \pi_n(TB \wedge E_{n-q^+})$  is the Thom-Pontrjagin map defined in Section 1.

Note as a corollary that  $\varphi_q$  is well-defined.

First we need the following

2.4. LEMMA. Let  $u_M$  be the Thom class  $T\nu_M \rightarrow TB \rightarrow E$  as above. Let  $v \in E^q(M^n)$ , and let  $j_n \in E_n(S^n)$  be induced by the unit  $1 \in \pi_0 E$ . Then

$$\langle v, [M^n] \rangle = \langle r^*(u_M \cdot v), j_n \rangle,$$

where  $r: S^n \rightarrow T\nu_M$  is as before.

*Proof.* By definition  $\langle v, [M^n] \rangle$  is the composition

$$S^n \xrightarrow{[M^n]} E \wedge M_+^n \xrightarrow{1 \wedge v} E \wedge \Sigma^q E \xrightarrow{\mu} \Sigma^q E.$$

By Corollary 1.4, this is the same as

$$S^n \xrightarrow{r} T\nu_M \xrightarrow{\Delta^*} T\nu_m \wedge M_+^n \xrightarrow{u_M \wedge 1} E \wedge M_+^n \xrightarrow{1 \wedge v} E \wedge \Sigma^q E \xrightarrow{\mu} \Sigma^q E.$$

But the above composition  $T\nu_M \rightarrow E$  is equal to  $u_E \cdot v$ , by definition. The result follows immediately.  $\square$

*Proof of Proposition 2.3.* Let

$$v \in E^q(B), \quad [M^n, \tilde{v}, h] \in \Omega_n(B, f; E_{n-q+}).$$

Then if we let  $\alpha \in \pi_n(TB \wedge E_{n-q+})$  be the composition

$$S^n \xrightarrow{r} T\nu_M \xrightarrow{\Delta^*} T\nu_M \wedge M_+^n \xrightarrow{T\tilde{v} \wedge h} TB \wedge E_{n-q+},$$

we have that  $((T^* \circ \psi_q(v)))([M^n, \tilde{v}, h]) = \langle \alpha^*(\Phi(v) \cdot \iota_{n-q}), j_n \rangle$ , where  $\Phi(v) \cdot \iota_{n-q}$  is given by the composition

$$TB \wedge E_{n-q+} \xrightarrow{\Phi(v) \wedge \iota_{n-q}} \Sigma^q E \wedge \Sigma^{n-q} E \xrightarrow{\mu} \Sigma^n E.$$

By Lemma 2.4,

$$\begin{aligned} \varphi_q([M^n, \tilde{v}, h]) &= \langle \tilde{v}^*(v) \cdot h^*(\iota_{n-q}), [M^n] \rangle \\ &= \langle r^*((u_M \cdot \tilde{v}^*(v)) \cdot h^*(\iota_{n-q})), j_n \rangle. \end{aligned}$$

Now  $\Phi(v)$  is the composition

$$TB \xrightarrow{\Delta^*} TB \wedge B_+ \xrightarrow{U \wedge v} E \wedge \Sigma^q E \xrightarrow{\mu} \Sigma^q E.$$

Hence  $\alpha^*(\Phi(v) \cdot \iota_{n-q}) = r^*(T\tilde{v}^*(U \cdot v) \cdot h^*(\iota_{n-q}))$ . Thus in order to complete the proof of Proposition 2.3 it remains to show that

$$T\tilde{v}^*(U \cdot v) = u_M \cdot \tilde{v}^*(v) \in E^q(T\nu_M).$$



Since  $u_M$  is the composition  $T\nu_M \xrightarrow{T\tilde{v}} TB \xrightarrow{U} E$ ,  $u_M \cdot \tilde{v}^*(v)$  is given by

$$T\nu_M \xrightarrow{\Delta^*} T\nu_M \wedge M_+^n \xrightarrow{T\tilde{v} \wedge \tilde{v}} TB \wedge B_+ \xrightarrow{U \wedge v} E \wedge \Sigma^q E \xrightarrow{\mu} \Sigma^q E$$

and  $\tilde{v}^*(U \cdot v)$  is given by

$$T\nu_M \xrightarrow{T\tilde{v}} TB \xrightarrow{\Delta^*} TB \wedge B_+ \xrightarrow{U \wedge v} E \wedge \Sigma^q E \xrightarrow{\mu} \wedge \Sigma^q E.$$

So we need only show that the following commutes:

$$\begin{array}{ccccccc} T\nu_M & \xrightarrow{T\tilde{v}} & TB & \xrightarrow{\Delta^*} & TB \wedge B_+ & & \\ \downarrow \parallel & & & & \downarrow \parallel & & \\ T\nu_M & \xrightarrow{\Delta^*} & T\nu_M \wedge M_+^n & \xrightarrow{t\tilde{v} \wedge \tilde{v}} & TB \wedge B_+ & & \end{array}$$

But the first line is induced by

$$M^n \rightarrow B \rightarrow B \times B,$$

and the second line by

$$M^n \xrightarrow{\Delta} M^n \times M^n \xrightarrow{\tilde{v} \times \tilde{v}} B \times B.$$

Since these two compositions are equal, the proposition follows. □

2.5. COROLLARY.  $I_n \cap E^q(B) = \text{Ker } \psi_q$ . □

Making use of the Thom isomorphism, we now study instead  $J_n = \Phi(I_n) \subset E^*(TB)$ . Define a map, which by abuse of notation we still call  $\psi_q$ , from  $E_q(TB)$  to  $\pi_n(TB \wedge E_{n-q^+})^*$  by the rule:  $\psi_q(v)(\alpha)$  is the composition

$$S^n \xrightarrow{\alpha} TB \wedge E_{n-q^+} \xrightarrow{v \wedge t_{n-q}} \Sigma^q E \wedge \Sigma^{n-q} E \xrightarrow{\mu} \Sigma^n E.$$

We have the following immediate consequence:

2.6. COROLLARY.  $J_n \cap E^q(TB) = \text{Ker } \psi_q$ . □

Using the fact that  $\pi_n(TB \wedge E_{n-q^+}) = TB_n(E_{n-q^+})$ , we have a map,

$$\psi_q^*: TB_n(E_{n-q^+}) \rightarrow E^q(TB)^*.$$

We last have a map  $\eta_q: E_q(TB) \rightarrow E^q(TB)^*$  given by

$$\eta_q(x)(v) = \langle v, x \rangle$$

for  $x \in E_q(TB)$ ,  $v \in E^q(TB)$ . Then the main result of this generalized Brown-Peterson process is the following, which is proven simply by checking the two composites on a homotopy level:

2.7. THEOREM. *The following diagram commutes:*

$$\begin{array}{ccc}
 TB_n(E_{n-q^+}) & \xrightarrow{\psi_q} & E^q(TB)^* \\
 \downarrow (i_{n-q})_* & & \uparrow \eta_q \\
 TB_n(\Sigma^{n-q}E) = TB_q(E) & \xrightarrow{\chi} & E_q(TB)
 \end{array}$$

where  $\chi$  is induced by the switch-map  $TB \wedge E \rightarrow E \wedge TB$ . □

2.8. COROLLARY. *If  $\eta_q$  is an isomorphism and  $E = TB$ , then*

$$J_n \cap E^q E = \text{Ker}((i_{n-q})^* \circ \chi^*),$$

where  $\chi: E_*E \rightarrow E_*E$  is the canonical anti-automorphism associated to the Hopf algebra  $E_*E$ . □

In particular, for  $TB = MO$ , i.e., for unoriented cobordism, the above calculation reduces to that of Brown and Peterson. In fact, since  $MO$  splits as a wedge of Eilenberg-Mac Lane spectra  $K\mathbb{Z}/2$ , if we restrict our attention to the  $K\mathbb{Z}/2$  summand containing the Thom class, one may easily verify that  $(T\nu_M)^*: H^*(K\mathbb{Z}/2) \rightarrow H^*(T\nu_M)$  (with  $\mathbb{Z}/2$  coefficients) is given by  $(T\nu_M)^*(a) = a \cdot u_M$ , where  $a \in A$ , the Steenrod algebra, and  $u_M \in H^0(T\nu_M)$  is the Thom class. Then if

$$J_n(0) = \bigcap_{M^n} \text{Ker}(T\nu_M)^*,$$

using the fact that  $a \in \text{Ker}(i_p)^*: A \rightarrow H^*(K(\mathbb{Z}/2, p))$  if and only if the element  $a$  has excess  $e(a) > p$ , we obtain the following result of Brown and Peterson's [BP]:

2.9. COROLLARY.  $J_n(0) = \{a \in A \mid \dim(\chi(a)) + e(\chi(a)) > n\}$ . □

3.  **$MU$ ,  $BP$ , and the Morava  $K$ -Theories.** In the remainder of this paper we restrict ourselves to the study of stably almost-complex manifolds, where  $B_{2k} = B_{2k+1} = BU(k)$ , and the resulting Thom spectrum is  $MU$ . Now  $MU$  localized at a prime  $p$  splits into a wedge of suspensions of  $BP$  summands. Unfortunately, neither  $MU$  nor  $BP$  satisfies condition 2.1(b) in general. Thus we are led to use  $E = K(m)$ , the Morava  $K$ -theories, as our generalized (co)homology

theories. In this section we collect some facts about  $K(m)$  and related spectra.

Fix a prime  $p$ . For the  $BP$  spectrum associated to  $p$  we have

$$\pi_*BP \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots], \quad \dim(v_i) = 2(p^i - 1),$$

where  $\mathbb{Z}_{(p)}$  represents the integers localized at  $p$ . The Morava  $K$ -theories are  $BP$ -module spectra related to  $BP$  by maps  $K_m: BP \rightarrow K(m)$ . We collect their basic properties in the following (see, for example, [RW2]):

**3.1. PROPOSITION.** (a) For  $p \neq 2$ ,  $K(m)$  is a commutative ring spectrum.

- (b)  $\pi_*K(m) \cong (\mathbb{Z}/p)[v_m, v_m^{-1}]$ .
- (c)  $K(m)_*(X \times Y) \cong K(m)_*(X) \otimes_{\pi_*K(m)} K(m)_*(Y)$  for spaces  $X, Y$ .
- (d) As a map of coefficient rings,

$$(K_m)_*(v_m) = v_m, \quad \text{and} \quad (K_m)_*(v_q) = 0, \quad q \neq m.$$

- (e)  $K(m)_*(X) \cong (K(m)^*(X))^*$  for  $X$  a space or a spectrum.
- (f) Let  $\underline{K(m)}_q$  be the  $q$ th space in the  $\Omega$ -spectrum for  $K(m)$ . Then there are homotopy equivalences for each  $q$ ,  $\underline{K(m)}_{q+2(p^m-1)} \rightarrow \underline{K(m)}_q$ . □

Note that 3.1(e) follows from the Universal Coefficient Theorem spectral sequence (see [A1]), since  $\pi_*K(m)$  is a “graded field” and hence  $K(m)_*(X)$  is free over  $\pi_*K(m)$ .

Next we introduce some intermediate theories lying between  $BP_*$  and  $K(m)_*$  which will be of use to us. Let  $E$  be a ring spectrum, and let  $x \in \pi_n E$ . Then multiplication on the left by  $x$  induces a map  $x: \Sigma^n E \rightarrow E$ . Let  $I(m) \subset \pi_*BP$  be the ideal defined by

$$I(0) = 0, \quad I(1) = (p), \quad I(m) = (p, v_1, \dots, v_{m-1}) \quad \text{for } m > 1.$$

**3.2. PROPOSITION [JW1].** There exist spectra  $P(m)$ ,  $m = 0, 1, 2, \dots$ , such that

- (a)  $P(0) = BP$ ;
- (b)  $\pi_*P(m) \cong BP/I(m) \cong \mathbb{Z}_{(p)}[v_m, v_{m+1}, \dots]$  for  $m \geq 1$ ;
- (c)  $P(m)$  is a left  $BP$ -module spectrum;
- (d)  $P(m+1)$  is related to  $P(m)$  by a stable cofibration

$$\Sigma^{2(p^m-1)}P(m) \xrightarrow{v_m} P(m) \xrightarrow{g_m} P(m+1);$$

(e)  $(g_m)_* : \pi_* P(m) \rightarrow \pi_* P(m+1)$  is given on generators by

$$(g_m)_*(v_i) = 0 \quad \text{if } i \leq m, \\ = v_i \quad \text{if } i > m.$$

(f) for  $p > 2$ ,  $P(m)$  is a commutative ring spectrum. □

Thus  $P(m+1)$  may be obtained from  $P(m)$  by “killing” the element  $v_m$  via the cofibration of 3.2(d). Proceeding in this manner, we may start from  $P(m)$  and kill the generators  $v_{m+1}, v_{m+2}, \dots$  of  $\pi_* P(m)$  to obtain in the limit the  $BP$ -module spectrum  $k(m)$ . We have then that  $\pi_* k(m) \cong (\mathbb{Z}/p)[v_m]$ . If we let  $T_m = \{1, v_m, v_m^2, \dots\}$  be the multiplicative set of nonnegative powers of the element  $v_m \in \pi_* k(m)$ , then we may obtain  $K(m)_*$  by localizing the homology theory  $k(m)_*$  with respect to  $T_m$  via the techniques described in [JW2].

Finally, we note that the maps

$$MU \rightarrow BP \rightarrow P(1) \rightarrow \dots \rightarrow P(m) \rightarrow k(m) \rightarrow K(m)$$

give  $MU$  an orientation with respect to the cohomology theories  $BP^*$ ,  $P(m)^*$ ,  $k(m)^*$ , and  $K(m)^*$ .

**4. Calculation of relations for stably almost-complex manifolds.** We now return to the generalized Brown-Peterson process and apply it to the case  $B_{2k} = B_{2k+1} = BU$  as before. By 3.1(e) and the remark at the end of the last section, the cohomology theory  $K(m)^*$  satisfies conditions 2.1(a') and 2.1(b) for stably almost-complex manifolds. By Corollary 2.6, we need to determine the kernel of the map  $\psi_q : K(m)^q(MU) \rightarrow MU_n(\underline{K(m)}_{n-q^+})^*$ . Dually, we need to determine the cokernel (and hence the image) of the map

$$\psi_q^* : MU_n \underline{K(m)}_{n-q^+} \rightarrow K(m)_q MU.$$

Here we are making use of 3.1(e). By Theorem 2.7, then, since  $\eta_q$  is an isomorphism, we need to calculate the image of

$$MU_n \underline{K(m)}_{n-q^+} \xrightarrow{(t_{n-q})^*} MU_n \Sigma^{n-q} K(m) = MU_q K(m) \xrightarrow{\chi} K(m)_q MU.$$

Since  $MU$  localized at  $p$  is made up of  $BP$ -summands, it suffices, modulo  $\chi$ , to calculate the image of the stabilization map

$$BP_* \underline{K(m)}_s \xrightarrow{(t_s)^*} BP_* K(m).$$

We make use of the following, where  $E(x_1, \dots, x_t)$  is the exterior algebra on the generators  $x_1, \dots, x_t$ .

4.1. LEMMA.  $K(m)_*P(m) \cong K(m)_*BP \otimes E(\tau_0, \dots, \tau_{m-1})$  as modules over  $\pi_*K(m)$ , where  $\dim(\tau_j) = 2p^j - 1$ .

*Proof.* The Atiyah-Hirzebruch spectral sequence for  $k(m)_*BP$  collapses, yielding

$$k(m)_*BP \cong H_*BP \otimes \pi_*k(m) \cong (\mathbb{Z}/p)[v_m; c_1, c_2, \dots]$$

as  $\mathbb{Z}/p$ -algebras, where  $\dim(c_j) = 2(p^j - 1)$ .

If we apply  $k(m)_*(\ )$  to the cofibration of 3.2(d), we obtain an exact sequence for  $q < m$ :

$$\begin{aligned} \dots \rightarrow k(m)_sP(q) \xrightarrow{v_q} k(m)_{s+r}P(q) \rightarrow k(m)_{s+r}P(q+1) \\ \rightarrow k(m)_{s-1}P(q) \rightarrow \dots \end{aligned}$$

where  $r = 2(p^q - 1)$ .

But multiplication by  $v_q$  is zero in  $k(m)_*(\ )$ . Hence  $k(m)_*P(q)$  injects in  $k(m)_*P(q+1)$ . Furthermore, when  $s = 1$  we obtain a new element  $\tau_q \in k(m)_{2q^q-1}P(q+1)$  which is external, as one easily checks inductively by using our knowledge of  $k(m)_*BP$  (recall that  $P(0) = BP$ ). Thus for  $q < m$ ,  $k(m)_*P(q+1) \cong k(m)_*P(q) \otimes E(\tau_q) \cong k(m)_*BP \otimes E(\tau_0, \dots, \tau_q)$ . Localizing now with respect to  $\{1, v_m, v_m^2, \dots\}$  gives the desired result.  $\square$

4.2. COROLLARY. *The map  $BP_*K(m) \rightarrow P(m)_*K(m)$  is injective.*  $\square$

With 4.2 in mind, we shall make use of the following commutative diagram:

$$(4.3) \quad \begin{array}{ccc} BP_*\underline{K(m)}_q & \xrightarrow{(i_q)_*} & BP_*K(m) \\ \downarrow & & \downarrow \\ P(m)_*\underline{K(m)}_q & \xrightarrow{(i_q)_*} & P(m)_*K(m), \end{array}$$

and calculate the image of  $(i_q)_*$  on  $P(m)_*$ -homology.

4.4. REMARK. Because of problems with the multiplication in the spectra  $P(m)$ ,  $k(m)$ , and  $K(m)$  at the prime 2  $[\mathbf{R}]$ , we restrict our attention to  $p$  odd from now on.

Wilson has calculated  $P(m)_*\underline{K(m)}_q$  for each  $q$ , by considering  $P(m)_*\underline{K(m)}_* = \{P(m)_*\underline{K(m)}_q\}$  as a Hopf ring. The general reference for Hopf rings is  $[\mathbf{RW1}]$ ; here we recall only that there are structure

maps

$$* : P(m)_* \underline{K(m)}_k \otimes P(m)_* \underline{K(m)}_k \rightarrow P(m)_* \underline{K(m)}_k \quad (\text{for each } k), \text{ and}$$

$$\circ : P(m)_* \underline{K(m)}_k \otimes P(m)_* \underline{K(m)}_n \rightarrow P(m)_* \underline{K(m)}_{k+n} \quad (\text{for all } n, k)$$

satisfying certain properties (associativity, distributivity, having a unit, etc.) The map  $*$  is induced by the loop-space multiplication on  $\underline{K(m)}_k$ , and  $\circ$  is induced by the multiplication

$$\mu : P(m) \wedge P(m) \rightarrow P(m) \quad \text{and} \quad m_{k,n} : \underline{K(m)}_k \wedge \underline{K(m)}_n \rightarrow \underline{K(m)}_{k+n}.$$

Using these two maps, the Hopf ring  $P(m)_* \underline{K(m)}_*$  is generated by elements  $e_1 \in P(m)_1 \underline{K(m)}_1$ ,  $a_{(i)} \in P(m)_{2p^i} \underline{K(m)}_1$  for  $i < m$ , and  $b_{(i)} \in P(m)_{2p^i} \underline{K(m)}_2$ , which we now describe. For  $q < 2p^m - 1$ ,  $\tilde{P}(m)_q \underline{K(m)}_1 \cong \tilde{H}_q(K(\mathbb{Z}/p, 1); \mathbb{Z}/p)$  since  $\underline{K(m)}_1 \simeq K(\mathbb{Z}/p, 1)$  through dimension  $2(p^m - 1)$ , and  $P(m) \simeq K\mathbb{Z}/p$  in stable dimensions less than  $2(p^m - 1)$ .  $H_1(K(\mathbb{Z}/p, 1); \mathbb{Z}/p)$  and  $H_{2p^i}(K(\mathbb{Z}/p, 1); \mathbb{Z}/p)$  are isomorphic to  $\mathbb{Z}/p$ ; use this isomorphism on the canonical generators to define  $e_1$  and  $a_{(i)}$ .  $P(m)_* CP^\infty$  is free over  $\pi_* P(m)$  on generators  $\beta_i \in P(m)_{2i} CP^\infty$ . Using these elements and the  $K(M)$ -orientation for  $CP^\infty$ , represented by a map  $CP^\infty \rightarrow \underline{K(m)}_2$ , one defines  $b_{(i)} \in P(m)_{2p^i} \underline{K(m)}_2$ .

For  $I = (i_0, i_1, \dots, i_{m-1})$  and  $J = (j_0, j_2, \dots)$  nonnegative finite sequences with  $i_k = 0$  or 1 and  $j_k < p^m$ , define

$$I J \quad \circ i_0 \quad \circ i_{m-1} \quad \circ j_0 \quad \circ j_1$$

$$ab = a_{(0)} \quad \circ \dots \circ a_{(m-1)} \quad \circ b_{(0)} \quad \circ b_{(0)} \quad \circ \dots$$

Then Wilson's theorem states that, as a  $\pi_* P(m)$ -algebra,  $P(m)_* \underline{K(m)}_*$  is described in terms of the above elements as follows. For  $j_0 < p^m - 1$ , each  $a^I b^J \circ e_1$  is an exterior generator; and depending on  $I$  and  $J$  each  $a^I b^J$  is either a polynomial or a truncated polynomial generator, all using the  $*$  product. Here,  $P(m)_* \underline{K(m)}_*$  is considered as graded over  $\mathbb{Z}/2(p^m - 1)$  instead of over  $\mathbb{Z}$ , by use of 3.1(f). The homotopy equivalence of 3.1(f) is given by the "periodicity operator"  $[v_m] \in \pi_0 \underline{K(m)}_{-2(p^m - 1)}$  as:

$$\underline{K(m)}_{q+r} \approx S^0 \wedge \underline{K(m)}_{q+r} \xrightarrow{[v_m]^{\wedge 1}} \underline{K(m)}_{-r} \wedge \underline{K(m)}_{q+2r} \xrightarrow{\mu} \underline{K(m)}_q,$$

where  $r = 2(p^m - 1)$ .

4.5. PROPOSITION (Wilson [W]). *The following relations hold in  $P(m)_* \underline{K(m)}_*$ , where  $\lambda : BP \rightarrow P(m)$  is the induced map from 3.2:*

- (a)  $e_1 \circ$ —is the homology suspension map.
- (b)  $e_1 \circ e_1 = b_{(0)}$ .

- (c)  $a_{(i)} \circ a_{(j)} = -a_{(j)} \circ a_{(i)}$ .  
 (d)  $\lambda_*(v_m)e_1 = [v_m] \circ b_{(0)}^{\circ p^m - 1} \circ e_1$ .  
 (e)  $[v_m] \circ b_{(k)}^{\circ p^m} = \sum_{i=0}^k \lambda_*(v_{m+i}^{p^{k-i}}) b_{(k-i)} \pmod{*}, k > 0$ .  $\square$

Our goal is to determine the image of the stabilization map  $P(m)_* \underline{K(m)}_q \rightarrow P(m)_* \underline{K(m)}$ . First we calculate the stable object  $P(m)_* \underline{K(m)}$ . Let  $R_m = \pi_* P(m)[v_m^{-1}] \cong (\mathbb{Z}/p)[v_m, v_m^{-1}, v_{m+1}, \dots]$ , and let  $E(x)$  and  $P(x)$  denote, respectively, the exterior and polynomial algebras on the generator  $x$ .

4.6. THEOREM. As  $\pi_* P(m)$ -modules,

$$P(m)_* \underline{K(m)} \cong E_{R_m}(\tilde{\tau}_0, \tilde{\tau}_1, \dots, \tilde{\tau}_{m-1}) \otimes P_{R_m}(\xi_1, \xi_2, \dots)$$

modulo the relations

$$\xi_k^{p^k} = v_m^{-1} \sum_{i=0}^k v_{m+i}^{p^{k-i}} \xi_{k+i},$$

where  $\dim(\tilde{\tau}_i) = 2p^i - 1$  and  $\dim(\xi_i) = 2(p^i - 1)$ .

*Proof.* The stabilization map  $P(m)_* \underline{K(m)}_q \rightarrow P(m)_* \underline{K(m)}$  is given, from 4.5(a), by  $\circ$ -multiplication with  $e_1$  infinitely often. Stabilization kills  $*$ -products and  $e_1$  stabilizes to  $1 \in P(m)_0 \underline{K(m)}$ , so we need only concern ourselves with elements of the form  $[v_n]^r \circ a^I b^J$ , where  $r \in \mathbb{Z}$  and  $I$  and  $J$  are as before, with the additional property that  $j_0 = 0$  (by 4.5(b)). By 4.5(d), all of these elements survive to  $P(m)_* \underline{K(m)}$ .

In particular, let  $\tilde{\tau}_i$  and  $\xi_j$  be the stable images of  $a_{(i)}$  and  $b_{(j)}$  respectively, for  $0 \leq i \leq m-1$  and  $j > 0$ . One may easily verify that for  $\alpha \in P(m)_* \underline{K(m)}_r$ ,  $\beta \in P(m)_* \underline{K(m)}_s$ ,

$$\iota_{r+s}(\alpha \circ \beta) = \iota_r(\alpha) \iota_s(\beta)$$

in  $P(m)_* \underline{K(m)}$ . Using this result we have that  $\tilde{\tau}_i \tilde{\tau}_j = -\tilde{\tau}_j \tilde{\tau}_i$  (from (4.5(c)), and  $a^I b^J$  stabilizes to  $\tilde{\tau}^I \xi^J$ , defined analogously.

By 4.5(d) we have that  $[v_m] \in P(m)_0 \underline{K(m)}_{2-2p^m}$  stabilizes to the same element as the image

$$v_m \in \pi_{2(p^m-1)} P(m) \rightarrow P(m)_{2(p^m-1)} \underline{K(m)}.$$

(That is to say, multiplication by  $v_m$  is the same on the left and on the right in  $P(m)_* \underline{K(m)}$ .) Hence the coefficient ring for  $P(m)_* \underline{K(m)}$  becomes  $\pi_* P(m)[v_m^{-1}] = R_m$ .

Finally, since stabilization is a  $\pi_*P(m)$ -module map, 4.5(e) stabilizes to the relation

$$v_m \xi_k^{p^k} = \sum_{i=0}^k v_{m+i}^{p^{k-i}} \xi_{k-i}, \quad \text{or} \quad \xi_k^{p^k} = v_m^{-1} \sum_{i=0}^k v_{m+i}^{p^{k-i}} \xi_{k+i}.$$

This finishes the proof of 4.6. □

From 4.6 we can tell how far each element of  $P(m)_*K(m)$  desuspends. Given  $\alpha \in P(m)_*K(m)$ , let  $d(\alpha)$  be the minimum  $q$  such that there is an  $\alpha_q \in P(m)_*K(m)_q$  with  $(i_q)_*(\alpha_q) = \alpha$ . Define  $\tilde{\tau}^I \tilde{\zeta}^J$  in analogy with  $a^I b^J$  (except that there is no  $\xi_0$ ).

4.7. COROLLARY.  $d(v_m^r \tilde{\tau}^I \tilde{\zeta}^J) = |I| + 2|J| - 2r(p^m - 1)$ , where

$$|I| = \sum_{s=0}^{m-1} i_s \quad \text{and} \quad |J| = \sum_{t=1}^{\infty} j_t (< \infty).$$

*Proof.* Since  $a^I \in P(M)_*K(m)_{|I|}$  and  $b^J \in P(m)_*K(m)_{|J|}$ , we need note only that  $d(v_m^r) = -2r(p^m - 1)$ . □

We now return our attention to  $BP_*K(m)$  and  $BP_*K(m)$ , by making use of 4.3. First we prove the following:

- 4.8. LEMMA. (a)  $\tilde{\tau}_1 \notin \text{Im } \lambda_* \subset P(m)_*K(m)$ .  
 (b)  $b_{(i)} \in \text{Im } \lambda_* \subset P(m)_*K(m)_2$ .

*Proof.* (a) Since  $\lambda_*(xy) = \lambda_*(x)\lambda_*(y)$  (see, for example, [Wü]), we have that if  $\lambda_*(\alpha) = \tilde{\tau}_i$ , then  $\alpha^2 = 0$ . But by the proof of 4.1,  $BP_*k(m)$  has no exterior elements, and the same holds, after localization, for  $BP_*K(m)$ . Hence no such  $\alpha$  exists.

(b) We have that  $BP_*CP^\infty$  is free over  $\pi_*BP$  on generators  $\tilde{\beta}_i \in BP_{2i}CP^\infty$ . By the commutativity of

$$\begin{array}{ccc} BP_*CP^\infty & \xrightarrow{\theta_*} & BP_*K(m)_2 \\ \downarrow \lambda_* & & \downarrow \lambda_* \\ P(m)_*CP^\infty & \xrightarrow{\theta_*} & P(m)_*K(m)_2 \end{array}$$

where  $\theta: CP^\infty \rightarrow K(m)_2$  is the orientation, since  $\lambda_*(\tilde{\beta}_i) = \beta_i$  and  $(\theta \circ \lambda)_*(\tilde{\beta}_i) = b_{(i)}$ , there is an element  $\tilde{b}_i \in BP_{2i}K(m)_2$  with  $\lambda_*(\tilde{b}_i) = b_{(i)}$ . □

By the commutativity of  $\lambda_*$  with the stabilization map, we conclude that there is an element  $\tilde{\xi}_i \in BP_{2(p^i-1)}K(m)$  with  $\lambda_*(\tilde{\xi}_i) = \xi_i$ .



By defining the function  $d$  in analogy with 4.7, the following is a consequence of 4.7 and 4.8:

$$4.9. \text{ COROLLARY. } d(v_m^r \xi^J) = 2|J| - 2r(p^m - 1). \quad \square$$

Using the fact that stabilization about is a  $\pi_*BP$ -module map, by 4.8 this completes the description of  $\text{Im } \iota_*: BP_*\underline{K}(m)_* \rightarrow BP_*K(m)_*$ .

#### REFERENCES

- [A1] J. F. Adams, *Lectures on Generalized Cohomology*, LNM 99, Springer-Verlag (1969), 1–138.
- [A2] ———, *Stable Homotopy and Generalized Homology*, University of Chicago Press, 1980.
- [BG] E. H. Brown, Jr. and S. Gitler, *A spectrum whose cohomology is a certain module over the Steenrod algebra*, *Topology*, **12** (1973), 283–295.
- [BP] E. H. Brown, Jr. and F. P. Peterson, *Relations among characteristic classes-I*, *Topology*, **3** Supp. 1 (1964), 39–52.
- [Ca] G. E. Carlsson, *G. B. Segal's Burnside ring conjecture for  $(\mathbb{Z}/2)^k$* , *Topology*, **22** (1983), 83–103.
- [C1] R. L. Cohen, *The immersion conjecture for differentiable manifolds*, *Ann. Math.*, **122** (1985), 237–328.
- [C2] ———, *Odd primary families in stable homotopy theory*, *Mem. Amer. Math. Soc.*, **242** (1981).
- [JW1] D. C. Johnson and W. S. Wilson, *BP operations and Morava's extraordinary K-theories*, *Math. Zeit.*, **144** (1975), 55–75.
- [JW2] ———, *Projective dimension and Brown-Peterson homology*, *Topology*, **12** (1973), 327–353.
- [Ma] M. Mahowald, *A new infinite family in  $2\pi_*^s$* , *Topology*, **16** (1977), 249–256.
- [Mi] H. R. Miller, *The Sullivan conjecture on maps from classifying spaces*, *Ann. Math.*, **120** (1984), 39–87.
- [R] D. C. Ravenel, *Localisation with respect to certain periodic cohomology theories*, *Amer. J. Math.*, **106** (1984), 351–414.
- [RW1] D. C. Ravenel and W. S. Wilson, *The Hopf ring for complex cobordism*, *J. Pure Appl. Algebra*, **9** (1977), 241–280.
- [RW2] ———, *The Morava K-theories of Eilenberg-MacLane spaces and the Connor-Floyd conjecture*, *Amer. J. Math.*, **102** (1980), 691–748.
- [Sh] D. H. Shimamoto, *An integral version of the Brown-Gitler spectrum*, *Trans. Amer. Math. Soc.*, **283** (1984), 383–421.
- [St] R. E. Stong, *Notes on cobordism theory*, *Mathematical Notes*, Vol. 7, Princeton University Press, 1968.
- [W] W. S. Wilson, *The Hopf ring for Morava K-theory*, *Publ. RIMS, Kyoto University*, **20** (1984), 1025–1036.

Received September 18, 1987 and in revised form May 30, 1989.

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

V. S. VARADARAJAN  
(Managing Editor)  
University of California  
Los Angeles, CA 90024-1555-05

HERBERT CLEMENS  
University of Utah  
Salt Lake City, UT 84112

THOMAS ENRIGHT  
University of California, San Diego  
La Jolla, CA 92093

R. FINN  
Stanford University  
Stanford, CA 94305

HERMANN FLASCHKA  
University of Arizona  
Tucson, AZ 85721

VAUGHAN F. R. JONES  
University of California  
Berkeley, CA 94720

STEVEN KERCKHOFF  
Stanford University  
Stanford, CA 94305

C. C. MOORE  
University of California  
Berkeley, CA 94720

MARTIN SCHARLEMANN  
University of California  
Santa Barbara, CA 93106

HAROLD STARK  
University of California, San Diego  
La Jolla, CA 92093

## ASSOCIATE EDITORS

R. ARENS

E. F. BECKENBACH  
(1906–1982)

B. H. NEUMANN

F. WOLF  
(1904–1989)

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA  
UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA, RENO  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY

UNIVERSITY OF OREGON  
UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF HAWAII  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the 1980 *Mathematics Subject Classification* (1985 Revision) scheme which can be found in the December index volumes of *Mathematical Reviews*. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California 90024-1555-05.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 50 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* (ISSN 0030-8730) is published monthly. Regular subscription rate: \$190.00 a year (12 issues). Special rate: \$95.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

---

The Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924 (ISSN 0030-8730) is published monthly. Second-class postage paid at Carmel Valley, California 93924, and additional mailing offices. Postmaster: send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Copyright © 1990 by Pacific Journal of Mathematics

<b>Chong Hsio Fang and Minking Eie, On the values of a zeta function at nonpositive integers</b> .....	201
<b>Howard D. Fegan, Brian F. Steer and L. Whiteway, Spectral symmetry of the Dirac operator for compact and noncompact symmetric pairs</b> .....	211
<b>William James Heinzer and David C. Lantz, Integral domains that lose ideals in overrings</b> .....	223
<b>Alexander Eben Koonce, Relations among generalized characteristic classes</b> .....	239
<b>M. S. Narasimhan and Günther Trautmann, Compactification of <math>M_{\mathbb{P}_3}(0, 2)</math> and Poncelet pairs of conics</b> .....	255
<b>James Alexander Reeds, III and Lawrence A. Shepp, Optimal paths for a car that goes both forwards and backwards</b> .....	367
<b>Ai-Nung Wang, Constant mean curvature surfaces on a strip</b> .....	395