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# SYMPLECTIC-WHITTAKER MODELS FOR $\mathrm{Gl}_n$

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**We consider the Klyachko models of admissible irreducible representations of the group $\mathrm{GL}_n(F)$ where $F$ is a non-Archimedean local field of characteristic 0. These are models which generalize the usual Whittaker model by allowing the inducing subgroup a symplectic component. We prove the uniqueness of the symplectic models and the disjointness for unitary representations of the different models. Moreover, for $n \leq 4$ we prove that all unitary irreducible representations admit a Klyachko model.**

**Introduction.** Let $F$ be a non-Archimedean local field of characteristic zero. This paper studies the realization of irreducible, admissible representation of $\mathrm{Gl}_n(F)$ in certain induced representations generalizing the Whittaker model. In contrast to generalizing by allowing degenerate Whittaker characters or smaller unipotent groups arising from some degenerate data (cf. **[Mo-Wa]**), we generalize the inducing subgroup by allowing a symplectic component.

Our investigation is motivated by results of A. A. Klyachko **[Kl]**, who exhibited a model, in the sense of I. M. Gel'fand, for $\mathrm{Gl}_n$ over a finite field. He found a set of representations (which we will refer to as models) which are disjoint, multiplicity free and exhaust the set of irreducible representations. The representations he considers form a family $\mathscr{M}_{n,k}$, $0 \leq k \leq [\frac{n}{2}]$. One extreme $\mathscr{M}_{n,0}$, is the Whittaker model, a representation induced off a character on the subgroup of unipotent, upper triangular matrices. When $n$ is even, the other extreme $\mathscr{M}_{n,n/2}$ is induced off the trivial character of $\mathrm{Sp}_n$, the symplectic group of $2n \times 2n$ matrices. The other "mixed" models $\mathscr{M}_{n,k}$, $0 < k < \frac{n}{2}$, are induced off characters of subgroups coming from smaller unipotent and symplectic groups. Since the Whittaker model for representations of $p$-adic $\mathrm{Gl}_n$ is of considerable importance, e.g. in the study of automorphic forms, it is natural to investigate the role of the other models in the $p$-adic case.

The natural category to study in the local field setting is the category of admissible representations. The Whittaker model $\mathscr{M}_{n,0}$ is the only model which has received attention. It was shown by I. M. Gel'fand and D. A. Kazhdan (**[Ge-Ka,1]**) that the Whittaker model is unique, meaning that for an irreducible representation $\pi$, $\mathrm{Hom}_{\mathrm{Gl}_n}(\pi, \mathscr{M}_{n,0})$ has dimension at most one.

The main results of this paper are:

(1) Uniqueness of the symplectic model.

(2) Unitary disjointness of the set of models, i.e. a unitary representation cannot embed in two different models.

The advent of unitary representations is natural in light of $\mathrm{Gl}_3$. In that case there is an irreducible representation without a model but the intriguing fact is that all irreducible unitary representations have unique models. This prompts focusing our attention on the questions of existence and uniqueness of models for unitary representations and leads to the remaining results of the paper.

(3) The description of the category of admissible representations of $\mathrm{Gl}_3$ with respect to models. In particular it is shown that every irreducible unitary representation admits a unique model and we describe the (essentially) only representation which does not admit a model.

(4) The existence and uniqueness of models for irreducible, unitary representations of $\mathrm{Gl}_4$.

The reason for the symplectic group playing such a role is not clear; however there are two properties it enjoys which are prominent in our results and those in **[Kl]**. The first is that $\mathrm{Sp}_n$ is the fixed point set of an involution on $\mathrm{Gl}_n$, which we use in (1). The second is that there is a bijection between the set of $\mathrm{Sp}_n$ double cosets of $\mathrm{Gl}_{2n}$ and the set of conjugacy classes of $\mathrm{Gl}_n$. Over the finite field with $q$ elements, this bijection has been central to recent work of Bannai, Kawanaka and Song (**[Ba-Ka-So]**), who prove that the character table of the Hecke algebra of $\mathrm{Sp}_n$ bi-invariant functions on $\mathrm{Gl}_{2n}$ is "almost" obtained from the character table of $\mathrm{Gl}_n$ by the substitution $q$ to $q^2$.

A word about the proofs. In the finite field case, no explicit descriptions or structure of the irreducible representations is used. In the $p$-adic case we depend heavily on the description of admissible and unitary representations due to I. N. Bernstein and A. V. Zelevinskii (**[Be-Ze,1]**, **[Ze]**) and M. Tadić (**[Ta,1]**). Using these and the yoga of Jacquet functors it is not difficult to inductively show that many representations have models, but this method will not show that a representation has a symplectic model. It is desirable to have a simple inductive statement for the existence of symplectic models. One of our goals is to determine to what extent this is possible. In the case of $\mathrm{Gl}_4$ we show that it is. There we consider a representation induced from representations with symplectic models as part of a family of induced representations depending on a complex parameter $s$. On these representations we define a functional by an integral and show that it converges if the real part of $s$ is sufficiently large. Then using the theory of Bernstein, developed for the analytic continuation of intertwining operators, we continue the functional to the original representation. This inductive statement in particular provides the symplectic models for certain complementary series representations of $\mathrm{Gl}_4$. Other unitary representations arise as Langlands quotients from square integrable data. For these to have symplectic models it must be shown that the functional descends to the unique irreducible quotient. The representations of $\mathrm{Gl}_4$ which require this attention are special cases of a unitary Langlands quotient representation of $\mathrm{Gl}_{2n}$ which is fundamental in the description of the unitary dual. Knowledge of the composition series of this induced representation is used to show that these irreducible quotients have symplectic models in general (Theorem 11.1). (H. Jacquet has recently obtained this result by similar methods.) This is the technical heart of the paper; the case of $\mathrm{Gl}_4$ illustrates the problems that will be encountered in the general case.

We now briefly describe the organization of this paper. Section 1 sets notation and conventions and reviews general background. The next two sections are devoted to proving the general results on uniqueness of symplectic models and unitary disjointness of models. Section 4 presents some results on symplectic orbits in certain flag varieties. The rest of the paper is devoted to specific groups $\mathrm{Gl}_2$ is dispatched in §5. In §6 we recall the classification of the unitary dual of $\mathrm{Gl}_n$ due to Tadić, and explicate it in the cases of $\mathrm{Gl}_3$ and $\mathrm{Gl}_4$ in §§7 and 10 respectively. Section 8 contains the proof that every irreducible, unitary representation of $\mathrm{Gl}_3$ has a unique model. Those admissible representations of $\mathrm{Gl}_3$ without models are described in §9. Section 11 shows that the unitary representations of $\mathrm{Gl}_4$ all have models.

We are grateful to various people for discussions which have proven valuable in the production of this paper. In particular we are grateful to M. Tadić for explaining some of his work and to H. Jacquet for his interest and receptivity.

## 1. Notation and terminology.

General references for notation and terminology are **[Be-Ze,1]** and **[Be-Ze,2]**.

Throughout, $F$ will denote a non-Archimedean local field of characteristic zero, i.e. a $p$-adic field. Unless stated otherwise, $\mathrm{GL}_n$ will denote $\mathrm{GL}_n(F)$.

The standard (upper triangular) parabolic subgroups of $\mathrm{Gl}_n$ are in one-to-one correspondence with partitions of $n$: $(n_1, \dots, n_k)$, $n_1 + \cdots + n_k = n$. $P_{n_1,\dots,n_k}$ denotes the associated group and $N_{n_1,\dots,n_k}$ its unipotent radical.

$J_n$ denotes the $2n \times 2n$ matrix $\left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right)$. We sometimes use $J$ to denote the associated symplectic form $J(x, y) = {}^t x J_n y$. The symplectic group $\mathrm{Sp}_n$ preserves this form.

Let $U_n$ denote the group of upper triangular unipotent matrices in $\mathrm{Gl}_n$; thus $U_n = N_{1,1,\dots,1}$. For $0 \leq k \leq [\frac{n}{2}]$, let $N_k$ be the subgroup of $U_n$ of matrices $(u_{ij})$ where for $i \neq j$, $u_{ij} = 0$ unless $i \leq n-2k \leq j$. With $U_{n-2k}$ embedded in the upper left, $\mathrm{Sp}_k$ in the lower right, let $M_k = (U_{n-2k} \times \mathrm{Sp}_k)N_k$.

$\nu$ denotes the character $g \to |\det g|$. $\delta_P$ denotes the modular function of the group $P$. A character of $\mathrm{Gl}_n$ is of the form $g \to \chi(\det g)$ for some character $\chi$ of $F^\times$. We sometimes write $\chi_n$ to indicate the group involved, but we will continue to write $\chi_n$ for the restriction to subgroups of $\mathrm{Gl}_n$.

Induction is always normalized, with ind (resp. Ind) denoting compact (resp. full) induction. Given representations $\sigma_i$ of $\mathrm{Gl}_{n_i}$, $i = 1, \dots, k$, extend $\sigma_{n_1} \otimes \cdots \otimes \sigma_{n_k}$ to $P_{n_1,\dots,n_k}$ so that it is trivial on $N_{n_1,\dots,n_k}$. Denote $\mathrm{Ind}_{P_{n_1,\dots,n_k}}^{\mathrm{Gl}_{n_1+\cdots+n_k}} \sigma_{n_1} \otimes \cdots \otimes \sigma_{n_k}$ by $\sigma_{n_1} \times \cdots \times \sigma_{n_k}$.

To a character $\theta$ of $N_{n_1,\dots,n_k}$ and representation $\pi$ of $\mathrm{Gl}_n$, we have the Jacquet functor $r_{n_1,\dots,n_k;\theta}(\pi)$ which is the quotient of the space of $\pi$, $V_\pi$, by the subspace spanned by $\{\pi(n)v - \theta(n)v | v \in V_\pi, n \in N_{n_1,\dots,n_k}\}$. It is naturally a $\mathrm{Gl}_{n_1} \times \cdots \times \mathrm{Gl}_{n_k}$ module. If $\theta \equiv 1$, we delete it from the notation and may simply write $(\pi)_N$ if there is no risk of confusion with regard to the subgroup $N$. $\tilde{r}$ will denote the normalized Jacquet functor (cf. **[Be-Ze, 2]**).

Let $\psi$ be any nontrivial, complex, additive character of $F$. Define the character $\psi_n$ of $U_n$ by $\psi_n(u_{ij}) = \psi(u_{12} + \cdots + u_{n-1n})$. Any character which is nontrivial on all the simple root groups in $U_n$ will be called nondegenerate or said to be a Whittaker character. The diagonal torus in $\mathrm{Gl}_n$ acts transitively on the set of Whittaker characters.

For $k \leq [\frac{n}{2}]$, define the set of models for $\mathrm{Gl}_n$ to be the representations

$$(1.1) \qquad \mathscr{M}_{n,k} = \mathrm{Ind}_{M_k}^{\mathrm{Gl}_n} \psi_n \otimes 1 \otimes 1.$$

When $n$ is understood, we simply write $\mathscr{M}_k$. $\mathscr{M}_0$ is called the Whittaker model. The Whittaker models for any two Whittaker characters are equivalent.

If $\pi$ is a representation, we denote by $\langle\pi\rangle$ (resp. $L(\pi)$) the unique irreducible submodule (resp. quotient module) of $\pi$, when it exists.

## 2. Uniqueness of symplectic models.

2.1. In this section we show that for an irreducible representation $\pi$, $\dim \mathrm{Hom}_{\mathrm{Gl}_{2n}}(\pi, \mathscr{M}_n) \leq 1$. The proof is a combination of the proof of the uniqueness of the Whittaker model in the $p$-adic case (**[Ge-Ka,1]**) and uniqueness of the symplectic model in the finite field case (**[Kl]**).

2.2. We collect here some results on polar decompositions. We are indebted to Daniel Shapiro for the proofs of these results.

Let $k$ be a field of characteristic different from 2, $\overline{k}$ its algebraic closure and $M$ (resp. $\overline{M}$) denote the set of $n \times n$ matrices with coefficients in $k$ (resp. $\overline{k}$). Similarly, let $\overline{G} = \mathrm{Gl}_{2n}(\overline{k})$, $\overline{\mathrm{Sp}} = \mathrm{Sp}_n(\overline{k})$, and $G$ and Sp will be the $k$ rational points of these groups. Let $\sigma$ denote an involution on $\overline{M}$, i.e. an anti-automorphism of order two.

Lemma 2.2.1. *For any $A \in G$, there exists a polynomial $f \in \overline{k}[t]$, such that $f(A)^2 = A$.*

*Proof.* If $R$ is a commutative ring with unit, in which 2 is invertible, it follows from the Taylor expansion of $(1+z)^{1/2}$ that $1+p$ is a square in $R$, for every nilpotent $p \in R$. If $b$ is a unit in $R$, let $\overline{t}$ be the image of $t$ in $R[t]/(t-b^2)^n$, $n \geq 1$. Since $\overline{s} = \overline{t} - \overline{b}^2$ is nilpotent in this ring, writing $\overline{t} = \overline{b}^2(1 + (\overline{b})^{-2}\overline{s})$ implies that $\overline{t}$ is a square.

Let $m(t)$ be the minimal polynomial of $A$.

$$m(t) = \prod_{1 \leq i \leq s} (t - a_i)^{n_i}. \tag{2.2.1}$$

Choose $b_i \in \overline{k}$ such that $a_i = b_i^2$. Then

$$\overline{k}[A] \cong \overline{k}[t]/(m(t)) \cong R_1 \oplus \cdots \oplus R_s, \tag{2.2.2}$$

where $R_i = \overline{k}[t]/(t - b_i^2)^{n_i}$. The conclusion follows easily □

Proposition 2.2.2. *For any $A \in \overline{G}$, there exist $S, T \in \overline{G}$ such that $\sigma(S) = S$, $\sigma(T) = T^{-1}$ and $A = ST$.*

*Proof.* By the lemma, there exists an $S \in \overline{F}$ such that $S^2 = A\sigma(A)$. As $S$ is a polynomial in $A\sigma(A)$, $\sigma(S) = S$. Set $T = S^{-1}A$. Then $\sigma(T) = \sigma(A)\sigma(A)\sigma(S)^{-1} = \sigma(A)S^{-1}$, and $T\sigma(T) = (S^{-1}A)(\sigma(A)S^{-1}) = S^{-1}(S^2)S^{-1} = I$. □

2.3. Let $J = J_n$. For $A \in \mathrm{Gl}_{2n}$ set $A^J = -J{}^tAJ$, where ${}^tA$ is the transpose of $A$.

PROPOSITION 2.3.1. *Let $k$ denote a local or global field of characteristic zero. There exist $P_1, P_2 \in \mathrm{Sp}_n$, such that $A^J = P_1AP_2$.*

*Proof.* By Proposition 2.2.2, there exist $S, T \in \overline{\mathrm{Gl}}_{2n}$, such that $T^J = T^{-1}$, $S^J = S$ and $A = ST$. Then $A^J = T^{-1}S = T^{-1}AT^{-1}$. Since $T \in \mathrm{Sp}_n$ if and only if $T \in \mathrm{Gl}_{2n}$ and $T^J = T^{-1}$, the proposition will follow if we can show there exists such a decomposition with $T \in \mathrm{Gl}_{2n}$.

The set

$$\mathscr{V}(A) = \{(P_1, P_2) | A^J = P_1AP_2,\ P_1, P_2 \in \overline{\mathrm{Sp}}_n\}, \tag{2.3.1}$$

is an algebraic subset of $\overline{\mathrm{Sp}}_n \times \overline{\mathrm{Sp}}_n$. Given $(P_1, P_2), (Q_1, Q_2) \in \mathscr{V}(A)$, set $R = Q_1P_1^{-1}$. As $P_1AP_2 = Q_1AQ_2$, it follows that $Q_2 = A^{-1}R^{-1}AP_2$, so that $R \in A\overline{\mathrm{Sp}}_nA^{-1}$. Define a left action of $\overline{\mathrm{Sp}}_n \cap A\overline{\mathrm{Sp}}_nA^{-1}$ on $\mathscr{V}(A)$ by $R(P_1, P_2) = (RP_1, A^{-1}RAP_2)$. $\mathscr{V}(A)$ is a left principal homogeneous space for this group.

$A\overline{\mathrm{Sp}}_nA^{-1}$ is the subgroup of $\overline{\mathrm{Gl}}_{2n}$ which leaves invariant the symplectic form associated to the matrix $J' = {}^tAJA^{-1}$. $\overline{\mathrm{Sp}}_n \cap A\overline{\mathrm{Sp}}_nA^{-1}$ is thus the group preserving the forms $J$ and $J'$; denote it by $\overline{\mathrm{Sp}}(J, J')$.

Since both forms are nondegenerate, an endomorphism $\Phi$ is defined by the condition that it satisfy $J'(x, y) = J(\Phi x, y)$. In the terminology of **[Kl]**, $\Phi$ is a symmetric operator and $\overline{\mathrm{Sp}}(J, J')$ is the centralizer of $\Phi$ in $\overline{\mathrm{Sp}}(J)$. By Corollary 5.6 of **[Kl]**, $\overline{\mathrm{Sp}}(J, J')$ is connected and there is an exact sequence

$$1 \to U \to \overline{\mathrm{Sp}}(J, J') \to S \to 1, \tag{2.3.2}$$

where $U$ is a unipotent group and $S$ is a product of symplectic groups. (The statement in **[Kl]** is for a finite field, but it is noted in the proof of Proposition 5.5 that the needed constructions are valid for any algebraically closed field.) Because $U$ is linear and $k$ has characteristic zero, $U$ is connected.

From (2.3.2) we obtain the sequence in Galois cohomology

$$H^1(k, U) \to H^1(k, \overline{\mathrm{Sp}}(J, J')) \to H^1(k, S), \tag{2.3.3}$$

which is exact at the middle term (cf. **[Sp]**, Proposition 2.2). Since $U$ is connected and unipotent, $H^1(k, U) = 0$ (**[Se]**, III, §2.1, Proposition 6). $S$ is a product of symplectic groups which have trivial first cohomology (**[Se]**, III, §1.2, Proposition 3), and thus $H^1(k, S) = 0$ (**[Sp]**), and $\mathscr{V}(A)$ has a rational point. □

2.4. In this section $k$ will now be a non-Archimedian local field of characteristic zero. Let $\overline{\mathscr{X}} = \overline{G} \times \overline{G}$ and $\mathscr{X} = G \times G$ and define an action on the left (resp. right) of $\overline{\mathrm{Sp}} \times \overline{\mathrm{Sp}}$ (resp. $\overline{G}$) on $\overline{\mathscr{X}}$ by coordinate (resp. diagonal) multiplication on the left (resp. right). Let $\mathscr{S}_1(\mathscr{X})$ denote the space of functions on $\mathscr{X}$ which are locally constant, constant on the orbits of $\mathrm{Sp} \times \mathrm{Sp}$ and compactly supported modulo the action of $\mathrm{Sp} \times \mathrm{Sp}$, i.e. for each $f \in \mathscr{S}_1$, there exists a compact set $C \subset \mathscr{X}$ such that $\operatorname{supp} f \subset (\mathrm{Sp} \times \mathrm{Sp})C$.

Define the involution $\sigma$ on $\overline{\mathscr{X}}$ by $\sigma(g_1, g_2) = ((g_2^{-1})^J, (g_1^{-1})^J)$. Let $\mathscr{S}_1(G)$ denote the space of locally constant functions on $G$ which are constant on the orbits of $\mathrm{Sp}$ acting by left multiplication and which are compactly supported modulo $\mathrm{Sp}$. We now have a symplectic version of Theorem 3 in **[Ge-Ka,1]**.

THEOREM 2.4.1. *Define the operator* $A$ *on* $\mathscr{S}_1(G)$ *by* $(Af)(g) = f((g^{-1})^J)$. *If* $C(f_1, f_2)$ *is a G-invariant, bilinear form on* $\mathscr{S}_1(G)$, *then* $C(f_1, f_2) = C(Af_2, Af_1)$.

*Proof*. The proof follows that of **[Ge-Ka,1]**. To use their Theorem 1′, we need only verify that the $\overline{\mathrm{Sp}} \times \overline{\mathrm{Sp}} \times \overline{G}$ orbits in $\overline{\mathscr{X}}$ are permuted by $\sigma$ and that the $\mathrm{Sp} \times \mathrm{Sp} \times G$ orbits in $\mathscr{X}$ are fixed by $\sigma$. The first condition is obvious.

Writing $(g_1, g_2) = (s_1, s_2^{-1})(1, s_2 g_2 g_1^{-1} s_1)(s_1^{-1} g_1)$, we see that the $\mathrm{Sp} \times \mathrm{Sp} \times G$ orbits may be identified with the $\mathrm{Sp}$ double cosets in $G$. We have

$$
\begin{aligned}
(2.4.1)\quad & \sigma(s_1 g_1 g, s_2 g_2 g) \\
& = ((s_2^{-1})^J (g_2^{-1})^J (g^{-1})^J, (s_1^{-1})^J (g_1^{-1})^J (g^{-1})^J), \\
& = (s_2, s_1)((g_2^{-1})^J, (g_1^{-1})^J)(g^{-1})^J,
\end{aligned}
$$

so that orbits are invariant. By Proposition 2.3.1, there exist $s_3$ and $s_4 \in \mathrm{Sp}$, such that

$$
(2.4.2)\qquad \sigma(1, s_1 g s_2) = (s_3 g^{-1} s_4, 1) = (1, s_4^{-1} g s_3^{-1})(s_3 g^{-1} s_4),
$$

so that the orbits are invariant by $\sigma$. □

Let $\pi$ be an irreducible, admissible representation of $G$ on a space $V$. Define the representation $\hat{\pi}$ on $V$ by $\hat{\pi}(g) = \pi((g^{-1})^J)$. By

Theorem 2 in **[Ge-Ka,1]**, $\hat{\pi}$ is equivalent to $\pi'$, the contragradient of $\pi$.

By Frobenius reciprocity (cf. **[Be-Ze,1]**, Theorem 2.28), $\pi$ admits an embedding in $\mathscr{M}_n$ if and only if it supports a nontrivial, $\mathrm{Sp}_n$ invariant linear functional; the embedding is unique up to scalar if and only if $\dim \mathrm{Hom}_{\mathrm{Sp}_n}(\pi, 1)$ equals one.

THEOREM 2.4.2. *Let $\pi$ be an irreducible, admissible representation of* $\mathrm{Gl}_{2n}$. *Then* $\dim \mathrm{Hom}_{\mathrm{Sp}_n}(\pi, 1) \leq 1$.

*Proof.* This a symplectic restatement of Theorem 4 and its corollary in **[Ge-Ka,1]**. In light of Theorem 2.4.3, their proof applies *mutatis mutandis.* □

**3. Unitary disjointness of models.** The main result of this section is the following theorem.

THEOREM 3.1. *Let $\pi$ be an irreducible, unitary representation of* $\mathrm{Gl}_n$. *Let $s_1$, $s_2$ be distinct integers, $0 \leq s_1, s_2 \leq [\frac{n}{2}]$. Then* $\mathrm{Hom}_{\mathrm{Gl}_n}(\pi, \mathscr{M}_{s_i})$ *is nonzero for at most one $i$.*

*Proof.* For simplicity denote $\mathscr{M}_i = \mathscr{M}_{s_i}$, $M_i = M_{s_i}$ and $\psi_i = \psi_{s_i}$ (see §1). Assume there are nontrivial maps $\pi \to \mathscr{M}_i$, $i = 1, 2$.

$\pi$ is equivalent to the Hermitian contragradient representation $\pi^+ = \bar{\pi}'$. By dualizing, obtain $\mathscr{M}_2' \to \pi' \cong \bar{\pi}$. Let $\iota_2 = \mathrm{Ind}_{M_2}^{\mathrm{Gl}_n} \psi_2^{-1}$. For $f_2 \in \iota_2$, $F \in \mathscr{M}_2'$, the pairing

$$\{f, F\} = \int_{M_2 \backslash \mathrm{Gl}_n} f(g) F(g)\, d\dot{g} \tag{3.1.1}$$

determines a map $\iota_1 \to \mathscr{M}_2'$, via $f \to \{f, \cdot\}$. Since $\iota_2 \cong \bar{\iota}_2$ (see §1), we obtain a nontrivial map $\iota_2 \to \pi$ (**[Be-Ze, 1]**); thus the composite

$$\iota_2 \to \pi \to \mathscr{M}_1 \tag{3.1.2}$$

is non trivial. By Frobenius reciprocity, this corresponds to an element of $\mathrm{Hom}_{M_1}(\iota_2, \psi_1)$.

Associated to $\iota_2$ is a unique isomorphism class of equivariant $l$-sheaves $\mathscr{F}$ on $M_2 \backslash \mathrm{Gl}_n$ (**[Be-Ze**, 1], Proposition 2.23). The right action of $M_1$ on $M_2 \backslash \mathrm{Gl}_n$ is constructive (**[Be-Ze**,1], Theorem A, 6.15) with locally closed orbits (ibid., Proposition 6.8(c)).

The restriction of $\mathscr{F}$ to the orbit $M_2 w M_1$ is associated to the representation $\mathrm{ind}_{(M_1 \cap w^{-1} M_2 w)}^{M_1} \psi_2^w$, where $\psi_2^w(g) = \psi_2(w g w^{-1})$,

$g \in M_1 \cap w^{-1}M_2 w$. Frobenius reciprocity gives

$$\begin{aligned}(3.1.3)\qquad \mathrm{Hom}_{M_1}(\mathrm{ind}^{M_1}_{(M_1\cap w^{-1}M_2 w)}\,\psi_2^w\,,\,\psi_1)\\ = \mathrm{Hom}_{M_1\cap w^{-1}M_2 w)}(\psi_2^w\,,\,\psi_1).\end{aligned}$$

The groups $M_1$ and $M_2$ are associated to symplectic forms with different ranks, as are $M_1$ and $w^{-1}M_2w$. Thus there exists $h \in M_1 \cap w^{-1}M_2w$ such that $\psi_2^w(h) \neq \psi_1(h)$; hence the right side of (3.3) is zero (cf. **[Kl]**, Proposition 1.3). Consequently there do not exist quasi-invariant distributions on $\mathscr{F}$ supported on a single orbit of $M_1$.

The proof of Theorem 6.9 in **[Be-Ze**,1**]** for invariant distributions can be trivially modified to apply to quasi-invariant distributions, the result being that if an $l$-group acts constructively on an $l$-sheaf $\mathscr{F}$ such that no orbit supports a non-zero quasi-invariant distribution, then there do not exist non-zero quasi-invariant distributions of $\mathscr{F}$. Therefore the composite (3.2) is zero and the theorem follows. □

3.2. *Disjointness of symplectic and Whittaker models.* In this section we drop the assumption of unitarity.

PROPOSITION 3.2.1. *Let $\pi$ be an irreducible, admissible representation. If $\pi$ has a Whittaker (resp. symplectic) model, then its contragradient $\pi'$ likewise has a Whittaker (resp. symplectic) model.*

*Proof.* Having a Whittaker model is equivalent to the existence of a nontrivial, $\psi_{2n}$-quasi-invariant distribution $T$. The contragradient $\pi'$ is equivalent to the representation obtained by composing $\pi$ with the automorphism $g \to {}^tg^{-1}$ (**[Ge-Ka**,1**]**, Theorem 2). This automorphism takes $U_{2n}$ to the opposite unipotent subgroup of lower triangular matrices. The opposition element $s_0$ of the Weyl group conjugates this back to $U_{2n}$. Therefore $u \to s_0{}^tu^{-1}s_0^{-1}$ preserves $U_{2n}$ and the representation $g \to \pi(s_0{}^tg^{-1}s_0^{-1})$ is equivalent to $\pi'$. Thus we have

$$\begin{aligned}(3.2.1)\qquad T(\pi(s_0{}^tu^{-1}s_0^{-1})f) &= \psi_{2n}(s_0{}^tu^{-1}s_0^{-1})T(f)\\ &= \psi_{2n}^{-1}(u)T(f).\end{aligned}$$

$\psi_{2n}^{-1}$ is a nondegenerate Whittaker; hence $\pi'$ has a Whittaker model.

The same argument applied to $g \to -J_{2n}{}^tg^{-1}J_{2n}$ gives the symplectic statement. □

THEOREM 3.2.2. *An irreducible, admissible representation cannot have both a Whittaker model and a symplectic model.*

*Proof.* If we have $\pi \to \operatorname{Ind}_{U_{2n}}^{\mathrm{Gl}_{2n}} \psi_{2n}$, we obtain $\pi' \to \operatorname{Ind}_{U_{2n}}^{\mathrm{Gl}_{2n}} \psi_{2n}$. As in §3.1, dualizing gives $\operatorname{ind}_{U_{2n}}^{\mathrm{Gl}_{2n}} \psi_{2n} \to \pi$ (cf. **[Ge-Ka, 1]** §3). Thus if $\pi$ has a symplectic model, we obtain the composite $\operatorname{ind}_{U_{2n}}^{\mathrm{Gl}_{2n}} \psi_{2n} \to \operatorname{Ind}_{\mathrm{Sp}_n}^{\mathrm{Gl}_{2n}} 1$.. □

**4. Orbits.** For applications in §11, we need descriptions of orbits in certain flag varieties. We prove here some general results.

4.1. $\mathrm{Sp}_n$ *Orbits in* $P_{2n-k,k}\backslash \mathrm{Gl}_{2n}$. To compute these orbits it suffices to consider the cases $k \leq n$.

Let $\mathscr{X}_k$ denote the variety of $k$-planes in $2n$-space. For $X_1, X_2 \in \mathscr{X}_k$, let $J'$, $J''$ be the restrictions of $J$ to $X_1$ and $X_2$ respectively. It follows from Witt's theorem that $X_1$ and $X_2$ are conjugate by a symplectic endomorphism if and only if the radicals of $J'$ and $J''$ have the same dimension. Thus $\mathscr{X}_k$ is the union of symplectic orbits

$$\mathscr{X}_k(r) = \{X \in \mathscr{X}_k \mid \dim \operatorname{Rad} J|_X = r\}. \tag{4.1.1}$$

$\mathscr{X}_k(r)$ is nonempty if and only if $k \equiv r(2)$.

PROPOSITION 4.1.1. *If* $k \leq \frac{n}{2}$ *set*

$$w_k = \begin{pmatrix} w'_k & 0 \\ 0 & w'_k \end{pmatrix}$$

*where* $w'_k$ *equals*

$$\begin{pmatrix} 0 & 0 & 1_k \\ 0 & 1_{n-2k} & 0 \\ 1_k & 0 & 0 \end{pmatrix} \tag{4.1.2}$$

*If* $k > \frac{n}{2}$ *set* $w_k = \left(\begin{smallmatrix} w'_k & 0 \\ 0 & w'_k \end{smallmatrix}\right)$ *where* $w'_k$ *equals*

$$\begin{pmatrix} 0 & 0 & 1_{n-k} \\ 0 & 1_{2k-n} & 0 \\ 1_{n-k} & 0 & \end{pmatrix}. \tag{4.1.3}$$

*Let*

$$\gamma_r = \begin{pmatrix} 1_r &&&&&&& \\ & 0 & 1_{\lambda-r} &&& 0 & 0 & \\ & 0 & 0 &&& 0 & 1_{\lambda_r} & \\ &&& 1_{n-k} &&&& \\ &&&& 1_r &&& \\ & 1_{\lambda-r} & 0 &&& 0 & 0 & \\ & 0 & 0 &&& 1_{\lambda-r} & 0 & \\ &&&&&&& 1_{n-k} \end{pmatrix}$$

*Then*

$$P_{2n-k,k}\backslash \mathrm{Gl}_{2n}/\mathrm{Sp}_n = \bigcup_{\substack{r\leq k \\ r\equiv k(2)}} P_{2n-k,k}w_k\gamma_r\,\mathrm{Sp}_n\,. \tag{4.1.5}$$

*Proof.* We may choose a representative for the orbit $\mathscr{X}_k(r)$ which is spanned by the set $\{f_1, \dots, f_r, f_{r+1}, \dots, f_\lambda, e_{r+1}, \dots, e_\lambda\}$, where $\lambda = (k+r)/2$ and $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ is the standard symplectic basis relative to $J$. A basis for the $k$-plane $X_0$ fixed by $P_{2n-k,k}$ is $\{f_{n-k+1}, \dots, f_n\}$. The image of $X_0$ under $w_k$ is the space spanned by $\{f_1, \dots, f_k\}$. $\gamma_r$ then maps this set to $\{f_1, \dots, f_\lambda, e_{r+1}, \dots, e_\lambda\}$. □

4.1.1. We specialize now to the case $k = n$ and describe the stabilizer of an orbit. This will be used in §11 in establishing the uniqueness of symplectic functionals on certain reducible representations.

PROPOSITION 4.1.1.1. *Let $\Sigma_r$ be the stabilizer of the* $\mathrm{Sp}_n$ *orbit of $P_{n,n}\gamma_r$. Then $\Sigma_r \cong (\mathrm{Gl}_r \times \mathrm{Sp}_{(n-r)/2} \times \mathrm{Sp}_{(n-r)/2})U_r'$, where $U_r'$ is unipotent. In particular, for the n-plane $X_r$, with basis $\{f_1, \dots, f_{(n-r)/2}, e_{r+1}, \dots, e_{(n-r)/2}\}$,* $\mathrm{Gl}_r \times \mathrm{Sp}_{(n-r)/2} \times \mathrm{Sp}_{(n-r)/2}$ *is realized as the matrices of the form*

$$\begin{pmatrix} g & 0 & 0 & 0 & 0 & 0 & \\ 0 & A & 0 & 0 & B & 0 & \\ 0 & 0 & A' & 0 & 0 & B' & \\ 0 & 0 & 0 & & {}^tg^{-1} & 0 & 0 \\ 0 & C & 0 & 0 & D & 0 & \\ 0 & 0 & C' & 0 & 0 & D' & \end{pmatrix}$$

*where* $g \in \mathrm{Gl}_r$, $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *and* $\left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right)$ *are in* $\mathrm{Sp}_{(n-r)/2}$, *and* $U_r'$ *is the group of matrices of the form*

$$\begin{pmatrix} 1_r & X & Y & Z \\ 0 & 1_{n-r} & {}^tZ & 0 \\ 0 & 0 & 1_r & 0 \\ 0 & 0 & -{}^tX & 1_{n-r} \end{pmatrix}, \tag{4.1.1.2}$$

*where* $Y$ *is symmetric.*

*Proof.* $\Sigma_r$ preserves the radical of $J$ restricted to $X_r$; hence it is contained in the symplectic parabolic subgroup $P_{\{f_1,\dots,f_r\}}$, fixing this isotropic subspace. The unipotent radical of $P_{\{f_1,\dots,f_r\}}$ is precisely $U_r'$; it clearly leaves $X_r$ invariant.

The Levi component of the parabolic is $\mathrm{Gl}_r \times \mathrm{Sp}_{n-r}$, realized as the matrices of the form

$$\begin{pmatrix} g & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & {}^tg^{-1} & 0 \\ 0 & c & 0 & d \end{pmatrix}, \tag{4.1.1.3}$$

where $g \in \mathrm{Gl}_r$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_{n-r}$. For such an element to fix $X_r$, the symplectic part must leave the span of $\{e_{r+1}, \dots, e_{(n-r)/2}, f_{r+1}, \dots, f_{(n-r)/2}\}$ invariant. Since the symplectic form restricted to this space is nondegenerate, the orthogonal complement is fixed. □

It is straightforward to compute the dimension of the stabilizers $\Sigma_r$. If $n$ is further assumed to be even we have the

COROLLARY 4.1.1.2. *Let* $n \equiv 0(2)$. *There is a single open* $\mathrm{Sp}_n$ *orbit in* $P_{n,n}\backslash \mathrm{Gl}_{2n}$ *given by the double coset* $P_{n,n}\gamma_0 \mathrm{Sp}_n$, *where*

$$\gamma_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{4.1.1.4}$$

4.2. $\mathrm{Sp}_{n/2} \times \mathrm{Sp}_{n/2}$ *Orbits in* $\mathrm{Sp}_n/(P_{n,n} \cap \mathrm{Sp}_n)$. Assume $n$ is even and set $P_{n,n}' = P_{n,n} \cap \mathrm{Sp}_n$. Acting on the right $P_{n,n}'$ preserves the span of $\{f_1, \dots, f_n\}$. Thus we consider the variety $P_{n,n}'\backslash \mathrm{Sp}_n$ of maximal isotropic subspaces.

**Proposition** 4.2.1. *There is a unique open* $\mathrm{Sp}_{n/2}\times\mathrm{Sp}_{n/2}$ *orbit in* $\mathrm{Sp}_n/P'_{n,n}$ *given by* $(\mathrm{Sp}_{n/2}\times\mathrm{Sp}_{n/2})\rho JP'_{n,n}$, *where*

$$\rho=\begin{pmatrix}1&0&0&1\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \tag{4.2.1}$$

*Proof*. Let $(V_n, \langle\ ,\ \rangle)$ denote an $n$-dimensional symplectic vector space with standard ordered basis $\{e_1, \dots, e_{n/2}, f_1, \dots, f_{n/2}\}$ associated to $J_{n/2}$. Set $W=V_n\oplus V_n$ and define a symplectic form on $W$ by

$$\langle(v_1, v_2), (v_1', v_2')\rangle=\langle v_1, v_1'\rangle-\langle v_2, v_2'\rangle. \tag{4.2.2}$$

Let $V_n^+$ (resp. $V_n^-$) be the embedding of $V_n$ on the first (resp. second) factor of $W$. Let $e_i^\pm$ (resp. $f_i^\pm$) be the images of $e_i$ (resp. $f_i$) in $V_n^\pm$. With respect to the basis $\{e_1^+, \dots, e_{n/2}^+, f_1^+, \dots, f_{n/2}^+, e_1^-, \dots, e_{n/2}^-, f_1^-, \dots, f_{n/2}^-\}$, the matrix of the form on $W$ is $\left(\begin{smallmatrix}J&0\\0&-J\end{smallmatrix}\right)$.

The transformation from $W$ to $V_{2n}$ defined by $e_i^+\to e_i$ $f_i^+\to f_i$, $e_i^-\to f_{n/2+i}$ and $f_i^-\to e_{(n/2)+i}$, $1\le i\le n/2$ is an isometry. The images of $V^+$ and $V^-$ are spanned by the images of $\{e_1, \dots, e_{n/2}, f_1, \dots, f_{n/2}\}$ and $\{e_{n/2+1}, \dots, e_n, f_{n/2+1}, \dots, f_n\}$ respectively.

According to Proposition 2.1 in **[PS-Ra]**, the only invariant of an $\mathrm{Sp}_{n/2}\times\mathrm{Sp}_{n/2}$ orbit in $P'_{n,n}\backslash\mathrm{Sp}_n$ is the dimension of the intersection of a representative $n$-plane with $V^+$ or $V^-$. Thus there is one open orbit which has a representative intersecting $V^+$ and $V^-$ only in $0$. A simple example of such a maximal isotropic subspace is given by the span of $\{e_i+f_{(n/2)+i}\}$. This space is the image of the span of $\{f_1, \dots, f_n\}$ by the matrix

$$\begin{pmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&1\\0&1&1&0\end{pmatrix}=J^{-1}\rho^{-1}. \tag{4.2.3}$$

Thus the open orbit in $P'_{n,n}\backslash\mathrm{Sp}_n$ is $P'_{n,n}J^{-1}\rho^{-1}(\mathrm{Sp}_{n/2}\times\mathrm{Sp}_{n/2})$. Inverting this gives the theorem. □

**5.** $\mathrm{Gl}_2$. In this case there are two models, the Whittaker model and the pure symplectic ($\mathrm{Sl}_2$) model.

In the notation of **[Ze]** the admissible representations of $\mathrm{Gl}_2$ are of two types: supercuspidal; $\langle\alpha_1\times\alpha_2\rangle$, where $\alpha_1$ and $\alpha_2$ are characters of $k^\times$. In general, supercuspidal representations have Whittaker models (**[Ge-Ka,2]**). The second type is irreducible if and only if

$\alpha_1 \neq \alpha_2\nu^{\pm 1}$. Whittaker models satisfy a hereditary property which says that the representation parabolically induced off representations with Whittaker models themselves have Whittaker models (cf. **[Ro]** Theorem 2, for the precise statement). Thus in the case $\alpha_1 \neq \alpha_2\nu^{\pm 1}$, these representations have Whittaker models.

The remaining cases are $\langle \alpha \times \alpha\nu^{\pm 1} \rangle$. These representations are the twists of the identity representation and Steinberg representation St. St is square integrable (mod center) and hence has a Whittaker model (**[Ze]** Example 9.3, Theorems 9.3, 9.7). The identity clearly has the symplectic model.

**6. The unitary dual of** $\mathrm{Gl}_n$. We now recall the classification of the irreducible, unitary representations of $\mathrm{Gl}_n$ due to M. Tadić (**[Ta,1]**).

Let $D_0(n)$ denote the set isomorphism classes of irreducible representation of $\mathrm{Gl}_n$ which are square integrable modulo center and $D_0 = \bigcup_{n\geq 0} D_0(n)$. Let $D(n)$ be the set of representations of the form $\nu^\alpha\delta$, where $\alpha$ is real and $\delta \in D_0$; $D = \bigcup_{n\geq 0} D(n)$, $M(D)$ is the collection of all finite (unordered) multisets on $D$.

Given $a = (\delta_1, \dots, \delta_n) \in M(D)$, $\delta_i = \nu^{\alpha_i}\delta_0^i$, $\delta_0^i \in D_0$, we may assume that $\alpha_1 \geq \cdots \geq \alpha_n$. The induced representation $\delta_1 \times \cdots \times \delta_n$ has a unique irreducible quotient module, $L(a)$.

Given an irreducible representation $\sigma$, let $\sigma^+$ denote its Hermitian (complex conjugate) contragradient. Set $\Pi(\sigma, \alpha) = \nu^\alpha\sigma \times \nu^{-\alpha}\sigma^+$, for $\alpha$ real. For a positive integer $n$ and $\delta \in D_0$, set $u(\delta, n) = L(\nu^p\delta, \nu^{p-1}\delta, \dots, \nu^{-p}\delta)$, where $p = (n-1)/2$. Thus if $\delta$ is a representation of $\mathrm{Gl}_m$, $u(\delta, n)$ is a representation of $\mathrm{Gl}_{nm}$. (We sometimes write $u(\delta_m, n)$.)

THEOREM 6.1 (*Tadić*). *Let* $B = \{u(\delta, n),\ \Pi(u(\delta, n), \alpha) | \delta \in D_0,\ 0 < \alpha < \frac{1}{2}\}$.

(i) *If* $\sigma_1, \dots, \sigma_r \in B$, *then* $\sigma_1 \times \cdots \times \sigma_r$ *is irreducible and unitary.*

(ii) *If* $\pi$ *is an irreducible unitarizable representation, then there exist* $\tau_1, \dots, \tau_s \in B$, *unique up to permutation, such that* $\pi = \tau_1 \times \cdots \times \tau_s$.

**7. The unitary dual of** $\mathrm{Gl}_3$. In this section we explicate Theorem 6.1 in the case of $\mathrm{Gl}_3$. Denote by $B_n$ the set $B$ of Theorem 6.1 for $\mathrm{Gl}_n$, i.e. the set of representations of $\mathrm{Gl}_m$, $m \leq n$, contained in $B$. Let $B_n'$ denote the set of elements of $B_n$ which are representations of $\mathrm{Gl}_n$. $B_3$ is the disjoint union of $B_1 = B_1'$, $B_2'$ and $B_3'$.

For $\mathrm{Gl}_2$, $B_2'$ is composed of:

(i) The supercuspidal representations and the Steinberg representation St. These are of the form $u(\delta_2, 1)$.

(ii) The unitary characters. These are of the form

$$L(\nu^{1/2}\delta_1 \times \nu^{-1/2}\delta_1), \qquad \delta_1 \in B_1$$

([**Ze**], §§9.1 and 3.2).

(iii) The complementary series $\Pi(\delta_1, \alpha)$, $\delta_1 \in B_1$, $\alpha \in (0, \frac{1}{2})$.

The rest of $B_2$ comes from $B_1$, viz.

(iv) $\delta_1 \times \delta_2$, $\delta_1, \delta_2 \in B_1$.

$B_3$ is the union of $B_2$ and $B'_3$, which contains:

(i′) The square integrable representations $\delta_3 = u(\delta_3, 1)$.

(ii″) The unitary characteris $u(\delta_1, 3) = L(\nu\delta_1 \times \delta_1 \times \nu^{-1}\delta_1)$, $\delta_1 \in B_1$.

The representations arising from $B_1$ and $B'_2$ are:

(iii′) $\chi_1 \times \delta_2$, $\chi_1 \in B_1$, $\delta_2 \in D_0(2)$.

(iv′) $\chi_1 \times \chi_2$, $\chi_i$ a unitary character of $\mathrm{Gl}_i$.

(v′) $\chi_1 \times \nu^{\alpha}\chi_2 \times \nu^{-\alpha}\chi_2$, $\chi_1, \chi_2 \in B_1$, $0 < \alpha < \frac{1}{2}$.

The remaining unitary representations of $\mathrm{Gl}_3$ arise from $B_1$:

(vi′) $\chi_1 \times \chi_2 \times \chi_3$, $\chi_1, \chi_2, \chi_3 \in B_1$.

**8. Models for** $\mathrm{Gl}_3$. For $\mathrm{Gl}_3$ there are only two models, the Whittaker model $\mathcal{M}_0$ and the mixed model $\mathcal{M}_1$. The main result of this section is the following.

THEOREM 8.1. *Let $\pi$ be an irreducible unitary representation of* $\mathrm{Gl}_3$. *Then $\pi$ can be uniquely embedded as a submodule of $\mathcal{M}_0$ or $\mathcal{M}_1$.*

*Proof*. By Theorem 3.1, $\pi$ cannot be realized in both models. Since the Whittaker model is unique, we need to show that every representation has a model and that the mixed model is unique. We do this by examining the catalog of representations compiled in the previous section, showing that they all have models and then examining those with mixed models to establish uniqueness in those cases.

The simplest cases to deal with are those with Whittaker models. We need two facts. The first is the hereditary property of Whittaker models quoted in §5. The other is that square integrable representations have Whittaker models, since in the terminology of [**Ze**] they are transposes of segments ([**Ze**], Theorem 9.3). Thus case (i′), (iii′), (v′) and (vi′) all have Whittaker models.

Case (ii′) is the unitary character $\chi_3$. Frobenius reciprocity gives $\mathrm{Hom}_{\mathrm{Gl}_3}(\chi_3, \mathcal{M}_1) = \mathrm{Hom}_{\mathrm{Sl}_2}(1, 1)$, thus the existence and uniqueness in this case. The remaining case (iv′) is $\chi_1 \times \chi_2$ where $\chi_i$ is a unitary character of $\mathrm{Gl}_i$. Inducing in stages, we have

$$\mathcal{M}_1 \cong \mathrm{Ind}_{P_{1,2}}^{\mathrm{Gl}_3}[\mathrm{Ind}_{1\times\mathrm{Sl}_2}^{\mathrm{Gl}_1\times\mathrm{Gl}_2} 1]\otimes 1. \tag{8.1}$$

Two guises of Frobenius reciprocity ([**Be-Ze,2**]), Proposition 1.9(b); [**Be-Ze,2**] Theorem 2.28) imply

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{Gl}_3}(\mathrm{Ind}_{P_{1,2}}^{\mathrm{Gl}_3}\chi_1\otimes\chi_2, \mathrm{Ind}_{1\times\mathrm{Sl}_2\times N_1}^{\mathrm{Gl}_3} 1) \\ &\quad = \mathrm{Hom}_{\mathrm{Gl}_1\times\mathrm{Gl}_2}(\tilde{r}_{1,2}(\mathrm{Ind}_{P_{1,2}}^{\mathrm{Gl}_3}\chi_1\otimes\chi_2), \mathrm{Ind}_{1\times\mathrm{Sl}_2}^{\mathrm{Gl}_1\times\mathrm{Gl}_2} 1), \\ &\quad = \mathrm{Hom}_{\mathrm{Sl}_2}(\tilde{r}_{1,2}(\mathrm{Ind}_{P_{1,2}}^{\mathrm{Gl}_3}\chi_1\otimes\chi_2)|_{\mathrm{Sl}_2}, 1). \end{aligned} \tag{8.2}$$

According to Theorem 1.2 ([**Ze**]), the $\mathrm{Gl}_1\times\mathrm{Gl}_2$ module

$$\tilde{r}_{1,2}(\mathrm{Ind}_{P_{1,2}}^{\mathrm{Gl}_3}\chi_1\otimes\chi_2) \tag{8.3}$$

has a filtration of length two with quotient module (closed orbit) $\chi_1\otimes\chi_2$ and submodule (open orbit) $\nu^{-1/2}\chi_2\otimes\mathrm{Ind}_{P_{1,1}}^{\mathrm{Gl}_2}(\chi_1\otimes\nu^{1/2}\chi_2)$. The last representation cannot support an $\mathrm{Sl}_2$ invariant functional since the second factor has a Whittaker model. Restricted to $\mathrm{Sl}_2$, the first representation is the identity, it has a unique $\mathrm{Sl}_2$ invariant functional and thus the quotient of $\tilde{r}_{1,2}(\mathrm{Ind}_{P_{1,2}}^{\mathrm{Gl}_3}\chi_1\otimes\chi_2)$ supports this functional. Hence (8.2) is one dimensional and $\chi_1\times\chi_2$ is uniquely embedded in $\mathcal{M}_1$. □

**9. Representations of $\mathrm{Gl}_3$ without models.** In this section we determine the admissible, irreducible representations of $\mathrm{Gl}_3$ which do not embed in either $\mathcal{M}_0$ or $\mathcal{M}_1$. It turns out that these are essentially the non-unitarizable representations, i.e. what remains after discarding the representations arising from twisting the inducing data in the set of representations that give the unitary dual.

9.1. Consider the representation $I = \mathrm{Ind}_{P_{2,1}}^{\mathrm{Gl}_3}\nu^{1/2}\otimes\nu^{-1}$. In the notation of [**Ze**], $I = \langle 1\times\nu\rangle\times\nu^{-1}$. By Proposition 2.1 and Corollary 2.3 in [**Ze**], $I$ is multiplicity free, as is $J = \nu^{-1}\times\langle 1\times\nu\rangle$, and they have the same composition factors. By transitivity of induction, $J$ embeds in $\nu^{-1}\times 1\times\nu$. Both of these have unique irreducible submodules $\langle J\rangle$ and $\langle\nu^{-1}\times 1\times\nu\rangle$, which are equal. $\langle J\rangle = 1$, the trivial representation ([**Ze**], Proposition 1.10, example 3.2). Thus we have an exact sequence

$$0\to\langle I\rangle\to I\to 1\to 0. \tag{9.1.1}$$

THEOREM 9.1.1. *The representation $\langle I\rangle$ has neither a Whittaker model nor a mixed model.*

*Proof*. Consider the case of the mixed model. By Frobenius reciprocity

$$(9.1.2) \qquad \begin{aligned} \mathrm{Hom}_{\mathrm{Gl}_3}(\langle I\rangle, \mathscr{M}_1) &= \mathrm{Hom}_{(1\times \mathrm{Sl}_2)N_{1,2}}(\langle I\rangle, 1), \\ &= \mathrm{Hom}_{\mathrm{Sl}_2}(r_{1,2}(\langle I\rangle)|_{\mathrm{Sl}_2}, 1). \end{aligned}$$

By exactness of $r_{1,2}$, $r_{1,2}(I)/r_{1,2}(\langle I\rangle) = 1$.

We describe $r_{1,2}(I)$ in detail. There are two orbits of $P_{1,2}$ on $P_{2,1}\backslash \mathrm{Gl}_3$, viz. the closed orbit which has stabilizer $P_{1,1,1}$, and the orbit $P_{2,1}w$, where

$$(9.1.3) \qquad w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The stabilizer of $P_{2,1}w$ is $\mathrm{Gl}_1 \times \mathrm{Gl}_2$. Orbital analysis ([**Ca**], 3.4) implies that $I$ has a $P_{1,2}$ submodule equivalent to

$$(9.1.4) \qquad R_1 = \mathrm{ind}_{\mathrm{Gl}_1\times \mathrm{Gl}_2}^{P_{1,2}} \nu^{-1} \otimes \nu^{1/2},$$

and corresponding $P_{1,2}$ quotient module

$$(9.1.5) \qquad R_2 = \mathrm{ind}_{P_{1,1,1}}^{P_{1,2}} \nu \otimes \nu^{1/2} \otimes \nu^{-3/2}.$$

Thus we have the exact sequence of $P_{1,2}$ modules

$$(9.1.6) \qquad 0 \to R_1 \to I \to R_2 \to 0,$$

and the exact sequence of $\mathrm{Gl}_1 \times \mathrm{Gl}_2$ modules

$$(9.1.7) \qquad 0 \to r_{1,2}(R_1) \to r_{1,2}(I) \to r_{1,2}(R_2) \to 0.$$

The center of $\mathrm{Gl}_1 \times \mathrm{Gl}_2$ acts on $r_{1,2}(R_1)$ and $r_{1,2}(R_2)$ by the characters

$$(9.1.8) \qquad \begin{pmatrix} s & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix} \to 1, \frac{|s|}{|t|},$$

respectively. Thus

$$(9.1.9) \qquad r_{1,2}(I) = r_{1,2}(R_1) \oplus r_{1,2}(R_2).$$

Let $f \in R_1$. From the relation

$$(9.1.10) \quad f\begin{pmatrix} s & x & y \\ 0 & & \\ & g & \\ 0 & & \end{pmatrix} = |s|^{-2}|\det g| f\begin{pmatrix} 1 & s^{-1}x & s^{-1}y \\ 0 & & \\ & 1 & \\ 0 & & \end{pmatrix},$$

we may, via the restriction to $N_{1,2}$, identify $R_1$ with the space of Schwartz functions on $F^2$. The action of $N_{1,2}$ becomes

$$\left(\begin{pmatrix} 1 & u & v \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} f\right)(x, y) = f(x+u, y+v). \tag{9.1.11}$$

Write $f = \sum_{i=1}^{n} c_i \chi_i$, where $c_i \in \mathbb{C}$ and $\chi_i$ is the characteristic function of the ball of some small radius $r$, entered at $(u_i, v_i)$. Let $\chi_0$ be the characteristic function of the ball of radius $r$ centered at $(0, 0)$. Then

$$\chi_i = \begin{pmatrix} 1 & u_i & v_i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \chi_0, \tag{9.1.12}$$

which equals $\chi_0$ in $r_{1,2}(R_1)$. Thus $f \equiv c\chi_0$, and $r_{1,2}(R_1)$ is the one dimensional representation $1 \otimes 1$. Restricted to $\mathrm{Sl}_2$, it is trivial.

Since the center of $\mathrm{Gl}_1 \times \mathrm{Gl}_2$ acts on $r_{1,2}(R_2)$ by a nontrivial character, the trivial representation does not occur there. Thus $r_{1,2}(\langle I\rangle)$ will have a nonzero $\mathrm{Sl}_2$ invariant functional if and only if $r_{1,2}(R_2)$ has one.

$N_{1,2}$ acts trivially on $R_2$, hence $R_2 = r_{1,2}(R_2)$, and restriction to $\mathrm{Gl}_1 \times \mathrm{Gl}_2$ gives

$$\begin{aligned} r_{1,2}(R_2) &= \operatorname{ind}_{\mathrm{Gl}_1 \times P_{1,1}}^{\mathrm{Gl}_1 \times \mathrm{Gl}_2} \nu \otimes \nu^{1/2} \otimes \nu^{-3/2} \\ &= \nu \otimes \operatorname{ind}_{P_{1,1,}}^{\mathrm{Gl}_2} \nu^{1/2} \otimes \nu^{-3/2}. \end{aligned} \tag{9.1.13}$$

Since

$$(\operatorname{ind}_{P_{1,1}}^{\mathrm{Gl}_2} \nu^{1/2} \otimes \nu^{-3/2})|_{\mathrm{Sl}_2} = \operatorname{ind}_{P'_{1,1}}^{\mathrm{Sl}_2} \nu \otimes \nu^{-1}, \tag{9.1.14}$$

we have

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{Sl}_2}(\operatorname{ind}_{\mathrm{Gl}_1 \times P_{1,1}}^{\mathrm{Gl}_1 \times \mathrm{Gl}_2} \nu \otimes \nu^{1/2} \otimes \nu^{-3/2}, 1) \\ &= \mathrm{Hom}_{\mathrm{Sl}_2}(\operatorname{ind}_{P'_{1,1}}^{\mathrm{Sl}_2} \nu \otimes \nu^{-1}, 1) \\ &= \mathrm{Hom}_{P'_{1,1}}(\nu^5 \otimes 1, 1), \end{aligned} \tag{9.1.15}$$

which is clearly zero. Thus $\langle I\rangle$ has no mixed model.

Now consider the Whittaker model. Note that $\langle I\rangle$ will have a Whittaker model if and only if $r_{1,1,1;\psi_3}(\langle I\rangle) \neq 0$. By exactness of $r_{1,1,1;\psi_3}$,

$$0 \to r_{1,1,1;\psi_3}(\langle I\rangle) \to r_{1,1,1;\psi_3}(I); \tag{9.1.16}$$

hence to show that $\langle I\rangle$ does not have a Whittaker model, it suffices to show that $r_{1,1,1;\psi_3}(I)=0$.

Consider $U_3$ acting on $P_{2,1}\backslash\mathrm{Gl}_3$. There is the orbit of $P_{2,1}$ with stabilizer $U_3$, and the orbits $P_{2,1}w_1$ and $P_{2,1}w_2$ where

$$w_1=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\quad\text{and}\quad w_2=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}. \tag{9.1.17}$$

The stabilizers $P_{w_1}$ and $P_{w_2}$ of these orbits are the matrices of the form

$$\begin{pmatrix}1&*&*\\0&1&0\\0&0&1\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}1&0&0\\0&1&*\\0&0&1\end{pmatrix}, \tag{9.1.18}$$

respectively. We have a filtration of $I$ by $U_3$ invariant subspaces $I\supset F_1\supset F_2$, where $F_2=\mathrm{ind}_{P_{w_2}}^{U_3}1$, $F_1/F_2=\mathrm{ind}_{P_{w_1}}^{U_3}1$, and $I/F_1=1$. Obviously there are no $U_3$ morphisms between 1 and $\psi_3$. Since $\psi_3$ is nontrivial on the inducing subgroups, $\mathrm{Hom}_{U_3}(F_2,\psi_3)$ and $\mathrm{Hom}_{U_3}(F_1/F_2,\psi_3)$ are both zero. Thus $r_{1,1,1;\psi_3}(I)=0$.

9.2. The classification of irreducible, admissible representations of $\mathrm{Gl}_n$ is given by Theorem 6.1 in **[Ze]**. Using the previous methods and the injectivity of the Whittaker map (**[Ja-Sh]**), it can be shown that, modulo twisting the inducing data by characters the counterexample presented is unique.

9.3. We compare the $p$-adic and finite field situations with respect to the counterexample. The representation $\bar{I}=1\times\nu\times\nu^{-1}$ is multiplicity free and has length four (**[Ze]**, Corollary 2.3). The finite field analogue of $\bar{I}$ is $I_f=\mathrm{Ind}_{P_{1,1,1}}^{\mathrm{Gl}_3}1$. There is a bijective correspondence between the irreducible representations that appear in $I_f$ and the irreducible representation of the group algebra $\mathbb{C}S_3$, with the degree of the latter giving the corresponding multiplicity (cf. **[Car]**, Theorem 10.1.2). $S_3$ has two distinct characters and a two dimensional representation. Thus $I_f$ has three irreducible constituents one appearing with multiplicity two.

**10. The unitary dual of** $\mathrm{Gl}_4$. We now enumerate the set of irreducible unitary representations of $\mathrm{Gl}_4$. In the notation introduced in §§6 and 8, the basic set of representations is $B_4=B_4'\cup B_3$. In the following, the $\delta_n$'s will be in $D_0(n)$, all $\alpha$'s are in the interval $(0,\frac{1}{2})$ and the $\chi_n$'s will be unitary characters of $\mathrm{Gl}_n$ (see §1 for conventions).

$B'_4$ consists of the following:

(i) $u(\delta_4, 1) = \delta_4$, the square integrable representations of $\mathrm{Gl}_4$.
(ii) $u(\delta_2, 2) = L(\nu^{1/2}\delta_2 \times \nu^{-1/2}\delta_2)$.
(iii) $u(\delta_1, 4)$, the representations of the form

$$L(\nu^{3/2}\chi_1 \times \nu^{1/2}\chi_1 \times \nu^{-1/2}\chi_1 \times \nu^{-3/2}\chi_1).$$

These representations are all characters ([**Ze**]).

The complementary series induced off $P_{2,2}$:
(iv-1) $\nu^\alpha\delta_2 \times \nu^{-\alpha}\delta_2$.
(iv-2) $\nu^\alpha\chi_2 \times \nu^{-\alpha}\chi_2$.

The representations induced off the parabolic subgroup $P_{1,3}$ are:
(v-1) $\chi_1 \times \delta_3$.
(v-2) $\chi_1 \times \chi_3$.

The representations induced off the parabolic subgroup $P_{1,1,2}$ are:
(vi-1) $\chi'_1 \times \chi''_1 \times \delta_2$.
(vi-2) $\chi'_1 \times \chi''_1 \times \chi_2$.
(vi-3) $\delta_2 \times \nu^\alpha\chi_1 \times \nu^{-\alpha}\chi_1$.
(vi-4) $\nu^\alpha\chi_1 \times \nu^{-\alpha}\chi_1 \times \chi_2$.

The representations induced off the Borel subgroup $P_{1,1,1,1}$ are:
(vii-1) $\chi'_1 \times \chi''_1 \times \nu^\alpha\chi'''_1 \times \nu^{-\alpha}\chi'''_1$.
(vii-2) $\chi'_1 \times \chi''_1 \times \chi'''_1 \times \chi''''_1$.
(vii-3) $\nu^{\alpha_1}\chi'_1 \times \nu^{-\alpha_1}\chi'_1 \times \nu^{\alpha_2}\chi''_1 \times \nu^{-\alpha_2}\chi''_1$.

The remaining representations are induced off $P_{2,2}$:
(viii-1) $\delta'_2 \times \delta''_2$.
(viii-2) $\delta_2 \times \chi_2$.
(viii-3) $\chi'_2 \times \chi''_2$.

**11. Models for unitary representations of** $\mathrm{Gl}_4$. In this section we consider the unitary representations of $\mathrm{Gl}_4$ with respect to the questions of existence and uniqueness of models. Besides the Whittaker model, there is a mixed model and a symplectic model. These cases lead to the technical heart of our investigation where we confront some of the significant problems which are encountered in proving that an irreducible unitary representation has a unique symplectic model. One of our goals is to determine to what extent a simple inductive statement, analogous to the hereditary property of Whittaker models, holds for symplectic models.

We prove the following general results.

THEOREM 11.1. *Let $\delta$ be an (arbitrary) irreducible admissible representation of* $\mathrm{Gl}_n$.

(a) *The representation* $\nu^{1/2}\delta \times \nu^{-1/2}\delta$ *admits a nontrivial* $\mathrm{Sp}_n$ *invariant functional.*

(b) *If* $\delta$ *is further assumed to be square integrable, then the functional is supported on the unique irreducible quotient* $L(\nu^{1/2}\delta \times \nu^{-1/2}\delta)$.

Lemma 11.2. *Let* $\pi_1$ *and* $\pi_2$ *be irreducible admissible representations of* $\mathrm{Gl}_n$. *The representation*

$$\mathrm{Ind}_{P_{n,n}}^{\mathrm{Gl}_{2n}} \pi_1 \otimes \pi_2 \otimes \delta_{P_{n,n}}^{s} \tag{11.1.1}$$

*has a filtration by* $\mathrm{Sp}_n$ *invariant subspaces with associated subquotient representations*

$$X_r(s) = \mathrm{ind}_{\Sigma_r}^{\mathrm{Sp}_n}(\pi_1 \otimes \pi_2)^{\gamma_r} \otimes \delta_{\Sigma_r}^{s+(r-n+1)/2}, \tag{11.1.2}$$

*where* $\Sigma_r$ *is the group described in Proposition* 4.1.1.1, *and the superscript* $\gamma_r$ *indicates composition with conjugation by* $\gamma_r$.

Corollary 11.3. *Except for a finite set of* $s$ *the representations* (11.1.1) *have at most one nontrivial* $\mathrm{Sp}_n$ *invariant functional.*

The remaining results pertain to $\mathrm{Gl}_4$. In this case, a functional is explicitly constructed on the representations of the corollary for $\mathrm{Re}\, s$ sufficiently large, where $\pi_1$ and $\pi_2$ are assumed to have symplectic functionals. The corollary then allows us to apply the method of Bernstein to analytically continue the functional to the cases of interest. We obtain the following.

Proposition 11.4. *If* $\pi_1$ *and* $\pi_2$ *are irreducible representations of* $\mathrm{Gl}_2$ *with symplectic invariant functionals, then for* $\mathrm{Re}\, s \gg 0$, *there exists a unique nontrivial* $\mathrm{Sp}_2$ *invariant functional on* $\mathrm{Ind}_{P_{2,2}}^{\mathrm{Gl}_4} \pi_1 \otimes \pi_2 \otimes \delta_{P_{2,2}}^{s}$ *given by a convergent integral. Moreover this functional may be analytically continued to the entire complex plane as a rational function in* $q^{-s}$.

Along the way, we examine the catalog of unitary representations of $\mathrm{Gl}_4$, determining first which of them have Whittaker or mixed models. From the results stated above our final result easily follows.

Theorem 11.5. *If* $\pi$ *is an irreducible unitary representation of* $\mathrm{Gl}_4$, *then* $\pi$ *can be realized in a unique way as a submodule of exactly one* *of the following representations: the Whittaker* $\mathscr{M}_0$, *the mixed model* $\mathscr{M}_1$ *or the symplectic model* $\mathscr{M}_2$.

11.1. *Simple cases for* $\mathrm{Gl}_4$. The previously observed facts about square integrable representations and the hereditary property of the Whittaker model allow us to conclude that cases (i), (iv-1), (v-1), (vi-1), (vi-3), (vii-1), (vii-2), (vii-3) and (viii-1) all have Whittaker models.

Case (iii), the unitary characters, obviously have symplectic models.

By Theorem 2.4.1, all these models are unique.

11.2. *Representations of* $\mathrm{Gl}_4$ *with the mixed model.* There are four cases which have mixed models: (v-2), (vi-2), (vi-4) and (viii-2).

11.2.1. *Case* (v-2). Noting that $\mathscr{M}_1$ may be written

$$\mathrm{Ind}_{P_{2,2}}^{\mathrm{Gl}_4}\,\mathrm{Ind}_{U_2\times \mathrm{Sl}_2}^{\mathrm{Gl}_2\times \mathrm{Gl}_2}\,\psi_2\otimes 1, \tag{11.2.1.1}$$

we have

$$\begin{aligned}(11.2.1.2)\quad &\mathrm{Hom}_{\mathrm{Gl}_4}(\mathrm{Ind}_{P_{1,3}}^{\mathrm{Gl}_4}\,\chi_1\otimes\chi_3,\ \mathrm{Ind}_{U_2\times \mathrm{Sl}_2\times N_2}^{\mathrm{Gl}_4}\,\psi_2\otimes 1\otimes 1)\\ &\quad = \mathrm{Hom}_{\mathrm{Gl}_2\times \mathrm{Gl}_2}(\tilde{r}_{2,2}(\mathrm{Ind}_{P_{1,3}}^{\mathrm{Gl}_4}\,\chi_1\otimes\chi_3),\ (\mathrm{Ind}_{U_2}^{\mathrm{Gl}_2}\,\psi_2)\otimes(\mathrm{Ind}_{\mathrm{Sl}_2}^{\mathrm{Gl}_2}1)).\end{aligned}$$

Proposition 1.5 ([**Ze**]) gives

$$\tilde{r}_{1,1,2}(\chi_1\otimes\chi_3) = \chi_1\otimes\tilde{r}_{1,2}(\chi_3) = \chi_1\otimes\nu^{-1}\chi_3\otimes\nu^{1/2}\chi_3. \tag{11.2.1.3}$$

The $\mathrm{Gl}_2\times\mathrm{Gl}_2$ representation

$$(\mathrm{Ind}_{P_{1,1}}^{\mathrm{Gl}_2}\,\nu^{-1/2}\chi_1\otimes\nu^{-1/2}\chi_3)\otimes\nu^{1/2}\chi_3 \tag{11.2.1.4}$$

corresponds to the closed $P_{2,2}$ orbit of $P_{1,3}\backslash\mathrm{Gl}_4$ and hence gives a quotient module of the orbit filtration of $\tilde{r}_{2,2}(\mathrm{Ind}_{P_{1,3}}^{\mathrm{Gl}_4}\,\chi_1\otimes\chi_3)$ ([**Ze**], Theorem 1.2).

The submodule of the representation is computed similarly. First

$$\tilde{r}_{1,2,1}(\chi_1\otimes\chi_3) = \chi_1\otimes\nu^{-1/2}\chi_3\otimes\nu\chi_3. \tag{11.2.1.5}$$

Conjugating by the coset representative

$$\begin{pmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{pmatrix}, \tag{11.2.1.6}$$

of the open $P_{2,2}$ orbit in $P_{1,3}\backslash\mathrm{Gl}_4$ to get $\nu^{-1/2}\chi_3\otimes\chi_1\otimes\nu^{1/2}\chi_3$, then inducing we obtain the submodule

$$\nu^{-1/2}\chi_3\otimes(\mathrm{Ind}_{P_{1,1}}^{\mathrm{Gl}_2}\nu^{-1/2}\chi_1\otimes\nu^{3/2}\chi_3). \tag{11.2.1.7}$$

Clearly there are no nontrivial morphisms from this representation to

$$(\mathrm{Ind}_{U_2}^{\mathrm{Gl}_2}\psi_2)\otimes(\mathrm{Ind}_{\mathrm{Sl}_2}^{\mathrm{Gl}_2}1), \tag{11.2.1.8}$$

On the other hand $\mathrm{Ind}_{P_{1,1}}^{\mathrm{Gl}_2}\nu^{-1/2}\chi_1\otimes\nu^{3/2}\chi_3$ has a unique Whittaker model. Whence the uniqueness of the mixed model for $\chi_1\times\chi_3$.

11.2.2. *Cases* (vi-2), (vi-4), (viii-1). The remaining representations are all induced off $P_{2,2}$ and are irreducible. For a representation of the form $\pi=\mathrm{Ind}_{P_{2,2}}^{\mathrm{Gl}_4}\pi_1\otimes\pi_2$, $\tilde{r}_{2,2}(\pi)$ has a filtration of $\mathrm{Gl}_2\times\mathrm{Gl}_2$ invariant subspaces $0=\tau_0\subset\tau_1\subset\tau_2\subset\tau_3=\pi$, such that $\tau_3/\tau_2$ is isomorphic to $F_1=\pi_1\otimes\pi_2$, $\tau_2/\tau_1$ is isomorphic to $F_{w_1}=\pi_2\otimes\pi_1$ and $\tau_1$ is isomorphic to

$$F_{w_2}=\mathrm{Ind}_{P_{1,1,1,1}}^{\mathrm{Gl}_2\times\mathrm{Gl}_2}w_2[\tilde{r}_{1,1}(\pi_1)\otimes\tilde{r}_{1,1}(\pi_2)], \tag{11.2.2.1}$$

where

$$w_1=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix},\quad w_2=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}. \tag{11.2.2.2}$$

Case (v-4) is of the form $\chi_1'\times\chi_1''\times\chi_2$. $\chi_1'\times\chi_1''$ is an irreducible representation of $\mathrm{Gl}_2$ with Whittaker model; $\chi_2$ has an $\mathrm{Sl}_2$ model. Hence

(11.2.2.3)
$$\dim\mathrm{Hom}_{\mathrm{Gl}_2\times\mathrm{Gl}_2}((\chi_1'\times\chi_1'')\otimes\chi_2,(\mathrm{Ind}_{U_2}^{\mathrm{Gl}_2}\psi_2)\otimes(\mathrm{Ind}_{\mathrm{Sl}_2}^{\mathrm{Gl}_2}1))=1.$$

Also

$$\mathrm{Hom}_{\mathrm{Gl}_2\times\mathrm{Gl}_2}(\chi_2\otimes(\chi_1'\times\chi_1''),(\mathrm{Ind}_{U_2}^{\mathrm{Gl}_2}\psi_2)\otimes(\mathrm{Ind}_{\mathrm{Sl}_2}^{\mathrm{Gl}_2}1))=0. \tag{11.2.2.4}$$

$\tilde{r}_{1,1}(\chi_1'\times\chi_1'')$ is glued from $\chi_1'\otimes\chi_1''$ and $\chi_1''\otimes\chi_1'$. $\tilde{r}_{1,1}(\chi_2)=\nu^{-1/2}\chi_2\otimes\nu^{1/2}\chi_2$. Hence $F_{w_2}$ has composition factors

$$(\mathrm{Ind}_{P_{1,1}}^{\mathrm{Gl}_2}\chi_1'\otimes\nu^{-1/2}\chi_2)\otimes(\mathrm{Ind}_{P_{1,1}}^{\mathrm{Gl}_2}\chi_1''\otimes\nu^{1/2}\chi_2), \tag{11.2.2.5}$$

and

$$(\mathrm{Ind}_{P_{1,1}}^{\mathrm{Gl}_2}\chi_1''\otimes\nu^{-1/2}\chi_2)\otimes(\mathrm{Ind}_{P_{1,1}}^{\mathrm{Gl}_2}\chi_1'\otimes\nu^{1/2}\chi_2). \tag{11.2.2.6}$$

Neither of these representations admits nontrivial homomorphisms into (11.2.1.7). Thus $\chi_1' \times \chi_1'' \times \chi_2$ has a unique mixed model.

The remaining cases are of the form $\delta_2 \times \chi_2$. The argument is similar to the previous cases. The quotient of $\tilde{r}_{2,2}(\delta_2 \times \chi_2)$ will be $\delta_2 \otimes \chi_2$, which has a unique map to (11.2.1.7). Thus it remains to show that the other filtration factors of $\tilde{r}_{2,2}(\delta_2 \times \chi_2)$ have no such morphisms. By disjointness of models, $F_{w_1} = \chi_2 \otimes \delta_2$ has no such map. To describe $F_{w_2}$, it is necessary to specify $\delta_2$.

As in the previous case $F_{w_2}$ is built from $\tilde{r}_{1,1}(\delta_2) \otimes \tilde{r}_{1,1}(\chi_2)$ so $\delta_2$ is either supercuspidal, the complementary series $\nu^{\alpha}\chi_1 \times \nu^{-\alpha}\chi_1$ or St, the Steinberg representation. If $\delta$ is supercuspidal, $\tilde{r}_{1,1}(\delta_2) = 0$. For the complementary series, $\tilde{r}_{1,1}(\nu^{\alpha}\chi_1 \times \nu^{-\alpha}\chi_1)$ is glued from $\nu^{\alpha}\chi_1 \otimes \nu^{-\alpha}\chi_1$ and $\nu^{-\alpha}\chi_1 \otimes \nu^{\alpha}\chi_1$. $\tilde{r}_{1,1}(\chi_2) = \nu^{-1/2}\chi_2 \otimes \nu^{1/2}\chi_2$. Thus $F_{w_2}$ is glued from $(\nu^{\alpha}\chi_1 \times \nu^{-1/2}\chi_2) \otimes (\nu^{-\alpha}\chi_1 \times \nu^{1/2}\chi_2)$ and $(\nu^{-\alpha}\chi_1 \times \nu^{-1/2}\chi_2) \otimes (\nu^{\alpha}\chi_1 \times \nu^{1/2}\chi_2)$. Since the characters are the only $\mathrm{Gl}_2$ representations with $\mathrm{Sl}_2$ models, we see that in each case the second tensor factor has no such model, since the central characters of these representations are $\nu^{1/2 \pm \alpha}(\chi_1 \otimes \chi_2)$, with $0 < \alpha < \frac{1}{2}$, which are not unitary.

For $\delta_2 = \mathrm{St}$, we have the exact sequence

$$0 \to 1 \to \nu^{-1/2} \times \nu^{1/2} \to \mathrm{St} \to 0. \tag{11.2.2.7}$$

$\tilde{r}_{1,1}$ is an exact functor; hence we get the exact sequence of $\mathrm{Gl}_1 \times \mathrm{Gl}_1$ representations

$$0 \to \nu^{-1/2} \otimes \nu^{1/2} \to \tilde{r}_{1,1}(\nu^{-1/2} \times \nu^{1/2}) \to \tilde{r}_{1,1}(\mathrm{St}) \to 0. \tag{11.2.2.8}$$

As $\tilde{r}_{1,1}(\mathrm{St}) = \nu^{1/2} \otimes \nu^{-1/2}$ ([**Ze**], Theorem 1.2), $F_{w_2} = (\nu^{1/2} \times \nu^{-1/2}\chi_2) \otimes (\nu^{-1/2} \times \nu^{1/2}\chi_2)$. If $\chi_2$ is nontrivial, the second factor is irreducible and has a Whittaker model. If $\chi_2$ is trivial, $\nu^{-1/2} \times \nu^{1/2}$ is reducible and supports no $\mathrm{Sl}_2$ invariant functional, for if it did, it would have an irreducible unitary character as a quotient but St is the unique irreducible quotient.

11.3. *Unitary representations and symplectic models.* We investigate symplectic models for certain representations of $\mathrm{Gl}_{2n}$. This will include giving the proofs of the general results stated at the beginning of §11 and finishing the proof of Theorem 11.5, by showing that the remaining three cases for $\mathrm{Gl}_4$ have symplectic models.

11.3.1. *Proof of Theorem* 11.1; *case* (ii). Case (ii) is $u(\delta, 2) = L(\nu^{1/2}\delta \times \nu^{-1/2}\delta)$, where $\delta$ is square integrable. That this has a symplectic functional is precisely Theorem 11.1(b). The proof naturally divides into two parts.

11.3.1.1. Consider in general the $\mathrm{Gl}_{2n}$ representation

$$u(\delta, 2) = L(\nu^{1/2}\delta \times \nu^{-1/2}\delta), \tag{11.3.1.1.1}$$

where $\delta \in D_0(n)$. Since $\delta$ has a Whittaker model, the full induced representation $\pi = \nu^{1/2}\delta \times \nu^{-1/2}\delta$ does so too.

Suppose that $\delta$ is square integrable. Then, in the terminology and notation of Zelevinsky ([**Ze**]), $\delta = \langle\Delta\rangle^t$, where $\Delta$ is a segment $\{\sigma, \nu\sigma, \dots, \nu^i\sigma\}$, with $\sigma$ supercuspidal, and $\delta$ is the unique irreducible quotient of

$$\sigma \times \nu\sigma \times \cdots \times \nu^k\sigma \tag{11.3.1.1.2}$$

(ibid., §9.1, Theorem 9.3). According to Lemma 3.2 in [**Ta**,2], in the Grothendieck ring of admissible representations of finite length,

$$\pi = u(\delta, 2) + (\langle\Delta_\cup\rangle^t \times \langle\Delta_\cap\rangle^t) \tag{11.3.1.1.3}$$

where

$$\Delta_\cap = \nu^{1/2}\Delta \cap \nu^{-1/2}\Delta, \quad \Delta_\cup = \nu^{1/2}\Delta \cup \nu^{-1/2}\Delta. \tag{11.3.1.1.4}$$

In particular, $\pi$ has length two. The submodule is an irreducible tempered representation, hence has a Whittaker model. Since a representation cannot have both a symplectic and Whittaker model (Theorem 3.2.2), we conclude that if $\pi$ has a map into the symplectic model, it must be supported on the irreducible quotient $u(\delta, 2)$.

We remark that the composition factors appearing in the induced representation which gives $u(\delta, n)$ (see §6) are now known. In the notation of [**Ta**,**3**], $u(\delta, n) = L(a)$, the unique irreducible quotient of a representation $\lambda(a)$, where $a$ is a multiset of segments. Then a necessary and sufficient condition for $L(b)$ to be a subquotient of $\lambda(a)$ is that $b \leq a$ in the Zelevinsky partial ordering ([**Ze**, §7]), so that the composition factors appearing in the Langlands, i.e. square integrable setting are exactly those which appear in the Zelevinsky, i.e. cuspidal setting. A proof of this result will be appearing in a forthcoming paper of Tadić.

11.3.1.2. We now show that $u(\delta, 2)$ has a symplectic functional by constructing one on $\nu^{1/2}\delta \times \nu^{-1/2}\delta$.

Consider the representation

$$I_s = \mathrm{Ind}_{P_{n,n}}^{\mathrm{Gl}_{2n}}(\pi \otimes \pi) \otimes \delta_{P_{n,n}}^s, \tag{11.3.1.2.1}$$

where $s$ is a complex parameter. For $f_s \in I_s$ and

$$p = \begin{pmatrix} g & * \\ 0 & {}^t g^{-1} \end{pmatrix} \in P'_{n,n}, \tag{11.3.1.2.2}$$

we have

$$(11.3.1.2.3)\quad \begin{aligned} f_s(p) &= \delta_{P_{n,n}}^{s+1/2}(p)(\pi(g)\otimes\pi({}^tg^{-1}))f_s(1)\\ &= |\det g|^{2ns+n}(\pi(g)\otimes\pi'(g))f(1)\\ &= \delta_{P'_{n,n}}^{(2ns+n)/(n+1)}(p)(\pi(g)\otimes\pi'(g))f(1).\end{aligned}$$

Thus restricting $f_s$ to $\mathrm{Sp}_n$ gives an element of

$$(11.3.1.2.4)\qquad \mathrm{Ind}_{P'_{n,n}}^{\mathrm{Sp}_n}(\pi\otimes\pi')\otimes\delta_{P'_{n,n}}^{s'},$$

where $s' = (2ns+n)/(n+1)-\frac{1}{2}$.

Let $l\colon \pi\otimes\pi' \to \mathbb{C}$ be the standard pairing. Then $l\circ f_s \in I'_{s'} = \mathrm{Ind}_{P'_{n,n}}^{\mathrm{Sp}_n}\delta_{P'_{n,n}}^{s'}$. $l$ is surjective and induction is an exact functor. Thus when $s'=\frac{1}{2}$, integration over $P'_{n,n}\backslash\mathrm{Sp}_n$ with respect to the quasi-invariant measure is a nontrivial, $\mathrm{Sp}_n$ invariant functional on $I'_s$. This value of $s'$ corresponds to $s = 1/2n$.

The restriction map $I_{1/2n}\to I'_{1/2}$ corresponds to a map between the finite sections of sheaves, induced from the restriction from $P_{n,n}\backslash\mathrm{Gl}_{2n}$ to the image of $P'_{n,n}\backslash\mathrm{Sp}_n$ in $P_{n,n}\backslash\mathrm{Gl}_{2n}$, which is closed. This is also surjective (**[Be-Ze,2]**, Propositions 1.8, 1.16, Proposition 2.23). The composite $I_{1/2n}\to I'_{1/2}\to\mathbb{C}$ is thus $\mathrm{Sp}_n$ invariant and nontrivial. Since

$$(11.3.1.2.5)\qquad \begin{aligned} I_{1/2n} &= \mathrm{Ind}_{P_{n,n}}^{\mathrm{Gl}_{2n}}(\pi\otimes\pi)\otimes\delta_{P_{n,n}}^{1/2n}\\ &= \mathrm{Ind}_{P_{n,n}}^{\mathrm{Gl}_{2n}}(\nu^{1/2}\pi\otimes\nu^{-1/2}\pi),\end{aligned}$$

we have shown that $\nu^{1/2}\pi\times\nu^{-1/2}\pi$ has an $\mathrm{Sp}_n$ invariant functional.

11.3.2. *Proofs of Lemma* 11.2 *and Corollary* 11.3. The $\mathrm{Sp}_n$ orbits of $P_{n,n}\backslash\mathrm{Gl}_{2n}$ are described in Proposition 4.1.1. Orbital analysis (cf. **[Ca]**) then gives a filtration of (11.1.2) of the form stated in Lemma 11.2.

The corollary will follow by showing that, except for a finite number of values of $s$, only one of the representation $X_r(s)$ carries a unique symplectic functional.

Conjugating the matrix (4.1.1.1) by $\gamma_r$ gives

$$(11.3.2.1)\qquad \begin{pmatrix} g & 0 & 0 & 0 & 0 & 0\\ 0 & A' & B' & 0 & 0 & 0\\ 0 & C' & D' & 0 & 0 & 0\\ 0 & 0 & 0 & {}^tg^{-1} & 0 & 0\\ 0 & 0 & 0 & 0 & A & B\\ 0 & 0 & 0 & 0 & C & D \end{pmatrix}.$$

Conjugating the unipotent element (4.1.1.2) by $\gamma_r$ gives

$$\begin{pmatrix} 1 & X_2 & Z_2 & Y & X_1 & Z_1 \\ 0 & 1 & 0 & {}^tZ_2 & 0 & 0 \\ 0 & 0 & 1 & -{}^tX_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & {}^tZ_1 & 1 & 0 \\ 0 & 0 & 0 & -{}^tX_1 & 0 & 1 \end{pmatrix}, \tag{11.3.2.2}$$

where $X = (X_1X_2)$, $Z = (Z_1Z_2)$ and $Y$ is symmetric. Mindful of the normalization in the induction, the inducing representation applied to an element of $\Sigma_r$ is

$$\begin{aligned} \pi_1 & \left[\begin{pmatrix} 1 & X_2 & Z_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} g & 0 & 0 \\ 0 & A' & B' \\ 0 & C' & D' \end{pmatrix}\right] \\ \otimes \pi_2 & \left[\begin{pmatrix} 1 & 0 & 0 \\ {}^tZ_1 & 1 & 0 \\ -{}^tX_1 & 0 & 1 \end{pmatrix} \begin{pmatrix} g^0 & 0 & \\ 0 & A & B \\ 0 & C & D \end{pmatrix}\right] |\det g|^{s+n/2}. \end{aligned} \tag{11.3.2.3}$$

The contragradient of $X_r(s)$ is

$$\mathrm{Ind}_{\Sigma_r}^{\mathrm{Sp}_n}[(\pi_1 \otimes \pi_2)^{\gamma_r}]' \otimes \delta_{\Sigma_r}^{-s-(r-n-1)/2}. \tag{11.3.2.4}$$

Let $U_r'(0)$ be the elements of the form (11.3.2.2) with $Y = 0$. Then

$$\mathrm{Hom}_{\mathrm{Sp}_n}(X_r(s), 1) = \mathrm{Hom}_{\mathrm{Sp}_n}(1, X_r(s)'). \tag{11.3.2.5}$$

which in turn equals

(11.3.2.6)

$$\mathrm{Hom}_{(\mathrm{Gl}_r \times \mathrm{Sp}_{(n-r)/2} \times \mathrm{Sp}_{(n-r)/2})U_r'(0)}((\pi_1' \otimes \pi_2')|\det g|^{-s+n/2+(2n-r+1)}, 1),$$

where the groups act according to (11.3.2.3). Applying Jacquet functors, (11.3.2.6) equals

(11.3.2.7)

$$\mathrm{Hom}_{\mathrm{Gl}_r \times \mathrm{Sp}_{(n-r)/2} \times \mathrm{Sp}_{(n-r)/2}}(((\pi_1')_{N(r)} \otimes (\pi_2')_{\overline{N}(r)}) \otimes |\det|^{-s+n/2+(2n-r+1)}, 1),$$

where $N(r)$ is the group of unipotent matrices appearing in (11.3.2.3) and $\overline{N}(r)$ is the opposite unipotent subgroup. $\mathrm{Gl}_r$ acts in $(\pi_2')_{\overline{N}(r)}$

via transpose inverse. Since $(\pi_2')_{\overline{N}(r)} = ((\pi_2)_{N(r)})'$ (**[Cas]**, Corollary 4.2.5), we find $\mathrm{Hom}_{\mathrm{Sp}_n}(X_r(s), 1)$ equal to

(11.3.2.8)
$$\mathrm{Hom}_{\mathrm{Gl}_r \times \mathrm{Sp}_{(n-r)/2} \times \mathrm{Sp}_{(n-r)/2}}((\pi_1')_{N(r)} \otimes ((\pi_2)_{N(r)})' \otimes |\det|^{-s+n/2+(2n-r+1)}, 1).$$

If $r \neq 0$, there is only one value of $s$ for which this groups can be nonzero. If $r = 0$, we have

$$\mathrm{Hom}_{\mathrm{Sp}_n}(X_r(s), 1) = \mathrm{Hom}_{\mathrm{Sp}_{n/2} \times \mathrm{Sp}_{n/2}}(\pi_1 \otimes \pi_2, 1). \tag{11.3.2.9}$$

This space has dimension one precisely when $\pi_1$ and $\pi_2$ both admit symplectic models.

11.3.3. *Proofs of Proposition* 11.4 *and Theorem* 11.5; *cases* (iv-2), (viii-3). Let $\pi_1$ and $\pi_2$ be irreducible with $\mathrm{Sl}_2$ invariant functionals $l_1$ and $l_1$. Consider the representation $I_s = \mathrm{Ind}_{P_{2,2}}^{\mathrm{Gl}_4} \pi_1 \otimes \pi_2 \otimes \delta_{P_{2,2}}^s$. Set $l = l_1 \otimes l_2$, and denote $P_{2,2}$ by $P$.

By Corollary 4.1.1.2 the open $\mathrm{Sp}_2$ orbit in $P\backslash \mathrm{Gl}_4$ is given by the coset $P\gamma\,\mathrm{Gl}_4$ where

$$\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{11.3.3.1}$$

For $f_s \in I_s$, $l \circ f_s(\gamma g) = l \circ f_s(\gamma)$ for $g \in \mathrm{Sl}_2 \times \mathrm{Sl}_2$, embedded in $\mathrm{Sp}_2$ so as to be $\gamma$ conjugate to the diagonal embedding in $\mathrm{Gl}_4$. Consider the integral

$$\lambda(f_s) = \int_{(\mathrm{Sl}_2 \times \mathrm{Sl}_2)\backslash \mathrm{Sp}_2} l \circ f_s(\gamma m)\, dm, \tag{11.3.3.2}$$

where $dm$ is a right invariant measure on $(\mathrm{Sl}_2 \times \mathrm{Sl}_2)\backslash \mathrm{Sp}_2$. If this converges, it will provide an $\mathrm{Sp}_2$ invariant functional on $I_s$. Letting $P' = P \cap \mathrm{Sp}_2$, by Proposition 4.2.1, the open dense $P'$ orbit in $(\mathrm{Sl}_2 \times \mathrm{Sl}_2)\backslash \mathrm{Sp}_2$ is $(\mathrm{Sl}_2 \times \mathrm{Sl}_2)\rho J P'$. Thus

$$\Lambda(f_s) = \int_{(\mathrm{Sl}_2 \times \mathrm{Sl}_2)\rho J P'} l \circ f_s(\gamma m)\, dm, \tag{11.3.3.3}$$

We may view the domain of integration as the $P'$ orbit $P_X'\backslash P'$ where

$$P_X' = \left\{ \begin{pmatrix} & b & & 0 \\ g & & & \\ & 0 & & c \\ 0 & & {}^t g^{-1} & \end{pmatrix} \middle| g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}_2 \right\}, \tag{11.3.3.4}$$

and thus

$$(11.3.3.5) \qquad \Lambda(f_s) = \int_{P'_X\backslash P'} l \circ f_s(\gamma\rho Jp)\, dp\,,$$

where $dp$ is a right invariant measure on $P'_X\backslash P'$. As

$$(11.3.3.6) \qquad P' = \left\{ \begin{pmatrix} g & gZ \\ 0 & {}^t g^{-1} \end{pmatrix} \middle|\, g \in \mathrm{Gl}_2\,,\ Z \in \mathrm{Sym}_2 \right\},$$

$P'_X\backslash P_X = \mathrm{Sl}_2\backslash\mathrm{Gl}_2 \times \mathrm{Sym}_2$. For a right invariant measure $dg$ on $\mathrm{Sl}_2\backslash\mathrm{Gl}_2$ and an additive invariant measure $dZ$ on $\mathrm{Sym}_2$, $dp = |g|^3\, dZ\, dg$. If $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sl}_2$, $g \in \mathrm{Gl}_2$ and $Z \in \mathrm{Sym}_2$,

$$(11.3.3.7) \qquad \begin{pmatrix} h & \begin{matrix} b & 0 \\ 0 & c \end{matrix} \\ 0 & {}^t h^{-1} \end{pmatrix} \begin{pmatrix} g & gZ \\ 0 & {}^t g^{-1} \end{pmatrix} = \begin{pmatrix} hg & 0 \\ 0 & {}^t(hg)^{-1} \end{pmatrix} \begin{pmatrix} 1 & Z + hg \left(\begin{smallmatrix} b & 0 \\ 0 & c \end{smallmatrix}\right) {}^t g^{-1} \\ 0 & 1 \end{pmatrix}.$$

Then by the invariance of $dZ$,

$$(11.3.3.8) \qquad \Lambda(f_s) = \int_{\mathrm{Sl}_2\backslash\mathrm{Gl}_2 \times \mathrm{Sym}_2} l \circ f_s\left(\gamma\rho J \begin{pmatrix} g & 0 \\ 0 & {}^t g^{-1} \end{pmatrix} \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix}\right) |g|^3\, dg\, dZ.$$

$g \to {}^t gJg$ maps $\mathrm{Sl}_2\backslash\mathrm{Gl}_2$ bijectively onto the nonzero $2 \times 2$ skew symmetric matrices. Identified with $F^\times$, the invariant measure on this $\mathrm{Gl}_2$ orbit is $d^\times\lambda = d\lambda/|\lambda|$, where $d\lambda$ is an additive measure. Thus

$$(11.3.3.9) \qquad \Lambda(f_s) = \int_{F^\times \times \mathrm{Sym}_2} l \circ f_s\left(\gamma\rho J D(\lambda) \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix}\right) |\lambda|^3\, d^\times\lambda\, dZ\,,$$

where

$$(11.3.3.10) \qquad D(\lambda) = \begin{pmatrix} \lambda & & & \\ & 1 & & \\ & & \lambda^{-1} & \\ & & & 1 \end{pmatrix}.$$

Set $\xi = \gamma\rho J$. Then

$$(11.3.3.11) \qquad \omega = \xi D(\lambda) \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix} \xi^{-1} = \begin{pmatrix} 1 + y\lambda & x\lambda & x\lambda & 1 - \lambda + y\lambda \\ -z & 1 - y & -y & -z \\ z & \lambda^{-1} - 1 + y & y + \lambda^{-1} & z \\ -\lambda y & -\lambda x & -\lambda x & \lambda - \lambda y \end{pmatrix},$$

where $Z = \left(\begin{smallmatrix} x & y \\ y & z \end{smallmatrix}\right)$. Define $h$ by

$$\text{(11.3.3.12)} \qquad \omega = \begin{pmatrix} 1 & & & \\ & -\lambda & & \\ & & 1 & \\ & & & -\lambda \end{pmatrix} h$$

Thus

$$\text{(11.3.3.13)} \qquad f_s(\omega\xi) = \left(\pi_1 \begin{pmatrix} 1 & 0 \\ 0 & -\lambda \end{pmatrix} \otimes \pi_2 \begin{pmatrix} 1 & 0 \\ 0 & -\lambda \end{pmatrix}\right) f_s(h\xi).$$

By right multiplication on $h\xi$ by elements of the maximal compact subgroup $K = \mathrm{Gl}_n(o)$, obtain $h = h_1 k_1$, $k_1 \in K$, where $h_1 = \left(\begin{smallmatrix} h_1^+ \\ h_1^- \end{smallmatrix}\right)$, $h_1^+$ and $h_1^-$ are $2 \times 4$ matrices and

$$\text{(11.3.3.14)} \qquad h_1^- = \begin{pmatrix} z & \lambda^{-1} + y & 1 & 0 \\ y & x & 0 & 1 \end{pmatrix}.$$

$h$ has an Iwasawa decomposition $\left(\begin{smallmatrix} A_1 & W \\ 0 & A_2 \end{smallmatrix}\right) k_2$, where $k_2 \in K$. According to Lemma 6.8 in **[PS-Ra]**, $|\det A_2| = \kappa_2(h_1^{-1})$, where $\kappa_2(A)$ is the maximum of the absolute values of the $2 \times 2$ minors of the $2 \times 4$ matrix $A$. We have $|\det a\omega| = 1$, $|\det h| = |\det h_1| = |\lambda|^{-2}$, and $|\det A_1||\det A_2| = |\lambda|^{-2}$. Thus for some $k \in K$, depending on $x$, $y$, $z$ and $\lambda$

(11.3.3.15)

$$\begin{aligned} f_s(\omega\xi) &= \left(\pi_1 \begin{pmatrix} 1 & 0 \\ 0 & -\lambda \end{pmatrix} \otimes \pi_2 \begin{pmatrix} 1 & 0 \\ 0 & -\lambda \end{pmatrix}\right) f_s(h_1 k) \\ &= (|\lambda|^{-2} \kappa_2(h_1^-)^{-2})^{s+1/2} \left(\pi_1 \begin{pmatrix} 1 & 0 \\ 0 & -\lambda \end{pmatrix} \otimes \pi_2 \begin{pmatrix} 1 & 0 \\ 0 & -\lambda \end{pmatrix}\right) f_s(k). \end{aligned}$$

Substituting this in the expression (11.3.2.9), we obtain $\Lambda(f_s)$ equal to

$$\text{(11.3.3.16)} \qquad \int_{F^\times \times F^3} |\lambda|^{-4s+1} \kappa_2(h_1^-)^{-(4s+2)} \cdot l\left(\pi_1 \begin{pmatrix} 1 & 0 \\ 0 & -\lambda \end{pmatrix} \otimes \pi_2 \begin{pmatrix} 1 & 0 \\ 0 & -\lambda \end{pmatrix}\right) f_s(k)\, d^\times\lambda\, dx\, dy\, dz.$$

Making the change of variable $\lambda \to \lambda^{-1}$ and $w = \lambda + y$, we obtain for $\Lambda(f_s)$

$$\text{(11.3.3.17)} \qquad \int_{F^4} |w - y|^{4s-2} \kappa_2 \begin{pmatrix} z & w & 1 & 0 \\ y & x & 0 & 1 \end{pmatrix}^{-(4s+2)} \cdot l\left(\pi \begin{pmatrix} 1 & 0 \\ 0 & -\lambda \end{pmatrix} \otimes \pi_2 \begin{pmatrix} 1 & 0 \\ 0 & -\lambda \end{pmatrix}\right) f_s(k)\, dw\, dx\, dy\, dz.$$

Since the inducing representations of $\mathrm{Gl}_2$ have symplectic models, either $\pi_i = \chi_i$ are unitary characters or $\pi_1 = \nu^{\alpha}\chi$ and $\pi_2 = \nu^{-\alpha}\chi$, so that

$$(11.3.3.18) \qquad \left| l\left(\pi_1\begin{pmatrix}1 & 0\\ 0 & -\lambda\end{pmatrix} \otimes \pi_2\begin{pmatrix}1 & 0\\ 0 & -\lambda\end{pmatrix}\right) f_s(k)\right| = |l f_s(k)|.$$

For $s \geq \frac{1}{2}$,

$$\begin{aligned}(11.3.3.19) \qquad |w-y|^{4s-2} &\leq \max\{|w|, |y|\}^{4s-2}\\ &\leq \kappa_2\begin{pmatrix} z & w & 1 & 0\\ y & x & 0 & 1\end{pmatrix}^{4s-2},\end{aligned}$$

so the integral of the absolute value of the integrand in (11.3.2.17) is bounded by a constant multiple of

$$(11.3.3.20) \qquad \int_{F^4} \kappa_2\begin{pmatrix} z & w & 1 & 0\\ y & x & 0 & 1\end{pmatrix}^{-4} dw\,dx\,dy\,dz.$$

Let $I(P, s) = \mathrm{Ind}_P^{\mathrm{Gl}_4}\delta_P^s$ and $I(\overline{P}, s) = \mathrm{Ind}_{\overline{P}}^{\mathrm{Gl}_4}\delta_{\overline{P}}^s$, where $\overline{P}$ is the parabolic subgroup opposite to $P$. Extend $\delta_P$ to $\mathrm{Gl}_4$ via the Iwasawa decomposition $\mathrm{Gl}_4 = PK$. We have the basic intertwining operator $A_s\colon I(P, s) \to I(\overline{P}, -s)$ defined by

$$(11.3.3.21) \qquad A_s(F_s)(g) = \int_{\overline{N}} F_s(\overline{n}g)\,d\overline{n},$$

where $\overline{N}$ is the unipotent radical of $\overline{P}$. From the Iwasawa decomposition of $\overline{n} = \left(\begin{smallmatrix}1 & 0\\ X & 1\end{smallmatrix}\right)$, with $F_s = \delta_P^{s+1/2}$ obtain

$$(11.3.3.22) \qquad A_s(\delta_P^{s+1/2})(1) = \int_{F^4} \kappa_2(X, 1_2)^{-4s-2}\,dX.$$

$A_s(\delta_P^{s+1/2})(1)$ converges for $\mathrm{Re}\, s > 1/4$ (cf. **[Bo-Wa]**). In particular for $s = 1/2$, we obtain (11.3.3.20). Thus for $s \geq 1/2$, we have constructed an $\mathrm{Sl}_2$ invariant functional on $\mathrm{Ind}_P^{\mathrm{Gl}_4}\chi_1 \otimes \chi_2 \otimes \delta_P^s$. The representations of interest correspond to $0 \leq s < \frac{1}{4}$.

We continue the functional $\Lambda_s$ using the method of Bernstein (cf. **[Ge-PS]**, pp. 126–129; **[Ka-Pa]**, p. 67). Let $V_s$ denote the space of the representation $I_s = \mathrm{Ind}_P^{\mathrm{Gl}_4}\chi \otimes \delta_P^s$, where $\chi$ is a unitary character. Then $V_s$ is naturally isomorphic to $V_0$ by restriction to $K$.

The action of $\mathrm{Gl}_4$ on $V_0$ via $I_s$ is given by

$$(I_s(g)\phi)(k) = \chi(p)\delta_P^{s+1/2}\phi(k'),$$

where $kg = pk'$. Let $V_0^*$ be the dual, $D = \mathbb{C}^{\times}$ as an irreducible variety and $\mathbb{C}[D] = \mathbb{C}[z, z^{-1}]$ the ring of regular functions on $D$.

Write $z = q^{-s}$, $-\pi/\log q \leq s < \pi/\log q$. Let $R = R_{V_0} \times R_s$ where $R_{V_0} = \{y_{v'}\}$ is a countable basis for $V_0$ and $R_s = \{g_v\}$ is a countable basis for $\mathrm{Sp}_2$. The family of systems $\Xi_s = \{I_s(g_v)y_{v'} - y_{v'} = 0;\ (v, v') \in R\}$, $s \in \mathbb{C}$, is polynomial in $z$ and by Corollary 11.3 it is a unique solution for $\operatorname{Re} s$ sufficiently large. According to Bernstein's theorem, there is a unique solution $\Lambda_s \in (V_0 \otimes \mathbb{C}(D))^*$, $\mathbb{C}(D)$ the function field of $D$. $\Lambda_s(y)$ is an $\mathrm{Sp}_2$ invariant functional which is a rational function of $q^{-s}$. In particular this will give nontrivial invariant functionals on the remaining representations.

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