ON THE ELIMINATION OF ALGEBRAIC INEQUALITIES

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Let \( S \) be a locally closed semi-algebraic subset of \( \mathbb{R}^n \). We find an irreducible equation of an algebraic set of \( \mathbb{R}^{n+1} \) projecting upon \( S \). Our methods are simple and explicit.

1. Introduction. The inequality \( x \geq 0 \) is often replaced by the proposition "\( x \) has a square root" or "\( \exists t \in \mathbb{R}, t^2 - x = 0 \)". This is the most immediate example of an elimination of one inequality. The general problem is to find an algebraic set projecting upon a given semi-algebraic set: it is a converse of the problem of the elimination of quantifiers.

Motzkin proved that every semi-algebraic subset of \( \mathbb{R}^n \) is the projection of an algebraic set in \( \mathbb{R}^{n+1} \). However this algebraic set is very complicated and generally reducible.

Andradas and Gamboa proved that any closed semi-algebraic subset of \( \mathbb{R}^n \) whose Zariski-closure is irreducible is the projection of an irreducible algebraic set in \( \mathbb{R}^{n+k} \).

In this paper we shall first improve Motzkin’s result by finding equations generally of minimal degree. Then we shall give a few results concerning irreducibility. One of the first examples of such a construction is due to Rohn and has been studied by Hilbert and Utkin:

If \( 4C_4C_2 = \varepsilon^2 \) is a plane curve of degree six (where \( \deg(C_2) = 2 \), \( \deg(C_4) = 4 \), \( \varepsilon \in \mathbb{R} \)), then it is the apparent contour of the quartic surface \( C_2z^2 - \varepsilon z + C_4 = 0 \).

2. The case of basic closed subsets. Let \( \mathbb{R}^+ = \{x \in \mathbb{R} | x \geq 0\} \) be the set of nonnegative numbers. Let \( x = (x_1, \ldots, x_N) \) be a “parameter” and \( t \) an “indeterminate”, so that we can speak of the roots of a polynomial \( P(x, t) \). In the same way, unless otherwise specified, the degree of \( P(x, t) \) will be its degree in \( t \).

Let us define the polynomials \( a_i(x) \) as follows:

\[
a_k(x_1, \ldots, x_{k+1}) = x_{k+1}(x_1 + x_2 + \cdots + x_k).
\]

It is easy to see that \( a_1(x) \geq 0, \ldots, a_n(x) \geq 0 \) if and only if all the \( x_i \) are nonnegative or all the \( x_i \) are nonpositive \((i = 1, \ldots, n + 1)\).
**Theorem 1.** Let \( P_1(x_1, u) = u - x_1 \).

\[
P_{n+1}(x_1, \ldots, x_{n+1}, u) = P_n(a_1(x), \ldots, a_n(x), (u - (x_1 + x_2 + \cdots + x_{n+1}))^2).
\]

Then the following properties are true:

(i) \( P_n \) is homogeneous of degree \( 2^{n-1} \).

(ii) If all the \( x_i \) are nonnegative

\[
P_n(x_1, \ldots, x_n, u) = 0 \Rightarrow 0 \leq u \leq 2 \sum_{i=1}^{n} x_i.
\]

(iii) If all the \( x_i \) are nonnegative, \( P_n(x_1, \ldots, x_n, t^2) \) has only real roots.

(iv) If \( P_n(x_1, \ldots, x_n, t^2) \) has a real root, then all the \( x_i \) are nonnegative.

(v) \[
P_n(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n, t) = [P_{n-1}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n, t)]^2
\]

(vi) \( P_n(x_1, \ldots, x_n, t^2) \) is irreducible and monic in each letter.

**Proof.** First, we prove (i), (ii), (iii), and (iv) by simultaneous induction: let us suppose (i), (ii), (iii) and (iv) verified for \( n \); we shall prove them for \( n + 1 \).

(i) Easy since the \( a_k \) are homogeneous of degree 2.

(ii) If \( u \) is a root of \( P_{n+1}(x_1, \ldots, x_{n+1}, u) = 0 \), then

\[
(u - (x_1 + \cdots + x_{n+1}))^2
\]

is a root of \( P_n(a_1(x), \ldots, a_n(x), u) = 0 \) by induction

\[
(u - (x_1 + \cdots + x_{n+1}))^2 \leq 2(a_1(x) + \cdots + a_n(x)) \leq (x_1 + \cdots + x_{n+1})^2
\]

whence \( 0 \leq u \leq 2(x_1 + \cdots + x_{n+1}) \), which shows (ii) and (iii).

(iv) If \( P_{n+1}(x_1, \ldots, x_{n+1}, t^2) = 0 \) has a real root, then

\[
P_n(a_1(x), \ldots, a_n(x), (t^2 - (x_1 + \cdots + x_{n+1}))^2)
\]

has a real root and by induction all the \( a_i(x) \) are nonnegative. Therefore, if all the \( x_i \) are nonpositive, \( P_n(a_1(x), \ldots, a_n(x), v) \) has a root which is greater than \( (x_1 + \cdots + x_{n+1})^2 \). By induction this is possible only if

\[
(x_1 + \cdots + x_{n+1})^2 = 2(a_1(x) + \cdots + a_n(x)),
\]

i.e., when all the \( x_i \) are equal to zero.
(v) By induction: suppose the formula true for \( n \), let us prove it for \( n + 1 \). Let us study the case \( j \geq 2 \) (the case \( j = 1 \) is similar).

Let

\[
x = (x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_{n+1}),
\]

\[
\hat{x} = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1}).
\]

We have:

\[
\begin{cases}
  a_i(x) = a_i(\hat{x}) & \text{if } i < j - 1, \\
  a_{j-1}(x) = 0, \\
  a_k(x) = a_{k-1}(\hat{x}) & \text{if } k \geq j.
\end{cases}
\]

Then,

\[
P_{n+1}(x, t) = P_n(a_1(x), \ldots, a_n(x), (t - (x_1 + \cdots + x_{n+1}))^2)
\]

\[
= P_n(a_1(\hat{x}), \ldots, a_{j-2}(\hat{x}), 0, a_{j-1}(\hat{x}), \ldots, a_{n-1}(\hat{x}),
\]

\[
(t - (x_1 + \cdots + x_{n+1}))^2)
\]

\[
= [P_{n-1}(a_1(\hat{x}), \ldots, a_{n-1}(\hat{x}), (t - (x_1 + \cdots + x_{n+1}))^2]^2
\]

\[
= [P_n(\hat{x}, t)]^2.
\]

(vi) By induction. Suppose \( P_n(x, t^2) \) irreducible. Let

\[
P_{n+1}(x_1, \ldots, x_{n+1}, t^2) = A(x, t) \cdot B(x, t),
\]

\( A \) and \( B \) monic in \( t \). Let us substitute 0 for \( x_{n+1} \) in this factorization; using (v) we get:

\[
(P_n(x_1, \ldots, x_n, t^2))^2 = A(x_1, \ldots, x_n, 0, t) \cdot B(x_1, \ldots, x_n, 0, t).
\]

Since \( P_n(x, t^2) \) is irreducible, and \( A \) and \( B \) are monic in \( t \), we get either:

\[
A(x_1, \ldots, x_n, 0, t) = B(x_1, \ldots, x_n, 0, t) = P_n(x_1, \ldots, x_n, t^2)
\]

or:

\[
A(x_1, \ldots, x_n, 0, t) = (P_n(x_1, \ldots, x_n, t^2))^2.
\]

In the first case, at any point where all the \( x_i \) are positive \( P_n \) has a simple root and then \( \partial A/\partial t \neq 0 \). Then (by the implicit function theorem) \( A \) has a root for \( x \) in a neighborhood of \( (x_1, \ldots, x_n, 0) \), which is impossible since \( P_{n+1} \) does not have such a root when \( x_{n+1} \) is negative. In the second case \( P_{n+1} \) and \( A \) have the same degree in \( t \), and since \( A \) and \( B \) are monic in \( t \), we obtain finally \( A(x, t) = P_{n+1}(x, t^2), B(x, t) = 1 \).

\( \square \)
REMARKS. We can compute easily $P_1$, $P_2$, $P_3$.

\[ P_1(x, t^2) = t^2 - x, \]
\[ P_2(x, y, t^2) = (t^2 - (x + y))^2 - xy, \]
\[ P_3(x, y, z, t^2) = [(t^2 - (x + y + z))^2 - (xy + yz + zx)] - xy(x + y). \]

If we use the elementary symmetric polynomials $s_1 = x + y + z + u$, $s_2$, $s_3$, $s_4 = xyzu$, we can even write $P_4$:

\[ P_4(x, y, z, u, t^2) = \]
\[ = [(t^2 - s_1)^2 - s_2]^2 - xy(x + y) - u(x + y + z)(xy + yz + zx)] \]
\[ = s_4(x + y)(x + y + z)(xy + yz + zx). \]

The main step in Motzkin's work (cf. [M1], [M2]) was to find "a real polynomial $U_d'(x_1, \ldots, x_d, t^2)$ such that $x_1 \geq 0, \ldots, x_d \geq 0$ if and only if, for some $t$, $U_d'(x_1, \ldots, x_d, t^2) = 0."" His polynomials are reducible, nonhomogeneous, have some complex roots even when all the $x_i$ are positive, and they are very complicated:

\[ U_d'(x, y, t^2) = [t^4(x - y)^6 - 2t^2(x - y)^2(x + y) + 1][(t^2 - y)^2 + (x - y)^2], \]
\[ \text{deg}_t(U_d') = 4, \text{but deg}_t(U_3') = 104, \text{deg}_t(U_4') = 12, 496, \text{deg}_t(U_5') = 7, 997, 472 \text{!!!} \]

The induction formula defining our polynomials $P_k$ was found by a geometrical construction (cf. [P1], [P2]):

The algebraic set $\nu_3: P_3(x, y, z, 1) = 0$ is such that the positive cone on it

\[ C^+(\nu_3) = \{ (x, y, z) \in \mathbb{R}^3 | \exists t > 0, \left( \frac{x}{t}, \frac{y}{t}, \frac{z}{t} \right) \in \nu_3 \} \]
\[ = \{ (x, y, z) \in \mathbb{R}^3 | \exists t > 0, P_3(x, y, z, t^2) = 0 \} = (\mathbb{R}^+)^3. \]

\[ \nu_3 \] is projectively equivalent to an algebraic set $\nu_3'$ whose vertical projection is a triangle. And it is not difficult, using $P_2$, to define such a set (see figure).

The following corollary is due to the cooperation of C. Andradas.

COROLLARY 1. There exists a real irreducible polynomial

\[ P_{n,m}(x_1, \ldots, x_n, y_1, \ldots, y_m, t^2) \]

having a real root iff all the $x_i$ are nonnegative and all the $y_j$ are positive.
Proof. Let us define $P_{n,m}$ by the formula:

$$P_{n,m}(x_i, y_j, t^2) = \left(y_1 \cdots y_m\right)^{2n+m-1} P_{n+m}(x_i, y_1, \ldots, y_{m-1}, 1/y_1 \cdots y_m, t^2).$$

Since the polynomials $P_n$ are monic in each variable we see that $P_{n,m}$ cannot have a real root if $y_1 \cdots y_m = 0$. The conclusion is easy. \hfill \square

For example we have: $P_{0,2}(b, c, t^2) = (btc^2 - b^2c - 1)^2 - b^2c$.

**Proposition 1.** Let $S$ be a semi-algebraic subset of $\mathbb{R}^M$ given by:

$$S = \{x \in \mathbb{R}^M | b_1(x) \geq 0, \ldots, b_n(x) \geq 0, c_1(x) > 0, \ldots, c_m(x) > 0\}.$$

There exists a real irreducible polynomial $P(x, t)$ such that:

$$x \in S \iff \exists t \in \mathbb{R}, \quad P(x, t) = 0.$$

**Proof.** Let $P$ be a nontrivial irreducible factor of

$$P_{n,m}(b_i(x), c_j(x), t^2).$$

Since $P_{n,m}$ has either only real roots or none, we see that $P$ has a real root iff $P_{n,m}$ has one. \hfill \square
3. The case of obtuse corners. Let us define a function \( g(t) \) and a polynomial \( Q_n(x, t) \) by the formula:

\[
g(t) = \frac{x_1}{t-x_1} + \cdots + \frac{x_n}{t-x_n} - 1 = \frac{Q_n(x, t)}{(t-x_1) \cdots (t-x_n)}.
\]

By symmetry we may suppose \( x_1 \leq x_2 \leq \cdots \leq x_n \).

The function \( g(t) \) has a root on any of the intervals \([-\infty, x_1[, \ldots, ]x_i x_{i+1}[, \ldots, ]x_n, \infty[ \) whose closure does not contain zero. To obtain all the other roots of \( Q_n(x, t) \), it is enough to take \( x_k \) as a root or order \( p - 1 \) if \( x_k \) appears \( p \) times in \( (x_1, \ldots, x_n) \), and take 0 as a root of order \( q \) if \( q \) of the \( x_k \) are equal to zero.

We also see that \( g'(t) \) never vanishes on these intervals.

Consequently \( \psi_n(x) = \sup\{t \in \mathbb{R} | Q_n(x, t) = 0\} \) is well defined, positive (resp. nonnegative) iff one of the \( x_i \) is positive (resp. nonnegative). \( \psi_n(x) \) is continuous because \( Q_n(x, t) \) has only real roots.

If \( \psi_n(x) \) is equal to one of the \( x_k \), all the \( x_k \) are nonpositive, and either the maximum of the \( x_k \) is 0, or the maximum of the \( x_k \) is attained by two or more \( x_k \). In the first case, if only one of the \( x_k \) is equal to 0, a direct computation shows that \( Q_n'(x, 0) \neq 0 \). In the second case, if the maximum of the \( x_k \) is attained by exactly two of the \( x_k \), we see that \( Q_n'(x, x_k) \neq 0 \). Then, using the implicit function theorem, we have:

**Proposition 2.** There exists a function \( \psi_n(x) \), semi-algebraic and continuous on \( \mathbb{R}^n \), positive (resp. nonnegative) if and only if one of the \( x_i \) is positive (resp. nonnegative). Furthermore \( \psi_n(x) \) is analytic everywhere except on \( E_1 \cup E_2 \)

\[
E_1 = \{(x) \in \mathbb{R}^n | \forall i, x_i \leq 0, \exists i_1, i_2, x_{i_1} = x_{i_2} = 0\},
\]

\[
E_2 = \{(x) \in \mathbb{R}^n | \forall i, x_i \leq 0, \exists i_1, i_2, i_3, x_{i_1} = x_{i_2} = x_{i_3} = \max_i (x_i)\}.
\]

This allows us to give a very simple proof of the following separation theorem of Mostowski (compare [B-C-R]).

**Corollary (Mostowski).** Let \( F \) be a closed semi-algebraic subset of \( \mathbb{R}^n \). There exists a continuous semi-algebraic function \( \psi \) zero on \( F \), analytic and positive outside \( F \).

**Proof.** We know that any closed semi-algebraic set \( F \) can be written \( F = \bigcup_1^N F_i \) with \( F_i = \{x \in \mathbb{R}^n | A_i^1(x) \geq 0, \ldots, A_i^k(x) \geq 0\} \). Let
\( f_i(x) = \psi_k(-A_1(x), \ldots, -A_k(x)). \) \( f_i \) is nonpositive on \( F_i \), analytic and positive outside \( F_i \). The function \( \psi(x) = \prod_1^N (f_i(x) + |f_i(x)|) \) has the desired property.

We need the following remark:

**Lemma.** Let \( C_1, \ldots, C_N \) be pairwise relatively prime elements in a factorial ring of characteristic zero. There exist positive integers \( d_1, \ldots, d_N \) such that the elements \( C_1, \ldots, C_N \) and \( d_iC_i - d_jC_j \) are pairwise relatively prime.

**Proof.** By induction. Suppose that for \( k < N \) there exist positive integers \( d_1, \ldots, d_k \) such that \( C_1, \ldots, C_N \) and \( d_iC_i - d_jC_j, \ i < j \leq k \), are pairwise relatively prime. Let \( P \) be the finite set of factors appearing in one of these polynomials. Let \( j \leq k \) be a fixed integer, and consider the polynomials \( nC_{k+1} - d_jC_j \). These polynomials are pairwise relatively prime, and then, except for a finite number of values for \( n \), they do not possess any factor belonging to \( P \). Take a positive integer \( d_{k+1} \) such that, for all \( j \leq k \), \( d_{k+1}C_{k+1} - d_jC_j \) does not possess any factor belonging to \( P \). Any common factor of \( d_{k+1}C_{k+1} - d_jC_j \) and \( d_{k+1}C_{k+1} - d_iC_i \) must be in \( P \), which is impossible.

**Proposition 3.** If the real polynomials \( A_1(x), \ldots, A_h(x), B_1(x), \ldots, B_k(x) \) are pairwise relatively prime, there exists a real irreducible polynomial \( R(x, t) \) which has a nonnegative root iff one \( A_i(x) \) is nonnegative or one \( B_j(x) \) is positive. It has a positive root iff one \( A_i(x) \) or one \( B_j(x) \) is positive.

**Proof.** By the lemma, we may suppose that the \( A_i, B_j \), and their differences are pairwise relatively prime. Let

\[
\psi_A(x) = \psi_h(A_1(x), \ldots, A_h(x)), \\
\psi_B(x) = \psi_k(B_1(x), \ldots, B_k(x)).
\]

\( \psi_A(x) \) and \( \psi_B(x) \) are analytic on \( \mathbb{R}^n \) except on a set of codimension two at most. Their minimal polynomials \( R_A(x, \psi_A(x)) = 0 \) and \( R_B(x, \psi_B(x)) = 0 \) are therefore irreducible. These polynomials, being factors of \( Q_A \) and \( Q_B \) respectively (in \( \mathbb{R}(x)[t] \)), have only real roots.

Consider now the following function defined for \( u \geq 0 \) or \( v \neq 0 \):

\[
\overline{\psi}(u, v) = \frac{u + v + \sqrt{u^2 + v^2}}{(u + \sqrt{u^2 + v^2})^2} (u^2 + v^2), \\
\overline{\psi}(0, 0) = 0.
\]
\( \overline{\psi} \) satisfies a real quadratic polynomial \( K(u, v, \overline{\psi}(u, v)) = 0 \) which has a nonnegative root if and only if \( u \geq 0 \) or \( v > 0 \); (if \( u \geq 0 \) or \( v > 0 \), \( \overline{\psi}(u, v) \) is a nonnegative root of this polynomial).

Let \( R_1(x, f) \) be the polynomial obtained by eliminating \( u \) and \( v \) of the following system (I):

\[
\begin{align*}
R_A(x, u) &= 0, \\
R_B(x, v) &= 0, \\
K(u, v, f) &= 0.
\end{align*}
\]

We see that \( R_1(x, \overline{\psi}(\psi_A(x), \psi_B(x))) = 0 \). Since \( \overline{\psi}(\psi_A(x), \psi_B(x)) \) is meromorphic in a dense connected open subset of \( \mathbb{R}^n \), there is an irreducible factor \( R(x, f) \) of \( R_1(x, f) \) such that \( R(x, \overline{\psi}(\psi_A(x), \psi_B(x))) = 0 \).

If \( R \) has a nonnegative root, the system (I) has a solution \( u, v, f \) with \( f_1 \) nonnegative. \( R_A \) and \( R_B \) having only real roots, \( u \) and \( v \) are real numbers. Finally we see that \( u \geq 0 \) or \( v > 0 \) which shows that \( \psi_A(x) \geq 0 \) or \( \psi_B(x) > 0 \). Conversely, if \( \psi_A(x) \geq 0 \) or \( \psi_B(x) > 0 \), \( \overline{\psi}(\psi_A(x), \psi_B(x)) \) is a nonnegative root of \( R(x, f) = 0 \).

We may also remark that, since \( R_A \) and \( R_B \) have only real roots, \( R_1 \) and \( R \) have the same property.

In the proof of our principal result, we shall only need the easier part of Proposition 3, when there is no \( B_j \). In this case the polynomial \( R(x, t) \) is monic in \( t \).

4. The principal result.

**Theorem.** If \( S \) is a locally closed semi-algebraic subset of \( \mathbb{R}^n \), there exists an irreducible real polynomial \( R(x, t) \) such that:

\[
x \in S \iff \exists t \in \mathbb{R}, \quad R(x, t) = 0.
\]

Furthermore, if \( S \) is closed, we can suppose \( R \) monic in \( t \).

**Proof.** Let \( S = F \cap U \), where \( F \) is closed and \( U \) open. We know that we can write \( F = \bigcap_{i=1}^N S_i \) with

\[
S_i = \{ x \in \mathbb{R}^n | A_1^i(x) \geq 0 \text{ or } \ldots \text{ or } A_n^i(x) \geq 0 \}
\]

where the \( A_j^i(x) \) are irreducible polynomials. (Cf. [A-G1] & [B-C-R] p. 26.). Similarly, we can write \( U = \bigcap_{i=1}^N S_i \) with:

\[
S_i = \{ x \in \mathbb{R}^n | A_1^i(x) > 0 \text{ or } \ldots \text{ or } A_n^i(x) > 0 \}.
\]
For each $l$ let $R_l(x, u_l)$ be the polynomial defined in Proposition 3. $R_l$ is irreducible, monic in $u_l$, and has only real roots. When $l \leq N_1$, $R_l$ has a nonnegative root iff $x \in S_l$. When $l > N_1$, $R_l$ has a positive root iff $x \in S_l$. The function $\psi_{S_l}(x)$ of Proposition 3 is noted $f_l$.

Let $\gamma$ be a root of $P_{N_1, N-N_1}(f_1, \ldots, f_N, \Gamma^2) = 0$ in an extension field of $\mathbb{R}(f_1, \ldots, f_N)$. Let $Q_1(x, \Gamma)$ be the polynomial obtained by eliminating the $u_i$ in the system (II):

$$
\begin{align*}
R_1(x, u_1) &= 0, \\
R_2(x, u_2) &= 0, \\
&\vdots \\
P_{N_1, N-N_1}(u_1, u_2, \ldots, u_N, \Gamma^2) &= 0.
\end{align*}
$$

We have $Q_1(x, \gamma) = 0$. Let $R(x, \Gamma)$ be an irreducible factor of $Q_1(x, \Gamma)$ such that $R(x, \gamma) = 0$.

Since $P_{N_1, N-N_1}$ is not monic, we must be careful with elimination theory. Let us introduce a new variable $u_{N+1}$, and consider the following system of homogeneous polynomials in the variables $u_1, \ldots, u_{N+1}$:

$$
\begin{align*}
R_1^b(x, u_1, u_{N+1}) &= 0, \\
R_2^b(x, u_2, u_{N+1}) &= 0, \\
&\vdots \\
P_{N_1, N-N_1}^b(u_1, \ldots, u_N, \Gamma^2, u_{N+1}) &= 0.
\end{align*}
$$

Let $Q_1(x, \Gamma)u_{N+1}^{M}$ be the polynomial obtained by successive elimination of the variables $u_N, u_{N-1}, \ldots, u_1$ in the system (II'). As it is well known for systems of homogeneous equations, this system has a nontrivial solution $(u_1, \ldots, u_N, u_{N+1})$ iff $Q_1(x, \Gamma) = 0$ (cf. [W]).

Since the polynomials $R_i(x, u_i)$ are monic in $u_i$, we see that any nontrivial root of (II') is such that $u_{N+1} \neq 0$. Therefore, the system (II) has a solution iff $Q_1(x, \Gamma) = 0$.

If $R(x, \Gamma)$ has a real root, the system (II) has a solution $u_1, \ldots, u_N, \Gamma$. Since the $R_i$ have only real roots, the $u_i$ are real and $P_{N_1, N-N_1}(u_1, \ldots, u_N, \Gamma^2)$ has a real root. Therefore, if $l \leq N_1$, $u_l$ is a nonnegative root of $R^l$; if $l > N_1$, $u_l$ is a positive root of $R^l$, which shows that $x \in S = \bigcap_1^{N_1} S_l$. Conversely, suppose $x \in S$. Since the two polynomials $R(x, \Gamma)$ and $P_{N_1, N-N_1}(f_1, \ldots, f_N, \Gamma^2)$ have a common root in an extension field of $\mathbb{R}(f_1, \ldots, f_N)$, their resultant relative to $\Gamma$ vanishes identically. $R(x, \Gamma)$ and $P_{N_1, N-N_1}(f_1(x), \ldots, f_N(x), \Gamma^2)$
have a common root. Since $x \in S$, $P_{N_1,N-N_1}(f_1(x), \ldots, f_N(x), \Gamma^2)$ has only real roots, therefore $R(x, \Gamma)$ has a real root. □

REMARKS. If $S = \bigcap_{l=1}^{N} S_l$, where each $S_l$ is a closed semi-algebraic set written with $m_l$ inequalities, the degree of our polynomial is $2^N m_1 \cdots m_N$. This degree is smaller than the one obtained in [P2] where the polynomials were solvable by square roots. It would be of interest to give a simple proof that this degree is optimal “in general”. (L. Bröcker has a proof using fan theory, valid for basic closed sets.) As in [P1], [P2] using the changing sign criterion, we obtain:

**COROLLARY.** Let $S$ be a locally closed semi-algebraic subset of $\mathbb{R}^n$ having some interior points. Then $S$ is the projection of an irreducible algebraic subset of $\mathbb{R}^{n+1}$.

This corollary is the generalisation to non closed sets of a result in [P1]. This earlier result was itself an improvement of the first paper of Andradas and Gamboa on the subject.

REFERENCES


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