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# ON THE ELIMINATION OF ALGEBRAIC INEQUALITIES

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**Let $S$ be a locally closed semi-algebraic subset of $\mathbb{R}^n$. We find an irreducible equation of an algebraic set of $\mathbb{R}^{n+1}$ projecting upon $S$. Our methods are simple and explicit.**

**1. Introduction**. The inequality $x \geq 0$ is often replaced by the proposition "$x$ has a square root" or "$\exists t \in \mathbb{R}$, $t^2 - x = 0$". This is the most immediate example of an elimination of one inequality. The general problem is to find an algebraic set projecting upon a given semi-algebraic set: it is a converse of the problem of the elimination of quantifiers.

Motzkin proved that every semi-algebraic subset of $\mathbb{R}^n$ is the projection of an algebraic set in $\mathbb{R}^{n+1}$. However this algebraic set is very complicated and generally reducible.

Andradas and Gamboa proved that any closed semi-algebraic subset of $\mathbb{R}^n$ whose Zariski-closure is irreducible is the projection of an irreducible algebraic set in $\mathbb{R}^{n+k}$.

In this paper we shall first improve Motzkin's result by finding equations generally of minimal degree. Then we shall give a few results concerning irreducibility. One of the first examples of such a construction is due to Rohn and has been studied by Hilbert and Utkin: If $4C_4C_2 = \varepsilon^2$ is a plane curve of degree six (where $\deg(C_2) = 2$, $\deg(C_4) = 4$, $\varepsilon \in \mathbb{R}$), then it is the apparent contour of the quartic surface $C_2z^2 - \varepsilon z + C_4 = 0$.

**2. The case of basic closed subsets**. Let $\mathbb{R}^+ = \{x \in \mathbb{R} | x \geq 0\}$ be the set of nonnegative numbers. Let $\mathbf{x} = (x_1, \dots, x_N)$ be a "parameter" and $t$ an "indeterminate", so that we can speak of the roots of a polynomial $P(\mathbf{x}, t)$. In the same way, unless otherwise specified, the degree of $P(\mathbf{x}, t)$ will be its degree in $t$.

Let us define the polynomials $a_i(\mathbf{x})$ as follows:

$$a_k(x_1, \dots, x_{k+1}) = x_{k+1}(x_1 + x_2 + \cdots + x_k).$$

It is easy to see that $a_1(\mathbf{x}) \geq 0, \dots, a_n(\mathbf{x}) \geq 0$ if and only if all the $x_i$ are nonnegative or all the $x_i$ are nonpositive $(i = 1, \dots, n+1)$.

THEOREM 1. *Let* $P_1(x_1, u) = u - x_1$.

$$P_{n+1}(x_1, \dots, x_{n+1}, u) \\ = P_n(a_1(\mathbf{x}), \dots, a_n(\mathbf{x}), (u - (x_1 + x_2 + \cdots + x_{n+1}))^2).$$

*Then the following properties are true*:

(i) $P_n$ *is homogeneous of degree* $2^{n-1}$.

(ii) *If all the* $x_i$ *are nonnegative*

$$P_n(x_1, \dots, x_n, u) = 0 \Rightarrow 0 \leq u \leq 2\sum_1^n x_i.$$

(iii) *If all the* $x_i$ *are nonnegative,* $P_n(x_1, \dots, x_n, t^2)$ *has only real roots.*

(iv) *If* $P_n(x_1, \dots, x_n, t^2)$ *has a real root, then all the* $x_i$ *are nonnegative.*

(v)

$$P_n(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n, t) \\ = [P_{n-1}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n, t)]^2$$

(vi) $P_n(x_1, \dots, x_n, t^2)$ *is irreducible and monic in each letter.*

*Proof*. First, we prove (i), (ii), (iii), and (iv) by simultaneous induction: let us suppose (i), (ii), (iii) and (iv) verified for $n$; we shall prove them for $n+1$.

(i) Easy since the $a_k$ are homogeneous of degree 2.

(ii) If $u$ is a root of $P_{n+1}(x_1, \dots, x_{n+1}, u) = 0$, then

$$(u - (x_1 + \cdots + x_{n+1}))^2$$

is a root of $P_n(a_1(\mathbf{x}), \dots, a_n(\mathbf{x}), v) = 0$ by induction

$$(u - (x_1 + \cdots + x_{n+1}))^2 \leq 2(a_1(\mathbf{x}) + \cdots + a_n(\mathbf{x})) \leq (x_1 + \cdots + x_{n+1})^2$$

whence $0 \leq u \leq 2(x_1 + \cdots + x_{n+1})$, which shows (ii) and (iii).

(iv) If $P_{n+1}(x_1, \dots, x_{n+1}, t^2) = 0$ has a real root, then

$$P_n(a_1(\mathbf{x}), \dots, a_n(\mathbf{x}), (t^2 - (x_1 + \cdots + x_{n+1}))^2)$$

has a real root and by induction all the $a_i(\mathbf{x})$ are nonnegative. Therefore, if all the $x_i$ are nonpositive, $P_n(a_1(\mathbf{x}), \dots, a_n(\mathbf{x}), v)$ has a root which is greater than $(x_1 + \cdots + x_{n+1})^2 \geq 2(a_1(\mathbf{x}) + \cdots + a_n(\mathbf{x}))$. By induction this is possible only if

$$(x_1 + \cdots + x_{n+1})^2 = 2(a_1(\mathbf{x}) + \cdots + a_n(\mathbf{x})),$$

i.e., when all the $x_i$ are equal to zero.

(v) By induction: suppose the formula true for $n$, let us prove it for $n+1$. Let us study the case $j \geq 2$ (the case $j=1$ is similar). Let

$$\begin{aligned} x &= (x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_{n+1}), \\ \hat{x} &= (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1}). \end{aligned}$$

We have:

$$\begin{cases} a_i(x) = a_i(\hat{x}) & \text{if } i < j-1, \\ a_{j-1}(x) = 0, \\ a_k(x) = a_{k-1}(\hat{x}) & \text{if } k \geq j. \end{cases}$$

Then,

$$\begin{aligned} P_{n+1}(x, t) &= P_n(a_1(x), \dots, a_n(x), (t-(x_1+\dots+x_{n+1}))^2) \\ &= P_n(a_1(\hat{x}), \dots, a_{j-2}(\hat{x}), 0, a_{j-1}(\hat{x}), \dots, a_{n-1}(\hat{x}), \\ &\qquad\qquad (t-(x_1+\dots+x_{n+1}))^2) \\ &= [P_{n-1}(a_1(\hat{x}), \dots, a_{n-1}(\hat{x}), (t-(x_1+\dots+x_{n+1}))^2]^2 \\ &= [P_n(\hat{x}, t)]^2. \end{aligned}$$

(vi) By induction. Suppose $P_n(x, t^2)$ irreducible. Let

$$P_{n+1}(x_1, \dots, x_{n+1}, t^2) = A(x, t) \cdot B(x, t),$$

$A$ and $B$ monic in $t$. Let us substitute 0 for $x_{n+1}$ in this factorization; using (v) we get:

$$(P_n(x_1, \dots, x_n, t^2))^2 = A(x_1, \dots, x_n, 0, t) \cdot B(x_1, \dots, x_n, 0, t).$$

Since $P_n(x, t^2)$ is irreducible, and $A$ and $B$ are monic in $t$, we get either:

$$A(x_1, \dots, x_n, 0, t) = B(x_1, \dots, x_n, 0, t) = P_n(x_1, \dots, x_n, t^2)$$

or:

$$A(x_1, \dots, x_n, 0, t) = (P_n(x_1, \dots, x_n, t^2))^2.$$

In the first case, at any point where all the $x_i$ are positive $P_n$ has a simple root and then $\partial A/\partial t \neq 0$. Then (by the implicit function theorem) $A$ has a root for $x$ in a neighborhood of $(x_1, \dots, x_n, 0)$, which is impossible since $P_{n+1}$ does not have such a root when $x_{n+1}$ is negative. In the second case $P_{n+1}$ and $A$ have the same degree in $t$, and since $A$ and $B$ are monic in $t$, we obtain finally $A(x, t) = P_{n+1}(x, t^2)$, $B(x, t) = 1$. $\square$

REMARKS. We can compute easily $P_1$, $P_2$, $P_3$.

$$P_1(x, t^2) = t^2 - x,$$
$$P_2(x, y, t^2) = (t^2 - (x+y))^2 - xy,$$
$$P_3(x, y, z, t^2) = [(t^2 - (x+y+z))^2 - (xy+yz+zx)]^2 - xyz(x+y).$$

If we use the elementary symmetric polynomials $s_1 = x+y+z+u$, $s_2$, $s_3$, $s_4 = xyzu$, we can even write $P_4$:

$$\begin{aligned}P_4(x, y, z, u, t^2) &= [((t^2 - s_1)^2 - s_2)^2 - xyz(x+y) - u(x+y+z)(xy+yz+zx)]^2 \\ &\quad - s_4(x+y)(x+y+z)(xy+yz+zx).\end{aligned}$$

The main step in Motzkin's work (cf. **[M1]**, **[M2]**) was to find "a real polynomial $U'_d(x_1, \dots, x_d, t^2)$ such that $x_1 \geq 0, \dots, x_d \geq 0$ if and only if, for some $t$, $U'_d(x_1, \dots, x_d, t^2) = 0$." His polynomials are reducible, nonhomogeneous, have some complex roots even when all the $x_i$ are positive, and they are very complicated:

$$U'_2(x, y, t^2) = [t^4(x-y)^6 - 2t^2(x-y)^2(x+y) + 1][(t^2-y)^2 + (x-y)^2],$$

$\deg_t(U'_2) = 4$, but $\deg_t(U'_3) = 104$, $\deg_t(U'_4) = 12, 496$, $\deg_t(U'_5) = 7, 997, 472$!!!

The induction formula defining our polynomials $P_k$ was found by a geometrical construction (cf. **[P1]**, **[P2]**):

The algebraic set $\nu_3$: $P_3(x, y, z, 1) = 0$ is such that the positive cone on it

$$\begin{aligned}C^+(\nu_3) &= \left\{(x, y, z) \in \mathbb{R}^3 | \exists t > 0, \left(\frac{x}{t}, \frac{y}{t}, \frac{z}{t}\right) \in \nu_3\right\} \\ &= \{(x, y, z) \in \mathbb{R}^3 | \exists t > 0, P_3(x, y, z, t^2) = 0\} = (\mathbb{R}^+)^3.\end{aligned}$$

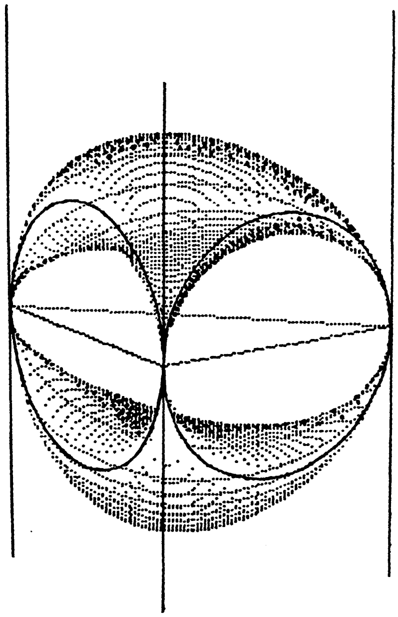
$\nu_3$ is projectively equivalent to an algebraic set $\nu'_3$ whose vertical projection is a triangle. And it is not difficult, using $P_2$, to define such a set (see figure).

The following corollary is due to the cooperation of C. Andradas.

COROLLARY 1. *There exists a real irreducible polynomial*

$$P_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m, t^2)$$

*having a real root iff all the $x_i$ are nonnegative and all the $y_j$ are positive.*



FIGURE

Surface $z^4 - 2(B_1 + B_2)z^2 + B_1^2 + B_2^2 = 0$ with $B_1 = x - x^2$, $B_2 = (1-x)y - y^2$

*Proof.* Let us define $P_{n,m}$ by the formula:

$$P_{n,m}(x_i, y_j, t^2) = (y_1 \dots y_m)^{2^{m+n-1}} P_{n+m}(x_i, y_1, \dots, y_{m-1}, 1/y_1 \cdots y_m, t^2).$$

Since the polynomials $P_n$ are monic in each variable we see that $P_{n,m}$ cannot have a real root if $y_1 \cdots y_m = 0$. The conclusion is easy. □

For example we have: $P_{0,2}(b, c, t^2) = (bct^2 - b^2c - 1)^2 - b^2c$.

PROPOSITION 1. *Let $S$ be a semi-algebraic subset of $\mathbb{R}^M$ given by:*

$$S = \{\mathbf{x} \in \mathbb{R}^M | b_1(\mathbf{x}) \geq 0, \dots, b_n(x) \geq 0, c_1(\mathbf{x}) > 0, \dots, c_m(\mathbf{x}) > 0\}.$$

*There exists a real irreducible polynomial $P(\mathbf{x}, t)$ such that:*

$$x \in S \Leftrightarrow \exists t \in \mathbb{R}, \qquad P(\mathbf{x}, t) = 0.$$

*Proof.* Let $P$ be a nontrivial irreducible factor of

$$P_{n,m}(b_i(\mathbf{x}), c_j(\mathbf{x}), t^2).$$

Since $P_{n,m}$ has either only real roots or none, we see that $P$ has a real root iff $P_{n,m}$ has one. □

**3. The case of obtuse corners.** Let us define a function $g(t)$ and a polynomial $Q_n(\mathbf{x}, t)$ by the formula:

$$g(t) = \frac{x_1}{t-x_1} + \cdots + \frac{x_n}{t-x_n} - 1 = \frac{Q_n(\mathbf{x}, t)}{(t-x_1)\cdots(t-x_n)}.$$

By symmetry we may suppose $x_1 \leq x_2 \leq \cdots \leq x_n$.

The function $g(t)$ has a root on any of the intervals $]-\infty, x_1[, \dots, ]x_i x_{i+1}[, \dots, ]x_n, \infty[$ whose closure does not contain zero. To obtain all the other roots of $Q_n(\mathbf{x}, t)$, it is enough to take $x_k$ as a root or order $p-1$ if $x_k$ appears $p$ times in $(x_1, \dots, x_n)$, and take 0 as a root of order $q$ if $q$ of the $x_k$ are equal to zero.

We also see that $g'(t)$ never vanishes on these intervals.

Consequently $\psi_n(\mathbf{x}) = \sup\{t \in \mathbb{R} | Q_n(\mathbf{x}, t) = 0\}$ is well defined, positive (resp. nonnegative) iff one of the $x_i$ is positive (resp. nonnegative). $\psi_n(\mathbf{x})$ is continuous because $Q_n(\mathbf{x}, t)$ has only real roots.

If $\psi_n(\mathbf{x})$ is equal to one of the $x_k$, all the $x_k$ are nonpositive, and either the maximum of the $x_k$ is 0, or the maximum of the $x_k$ is attained by two or more $x_k$. In the first case, if only one of the $x_k$ is equal to 0, a direct computation shows that $Q_n'(\mathbf{x}, 0) \neq 0$. In the second case, if the maximum of the $x_k$ is attained by exactly two of the $x_k$, we see that $Q_n'(\mathbf{x}, x_k) \neq 0$. Then, using the implicit function theorem, we have:

PROPOSITION 2. *There exists a function $\psi_n(\mathbf{x})$, semi-algebraic and continuous on $\mathbb{R}^n$, positive (resp. nonnegative) if and only if one of the $x_i$ is positive (resp. nonnegative). Furthermore $\psi_n(\mathbf{x})$ is analytic everywhere except on $E_1 \cup E_2$*

$$E_1 = \{(\mathbf{x}) \in \mathbb{R}^n | \forall i, x_i \leq 0, \exists i_1, i_2, x_{i_1} = x_{i_2} = 0\},$$
$$E_2 = \left\{(\mathbf{x}) \in \mathbb{R}^n | \forall i, x_i \leq 0, \exists i_1, i_2, i_3, x_{i_1} = x_{i_2} = x_{i_3} = \max_i(x_i)\right\}.$$

This allows us to give a very simple proof of the following separation theorem of Mostowski (compare **[B-C-R]**).

COROLLARY (*Mostowski*). *Let $F$ be a closed semi-algebraic subset of $\mathbb{R}^n$. There exists a continuous semi-algebraic function $\psi$ zero on $F$, analytic and positive outside $F$.*

*Proof*. We know that any closed semi-algebraic set $F$ can be written $F = \bigcup_1^N F_i$ with $F_i = \{\mathbf{x} \in \mathbb{R}^n | A_1^i(\mathbf{x}) \geq 0, \dots, A_{k_i}^i(\mathbf{x}) \geq 0\}$. Let

$f_i(\mathbf{x}) = \psi_{k_i}(-A_1^i(\mathbf{x}), \dots, -A_{k_i}^i(\mathbf{x}))$. $f_i$ is nonpositive on $F_i$, analytic and positive outside $F_i$. The function $\psi(\mathbf{x}) = \prod_1^N (f_i(\mathbf{x}) + |f_i(\mathbf{x})|)$ has the desired property. □

We need the following remark:

LEMMA. *Let $C_1, \dots, C_N$ be pairwise relatively prime elements in a factorial ring of characteristic zero. There exist positive integers $d_1, \dots, d_N$ such that the elements $C_1, \dots, C_N$ and $d_iC_i - d_jC_j$ are pairwise relatively prime.*

*Proof*. By induction. Suppose that for $k < N$ there exist positive integers $d_1, \dots, d_k$ such that $C_1, \dots, C_N$ and $d_iC_i - d_jC_j$, $i < j \leq k$, are pairwise relatively prime. Let $P$ be the finite set of factors appearing in one of these polynomials. Let $j \leq k$ be a fixed integer, and consider the polynomials $nC_{k+1} - d_jC_j$. These polynomials are pairwise relatively prime, and then, except for a finite number of values for $n$, they do not possess any factor belonging to $P$. Take a positive integer $d_{k+1}$ such that, for all $j \leq k$, $d_{k+1}C_{k+1} - d_jC_j$ does not possess any factor belonging to $P$. Any common factor of $d_{k+1}C_{k+1} - d_jC_j$ and $d_{k+1}C_{k+1} - d_iC_i$ must be in $P$, which is impossible. □

PROPOSITION 3. *If the real polynomials $A_1(\mathbf{x}), \dots, A_h(\mathbf{x})$, $B_1(\mathbf{x}), \dots, B_k(\mathbf{x})$ are pairwise relatively prime, there exists a real irreducible polynomial $R(\mathbf{x}, t)$ which has a nonnegative root iff one $A_i(\mathbf{x})$ is nonnegative or one $B_j(\mathbf{x})$ is positive. It has a positive root iff one $A_i(\mathbf{x})$ or one $B_j(\mathbf{x})$ is positive.*

*Proof*. By the lemma, we may suppose that the $A_i$, $B_j$, and their differences are pairwise relatively prime. Let

$$\psi_A(\mathbf{x}) = \psi_h(A_1(\mathbf{x}), \dots, A_h(\mathbf{x})),$$
$$\psi_B(\mathbf{x}) = \psi_k(B_1(\mathbf{x}), \dots, B_k(\mathbf{x})).$$

$\psi_A(\mathbf{x})$ and $\psi_B(\mathbf{x})$ are analytic on $\mathbb{R}^n$ except on a set of codimension two at most. Their minimal polynomials $R_A(\mathbf{x}, \psi_A(\mathbf{x})) = 0$ and $R_B(\mathbf{x}, \psi_B(\mathbf{x})) = 0$ are therefore irreducible. These polynomials, being factors of $Q_A$ and $Q_B$ respectively (in $\mathbb{R}(x)[t]$), have only real roots.

Consider now the following function defined for $u \geq 0$ or $v \neq 0$:

$$\overline{\psi}(u, v) = \frac{u + v + \sqrt{u^2 + v^2}}{(u + \sqrt{u^2 + v^2})^2}(u^2 + v^2),$$
$$\overline{\psi}(0, 0) = 0.$$

$\overline{\psi}$ satisfies a real quadratic polynomial $K(u, v, \overline{\psi}(u, v)) = 0$ which has a nonnegative root if and only if $u \geq 0$ or $v > 0$; (if $u \geq 0$ or $v > 0$, $\overline{\psi}(u, v)$ is a nonnegative root of this polynomial).

Let $R_1(\mathbf{x}, f)$ be the polynomial obtained by eliminating $u$ and $v$ of the following system (I):

$$\text{(I)} \qquad \begin{cases} R_A(\mathbf{x}, u) = 0, \\ R_B(\mathbf{x}, v) = 0, \\ K(u, v, f) = 0. \end{cases}$$

We see that $R_1(\mathbf{x}, \overline{\psi}(\psi_A(\mathbf{x}), \psi_B(\mathbf{x}))) = 0$. Since $\overline{\psi}(\psi_A(\mathbf{x}), \psi_B(\mathbf{x}))$ is meromorphic in a dense connected open subset of $\mathbb{R}^n$, there is an irreducible factor $R(\mathbf{x}, f)$ of $R_1(\mathbf{x}, f)$ such that $R(\mathbf{x}, \overline{\psi}(\psi_A(\mathbf{x}), \psi_B(\mathbf{x}))) = 0$.

If $R$ has a nonnegative root, the system (I) has a solution $u, v, f_1$ with $f_1$ nonnegative. $R_A$ and $R_B$ having only real roots, $u$ and $v$ are real numbers. Finally we see that $u \geq 0$ or $v > 0$ which shows that $\psi_A(\mathbf{x}) \geq 0$ or $\psi_B(\mathbf{x}) > 0$. Conversely, if $\psi_A(\mathbf{x}) \geq 0$ or $\psi_B(\mathbf{x}) > 0$, $\overline{\psi}(\psi_A(\mathbf{x}), \psi_B(\mathbf{x}))$ is a nonnegative root of $R(\mathbf{x}, f) = 0$.□

We may also remark that, since $R_A$ and $R_B$ have only real roots, $R_1$ and $R$ have the same property.

In the proof of our principal result, we shall only need the easier part of Proposition 3, when there is no $B_j$. In this case the polynomial $R(\mathbf{x}, t)$ is monic in $t$.

## 4. The principal result.

THEOREM. *If $S$ is a locally closed semi-algebraic subset of $\mathbb{R}^n$, there exists an irreducible real polynomial $R(\mathbf{x}, t)$ such that*:

$$\mathbf{x} \in S \Leftrightarrow \exists t \in \mathbb{R}, \qquad R(\mathbf{x}, t) = 0.$$

*Furthermore, if $S$ is closed, we can suppose $R$ monic in $t$.*

*Proof*. Let $S = F \cap U$, where $F$ is closed and $U$ open. We know that we can write $F = \bigcap_1^{N_1} S_l$ with

$$S_l = \{\mathbf{x} \in \mathbb{R}^n | A_1^l(\mathbf{x}) \geq 0 \text{ or } \cdots \text{ or } A_{n_l}^l(\mathbf{x}) \geq 0\}$$

where the $A_i^l(\mathbf{x})$ are irreducible polynomials. (Cf. **[A-G1]** & **[B-C-R]** p. 26.). Similarly, we can write $U = \bigcap_{N_1+1}^{N} S_l$ with:

$$S_l = \{\mathbf{x} \in \mathbb{R}^n | A_1^l(\mathbf{x}) > 0 \text{ or } \ldots \text{ or } A_{n_l}^l(\mathbf{x}) > 0\}.$$

For each $l$ let $R_l(\mathbf{x}, u_1)$ be the polynomial defined in Proposition 3. $R_l$ is irreducible, monic in $u_l$, and has only real roots. When $l \leq N_1$, $R_l$ has a nonnegative root iff $\mathbf{x} \in S_l$. When $l > N_1$, $R_l$ has a positive root iff $\mathbf{x} \in S_l$. The function $\psi_{S_l}(\mathbf{x})$ of Proposition 3 is noted $f_l$. Let $\gamma$ be a root of $P_{N_1, N-N_1}(f_1, \ldots, f_N, \Gamma^2) = 0$ in an extension field of $\mathbb{R}(f_1, \ldots, f_N)$. Let $Q_1(\mathbf{x}, \Gamma)$ be the polynomial obtained by eliminating the $u_i$ in the system (II):

$$\text{(II)} \qquad \begin{cases} R_1(\mathbf{x}, u_1) = 0, \\ R_2(\mathbf{x}, u_2) = 0, \\ \quad \vdots \\ P_{N_1, N-N_1}(u_1, u_2, \ldots, u_N, \Gamma^2) = 0. \end{cases}$$

We have $Q_1(\mathbf{x}, \gamma) = 0$. Let $R(\mathbf{x}, \Gamma)$ be an irreducible factor of $Q_1(\mathbf{x}, \Gamma)$ such that $R(\mathbf{x}, \gamma) = 0$.

Since $P_{N_1, N-N_1}$ is not monic, we must be careful with elimination theory. Let us introduce a new variable $u_{N+1}$, and consider the following system of homogeneous polynomials in the variables $u_1, \ldots, u_{N+1}$:

$$\text{(II}'\text{)} \qquad \begin{cases} R_1^h(\mathbf{x}, u_1, u_{N+1}) = 0, \\ R_2^h(\mathbf{x}, u_2, u_{N+1}) = 0, \\ \vdots \\ P^h_{N_1, N-N_1}(u_1, \ldots, u_N, \Gamma^2, u_{N+1}). \end{cases}$$

Let $Q_1(\mathbf{x}, \Gamma)u_{N+1}^M$ be the polynomial obtained by successive elimination of the variables $u_N, u_{N-1}, \ldots, u_1$ in the system (II$'$). As it is well known for systems of homogeneous equations, this system has a nontrivial solution $(u_1, \ldots, u_N, u_{N+1})$ iff $Q_1(\mathbf{x}, \Gamma) = 0$ (cf. **[W]**).

Since the polynomials $R_l(\mathbf{x}, u_l)$ are monic in $u_l$, we see that any nontrivial root of (II$'$) is such that $u_{N+1} \neq 0$. Therefore, the system (II) has a solution iff $Q_1(\mathbf{x}, \Gamma) = 0$.

If $R(\mathbf{x}, \Gamma)$ has a real root, the system (II) has a solution $u_1, \ldots, u_N, \Gamma$. Since the $R_i$ have only real roots, the $u_i$ are real and $P_{N_1, N-N_1}(u_1, \ldots, u_N, \Gamma^2)$ has a real root. Therefore, if $l \leq N_1$, $u_l$ is a nonnegative root of $R_l$; if $l > N_1$, $u_l$ is a positive root of $R_l$, which shows that $\mathbf{x} \in S = \bigcap_1^{N_1} S_l$. Conversely, suppose $\mathbf{x} \in S$. Since the two polynomials $R(\mathbf{x}, \Gamma)$ and $P_{N_1, N-N_1}(f_1, \ldots, f_N, \Gamma^2)$ have a common root in an extension field of $\mathbb{R}(f_1, \ldots, f_N)$, their resultant relative to $\Gamma$ vanishes identically. $R(\mathbf{x}, \Gamma)$ and $P_{N_1, N-N_1}(f_1(\mathbf{x}), \ldots, f_N(\mathbf{x}), \Gamma^2)$

have a common root. Since $\mathbf{x} \in S$, $P_{N_1, N-N_1}(f_1(\mathbf{x}), \ldots, f_N(\mathbf{x}), \Gamma^2)$ has only real roots, therefore $R(\mathbf{x}, \Gamma)$ has a real root. $\square$

REMARKS. If $S = \bigcap_1^N S_l$, where each $S_l$ is a closed semi-algebraic set written with $m_l$ inequalities, the degree of our polynomial is $2^N m_1 \cdots m_N$. This degree is smaller than the one obtained in **[P2]** where the polynomials were solvable by square roots. It would be of interest to give a simple proof that this degree is optimal "in general". (L. Bröcker has a proof using fan theory, valid for basic closed sets.) As in **[P1]**, **[P2]** using the changing sign criterion, we obtain:

COROLLARY. *Let $S$ be a locally closed semi-algebraic subset of $\mathbb{R}^n$ having some interior points. Then $S$ is the projection of an irreducible algebraic subset of $\mathbb{R}^{n+1}$.*

This corollary is the generalisation to non closed sets of a result in **[P1]**. This earlier result was itself an improvement of the first paper of Andradas and Gamboa on the subject.

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