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**THE C^* -ALGEBRAS GENERATED BY PAIRS OF SEMIGROUPS
OF ISOMETRIES SATISFYING CERTAIN COMMUTATION
RELATIONS**

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Arising in the computation of the Arveson-Powers index for *-endomorphisms of $\mathfrak{B}(\mathfrak{H})$ is the notion of a pair of one-parameter semigroups of isometries $\mathcal{U} = \{U_t: t \in \Gamma^+\}$ and $\mathcal{S} = \{S_t: t \in \Gamma^+\}$ satisfying the commutation relations $S_t^* U_t = e^{-\lambda t} I$, for Γ the set of real numbers. If Γ is any subgroup of \mathbb{R} we show that the C^* -algebra \mathfrak{A}_Γ generated by \mathcal{U} and \mathcal{S} is canonically unique. \mathfrak{A}_Γ is simple if and only if Γ is dense in \mathbb{R} .

I. Introduction. According to the von Neumann-Wold decomposition for an isometry V acting on a Hilbert space \mathfrak{H} , \mathfrak{H} may be decomposed into an orthogonal direct sum of reducing Hilbert subspaces $\mathfrak{H}_1, \mathfrak{H}_2$ for V , where $V|_{\mathfrak{H}_1}$ is a unitary operator and $V|_{\mathfrak{H}_2}$ is a pure isometry. In [6], L. A. Coburn characterized the C^* -algebra $C^*(V)$ generated by an isometry. If V is completely unitary then as is well known, $C^*(V)$ is isometrically *-isomorphic to $C(\sigma(V))$, the algebra of complex-valued continuous functions on the spectrum of V . If V has a non-trivial pure isometric part, $C^*(V)$ contains a closed two-sided ideal which is isomorphic to the compact operators \mathcal{K} . The quotient algebra $C^*(V)/\mathcal{K}$ is isomorphic to the algebra of continuous functions on the circle, [6].

Generalizations of this result (see [4], [7]–[10], [12]) made by Coburn and other authors have taken various forms. For example, the study of C^* -algebras generated by a semigroup of isometries has led to interesting developments in the theory of an index for algebras of operators. This theory is modelled on the theory of Fredholm operators in $\mathfrak{B}(\mathfrak{H})$, and has led to some interesting connections between the notions of topological and analytic index, [8]–[10].

In [12], R. G. Douglas analyzed the structure of the C^* -algebras \mathfrak{A}_Γ generated by one-parameter semigroups of isometries $\mathcal{V}_\Gamma = \{V_\gamma: \gamma \in \Gamma^+\}$, where Γ is a subgroup of the real numbers. Without making any assumptions about the continuity of the mapping $\gamma \rightarrow V_\gamma$, Douglas showed that the C^* -algebra \mathfrak{A}_Γ is canonically unique. This analysis

was carried out via a characterization of the (commutative) quotient algebras $\mathfrak{A}_\Gamma/C_\Gamma$, where C_Γ is the closed two-sided ideal generated by the commutators in \mathfrak{A}_Γ . He determined also that \mathfrak{A}_Γ and $\mathfrak{A}_{\Gamma'}$ are isomorphic if and only if the corresponding groups Γ, Γ' are order isomorphic. (A similar analysis, using K -theoretic techniques, has recently been carried out on the commutator ideals, [13], see also [19].) This uniqueness result stands in marked contrast to the abundance of isometric representations of the semigroups Γ^+ , as shown in [14].

The Cuntz algebras O_n , $n \in (\infty, 2, 3, \dots)$, are a highly non-commutative generalization of $C^*(V)$. For $n < \infty$, O_n is defined as the C^* -algebra generated by n isometries S_1, \dots, S_n on a Hilbert space which satisfy the relations $S_i^*S_j = \delta_{ij}I$, and $\sum_{i=1}^n S_iS_i^* = I$. These identities characterize O_n uniquely, up to isomorphism. O_n is a simple C^* -algebra; in fact, it possesses the remarkable property that for any non-zero X in O_n , there are $A, B \in O_n$ satisfying $AXB = I$, [11, Theorem 1.13] (see also Theorem 3.9 below).

If one replaces the second equation above with the inequality $\sum_{i=1}^n S_iS_i^* < I$, then the C^* -algebra generated by the polynomials in the S_i 's is an extension of O_n by the compact operators ([11, Proposition 3.1], see also Theorem 2.4 below). Taking $n = 1$, the C^* -algebra generated by a (non-unitary) isometry fits into this framework.

In this work we study a problem which is a combination, in a sense, of the two generalizations discussed briefly above. For a subgroup Γ of \mathbb{R} , let $\mathcal{U}_\Gamma = \{U_\gamma: \gamma \in \Gamma^+\}$ and $\mathcal{S}_\Gamma = \{S_\gamma: \gamma \in \Gamma^+\}$ be a pair of semigroups of isometries on a separable Hilbert space. We assume that \mathcal{U}_Γ and \mathcal{S}_Γ are related by the Weyl commutation relations

$$(1) \quad S_\gamma^*U_\gamma = e^{-\lambda\gamma}I, \quad \gamma \in \Gamma^+,$$

for some fixed $\lambda > 0$. Here again we make no assumptions about the continuity of the mappings $\gamma \rightarrow S_\gamma$ and $\gamma \rightarrow U_\gamma$. We should point out that from (1) it follows that each S_γ must contain a nontrivial pure isometric part, for $\gamma > 0$, since the assertion that S_γ is unitary leads to the equation $1 = \|U_\gamma\| = \|e^{-\lambda\gamma}S_\gamma\| = e^{-\lambda\gamma}$, which is absurd. By symmetry, U_γ also contains a pure isometric part. We show below in Theorem 2.4 that if Γ is a discrete subgroup of \mathbb{R} , then the C^* -algebra \mathfrak{A}_Γ generated by all operators U_γ, S_γ , for $\gamma \in \Gamma^+$, is an extension, as above, of the algebra O_2 by the ideal of compact operators. If Γ is dense, then \mathfrak{A}_Γ is simple: in fact, \mathfrak{A}_Γ is strongly simple in the sense shared by the Cuntz algebras that for any $X \neq 0$ there are operators A, B in \mathfrak{A}_Γ such that $AXB = I$ (Theorem 3.9). We also show that the C^* -algebras \mathfrak{A}_Γ are canonically unique, Theorem 3.12. Our methods

of proof of these results rely heavily on some techniques used by J. Cuntz, [11], and R. G. Douglas, [12].

The principal motivation for studying this algebra comes from the recent work of R. T. Powers and the author, [17], relating the index theories of Powers and W. B. Arveson on E_0 -semigroups of $*$ -endomorphism of $\mathfrak{B}(\mathfrak{H})$, [1]–[3], [15]–[17]. Let $\alpha = \{\alpha_t: t \geq 0\}$ be a one-parameter semigroup of $*$ -endomorphisms of $\mathfrak{B}(\mathfrak{H})$. Then α is an E_0 -semigroup if each α_t is unital, if $\alpha_t(\mathfrak{B}(\mathfrak{H}))$ is properly contained in $\mathfrak{B}(\mathfrak{H})$, and if the mapping $t \rightarrow \alpha_t(A)$ is continuous in the weak operator topology for all A in $\mathfrak{B}(\mathfrak{H})$. A strongly continuous one-parameter semigroup $\mathcal{U} = \{U_t: t \geq 0\}$ of operators (not necessarily isometries) in $\mathfrak{B}(\mathfrak{H})$ is said to *intertwine* α , [1], if for all $t \geq 0$ and for all A in $\mathfrak{B}(\mathfrak{H})$, $U_t A = \alpha_t(A)U_t$. It may occur that α has no intertwining semigroups, [16]. However, when intertwining semigroups \mathcal{U} and \mathcal{S} do exist, it follows, [2], that there is a complex number $c(\mathcal{U}, \mathcal{S})$ such that, for all t ,

$$(2) \quad S_t^* U_t = \exp(tc(\mathcal{U}, \mathcal{S}))I.$$

Modifying \mathcal{S} and \mathcal{U} through multiplication by scalar-valued semigroups, one may assume that \mathcal{U} and \mathcal{S} are semigroups of isometries satisfying (1), [17].

Let \mathcal{U}_α be the family of all strongly continuous intertwining semigroups of α . Arveson’s index for α is obtained by calculating the dimension of the Hilbert space completion of the space of functions $\{f: \mathcal{U}_\alpha \rightarrow \mathbb{C}: f \text{ is finitely non-zero and } \sum_{\mathcal{S} \in \mathcal{U}_\alpha} f(\mathcal{S}) = 0\}$ in the positive semidefinite inner product $(f, g) = \sum_{\mathcal{U}, \mathcal{S} \in \mathcal{U}_\alpha} f(\mathcal{U})\overline{g(\mathcal{S})}c(\mathcal{U}, \mathcal{S})$. The Powers’ index is obtained by calculating the multiplicity of a certain representation of the dense $*$ -subalgebra $\mathfrak{D}(\delta)$ of $\mathfrak{B}(\mathfrak{H})$, where $\mathfrak{D}(\delta)$ is the domain of the infinitesimal generator δ of the one-parameter semigroup α , [15]. The key problem involved in showing that these two versions of index agree is to analyze the structure of a pair of strongly continuous flows of isometries satisfying (1) (see [17] for a proof of the existence of these flows and an analysis of their structure).

We end this section by remarking that W. B. Arveson has defined and analyzed the structure of a separable C^* -algebra, called the *spectral C^* -algebra*, associated with an E_0 -semigroup α of endomorphisms. These algebras, which are, along with the index, an outer conjugacy invariant for E_0 -semigroups, are constructed from the product systems E corresponding to α , [3]. As noted by Arveson, this family

of algebras contains the Wiener-Hopf C^* -algebra as a degenerate case in much the same way that the Toeplitz C^* -algebra studied by Coburn is the degenerate case of the Cuntz algebras.

II. The discrete case. In this section we consider the structure of the C^* -algebra $C^*(U_t, S_t)$ generated by a pair of isometries U_t, S_t acting on a separable Hilbert space and satisfying the relation (1), for fixed t . As we shall see in the next section, the proof of the simplicity of \mathfrak{A}_Γ , for Γ a dense subgroup of \mathbb{R} , depends greatly on the special case considered here.

We begin this section by introducing some notation which shall be used throughout the paper. We denote by $\mathcal{U} = \{U_t : t \geq 0\}$ and by $\mathcal{S} = \{S_t : t \geq 0\}$ a pair of semigroups of isometries on a separable Hilbert space \mathfrak{H} which satisfy, for a fixed positive $\lambda > 0$, the commutation relations (1). An explicit construction in [17] shows that such pairs do indeed exist. Let \mathcal{P} be the $*$ -algebra of polynomials in the operators $U_t, S_t, t \geq 0$. Using (1) and the fact that U_t, S_t are isometries, one may always write any polynomial $P \in \mathcal{P}$ as a linear combination of terms of the form

$$(3) \quad A = U_{l_1} S_{l_2} \cdots U_{l_{2\mu-1}} S_{l_{2\mu}} S_{r_{2\nu}}^* U_{r_{2\nu-1}}^* \cdots S_{r_2}^* U_{r_1}^*$$

for non-negative real numbers l_i, r_j . We say that a term in this form is a *word in reduced form*. Associated with A are its (left and right) lengths, $l(A), r(A)$, where $l(A) = \sum_{i=1}^{2\mu} l_i$ and $r(A) = \sum_{j=1}^{2\nu} r_j$. As we shall see (Lemma 3.1) a polynomial P has one and only one expression as a linear combination of words in reduced form (where we agree to use the semigroup laws $U_t U_s = U_{t+s}, S_t S_s = S_{t+s}$ to combine the terms in A as much as possible), so that the length functions are well-defined on reduced words. We say that a word A is even if $l(A) = r(A)$. By $\Phi_0(P)$ we denote the summand of P consisting of linear combinations of all even words of P . P is said to be even if $\Phi_0(P) = P$. Let \mathcal{P}_0 be the subspace of all even polynomials in \mathcal{P} . Using the commutation relations (1) one sees that \mathcal{P}_0 is actually a $*$ -subalgebra of \mathcal{P} .

DEFINITION 2.1. For $t > 0$, let F_t be the even polynomial

$$F_t = [U_t U_t^* + S_t S_t^* - e^{-\lambda t} (U_t S_t^* + S_t U_t^*)] / (1 - e^{-2\lambda t}).$$

Let $F_0 = I$. For $t \geq 0$, let $J_t = 1 - F_t$.

Using the commutation relations and the isometric properties of U and \mathcal{S} , Lemma 2.2.1 below is easily verified. The other assertions follow directly from 2.2.1.

LEMMA 2.2. *The operators F_t, J_t are projections in \mathscr{P} satisfying the following identities, for $s \geq t \geq 0$;*

- (1) $F_t U_s = U_s$, and $F_t S_s = S_s$,
- (2) $J_t U_s = 0 = J_t S_s$,
- (3) $F_t F_s = F_s F_t = F_s$, and
- (4) $J_t J_s = J_s J_t = J_s$.

LEMMA 2.3. *$J_t \neq 0$, for $t > 0$.*

Proof. It suffices to show that for some isometry W in \mathscr{P} , $W^* F_t W \neq I$, since $I = F_t + J_t$. Let $W = U_{t/2} S_{t/2}$, then $W^* U_t = e^{-\lambda t/2} I = W^* S_t$, so

$$W^* F_t W = [e^{-\lambda t}(2 - 2e^{-\lambda t}) / (1 - e^{-2\lambda t})] I \neq I. \quad \square$$

We may now determine the structure of the algebra $C^*(U_t, S_t) = \mathfrak{A}_t$. We shall show below that this algebra is *not simple*. To see this, define positive numbers $a = a_t = \frac{1}{2}(\sqrt{1 + e^{-\lambda t}} + \sqrt{1 - e^{-\lambda t}})$ and $b = b_t = \frac{1}{2}(\sqrt{1 + e^{-\lambda t}} - \sqrt{1 - e^{-\lambda t}})$, and define operators

$$(4) \quad T_{t,1} = (aU_t - bS_t) / (a^2 - b^2) \quad \text{and} \quad T_{t,2} = (aS_t - bU_t) / (a^2 - b^2).$$

\mathfrak{A}_t is clearly generated as a C^* -algebra by the operators $T_{t,i}$, $i = 1, 2$, and it is straightforward to show that the $T_{t,i}$ are isometries which satisfy the following identities:

$$(5.1) \quad T_{t,1}^* T_{t,2} = 0 = T_{t,2}^* T_{t,1},$$

$$(5.2) \quad T_{t,1} T_{t,1}^* + T_{t,2} T_{t,2}^* = F_t.$$

Hence we may apply [11, Proposition 3.1] to obtain the following result.

THEOREM 2.4. *For $t > 0$, let \mathfrak{A}_t be the C^* -subalgebra of $\mathfrak{B}(\mathfrak{H})$ generated by the isometries U_t and S_t . Then the projection J_t generates a two-sided closed ideal in \mathfrak{A}_t isomorphic to the C^* -algebra of compact operators \mathcal{K} , and $\mathfrak{A}_t / \mathcal{K}$ is isomorphic to the Cuntz algebra O_2 .*

As one might suspect from this result, the Cuntz algebra O_2 plays a significant role in understanding the structure of the C^* -algebras \mathfrak{A}_Γ .

III. Simplicity of \mathfrak{A}_Γ for semigroups Γ . In this section we show that if Γ is a dense subgroup of the real numbers, then the C^* -algebra \mathfrak{A}_Γ generated by the semigroups of isometries \mathcal{U}_Γ and \mathcal{S}_Γ is simple.

(Unless stated otherwise, we take $\Gamma = \mathbb{R}$ in this section.) Our main tool is to construct a conditional expectation from \mathfrak{A}_Γ to the C^* -subalgebra of \mathfrak{A}_Γ generated by the even polynomials \mathscr{P}_0 . In order to show that this construction is well-defined, we need the following lemma.

LEMMA 3.1. *Any polynomial $P \in \mathscr{P}$ has a unique expression as a linear combination of words in reduced form.*

Proof. To prove the lemma it suffices to show that if $P = 0$ is a linear combination $\sum_{i=0}^q c_i A_i$ of words in reduced form, then each coefficient c_i must be 0. If not, let $l = \min_i \{l(A_i), r(A_i)\}$. Without loss of generality we may assume $l = l(A_i)$, for some i . Next let r ($\geq l$) be the minimum length $r(A_j)$, where j ranges over all indices such that $l(A_j) = l$. We may assume $l(A_0) = l$ and $r = r(A_0)$. Using the semigroup properties $U_s U_t = U_{s+t}$, $S_s S_t = S_{s+t}$, we may construct partitions $\{0, l_1, l_1 + l_2, \dots, l_1 + \dots + l_n\}$ of $[0, l]$ and $\{0, r_1, r_1 + r_2, \dots, r_1 + \dots + r_m\}$ of $[0, r]$ such that every term A_i of P having lengths $l(A_i) = l$ and $r(A_i) = r$ may be written as a scalar multiple of a word of the form

$$(6) \quad W_{l_1, a_1} \cdots W_{l_n, a_n} W_{r_m, b_m}^* \cdots W_{r_1, b_1}^*$$

for $a_i, b_j \in \{1, 2\}$ and $W_{t,1} = U_t$, $W_{t,2} = S_t$, for any $t \geq 0$.

Now if A_k is any summand of P such that $l(A_k) > l$ or $r(A_k) > r$, then $C = X^* A_k Y$ is a scalar multiple of a word in reduced form with $l(C) > 0$ or $r(C) > 0$, for X any word of the form $W_{l_1, a_1} \cdots W_{l_n, a_n}$ and Y any word of the form $W_{r_1, b_1}^* \cdots W_{r_m, b_m}^*$. Using Lemma 2.2.2, there is a positive scalar t_k sufficiently small such that $J_t C = 0$ or $C J_t = 0$ for $0 < t \leq t_k$. Let t be the minimum of the lengths t_k , where k ranges over the summands of P such that $l(A_k) > l$ or $r(A_k) > r$.

Consider the operators $Z_{t,1} = U_t - e^{-\lambda t} S_t$ and $Z_{t,2} = S_t - e^{-\lambda t} U_t$. It is straightforward to show that the $Z_{t,i}$ are scalar multiples of isometries and satisfy $Z_{t,1}^* W_{t,2} = 0$, $Z_{t,2}^* W_{t,1} = 0$, and $Z_{t,i}^* W_{t,i} = (1 - e^{-\lambda t}) I$. We may suppose that A_0 has the form (6). Let $X = Z_{l_1, a_1} \cdots Z_{l_n, a_n} J_t$, $Y = Z_{r_1, b_1}^* \cdots Z_{r_m, b_m}^* J_t$. Then $X^* A_0 Y$ is a non-zero scalar multiple of J_t , but $X^* A_j Y = 0$ for all other j . But then $0 = X^* P Y = X^* A_0 Y$, a contradiction, which yields the result. \square

Using the uniqueness result above, and following [11], we note that if $\tilde{\mathscr{U}} = \{\tilde{U}_t: t \geq 0\}$ and $\tilde{\mathscr{S}} = \{\tilde{S}_t: t \geq 0\}$ are a pair of semigroups

of isometries on a separable Hilbert space $\tilde{\mathfrak{H}}$ which satisfy (1), then the algebra $\tilde{\mathcal{P}}$ of polynomials in the operators in $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{S}}$ is algebraically isomorphic to \mathcal{P} . Hence we may define a norm $\| \cdot \|_0$ on \mathcal{P} by setting, for $P \in \mathcal{P}$,

$$\|P\|_0 = \sup\{\|\pi(P)\| : \pi \text{ is a separable representation of } \mathcal{P}\}.$$

We shall denote by \mathcal{L} the C^* -algebra obtained by completing \mathcal{P} in the $\| \cdot \|_0$ -norm, and by \mathcal{L}_0 we shall denote the completion of the subalgebra \mathcal{P}_0 of even polynomials in \mathcal{P} , see [11, 1.9].

The result above also shows that there is a unique way of extending the mappings $U_t \rightarrow e^{i\gamma t}U_t$ and $S_t \rightarrow e^{i\gamma t}S_t$ to $*$ -homomorphisms α_γ of \mathcal{P} , for all $\gamma \in \mathbb{R}$. We observe that $\alpha_\gamma(P) = P$ for all $\gamma \in \mathbb{R}$ if, and only if, $P \in \mathcal{P}_0$. We also note that the mappings α_γ are in fact $*$ -automorphisms of \mathcal{P} , since clearly $\alpha_{-\gamma} \circ \alpha_\gamma = \text{id} = \alpha_\gamma \circ \alpha_{-\gamma}$. Moreover, if π is a separable $*$ -representation of \mathcal{P} then so is $\pi \circ \alpha_\gamma$, whence $\|P\|_0 = \|\alpha_\gamma(P)\|_0$ for all $p \in \mathcal{P}$. Hence there is a unique extension of α_γ (which we also denote by α_γ) to a $*$ -automorphism of \mathcal{L} , and from the obvious group law $\alpha_\gamma \circ \alpha_{\gamma_0} = \alpha_{\gamma+\gamma_0}$ on \mathcal{P} , the family $\alpha = \{\alpha_\gamma : \gamma \in \mathbb{R}\}$ is a one-parameter group of automorphisms of \mathcal{L} . α is in fact a strongly continuous family; clearly $\|\alpha_\gamma(P) - P\|_0 \rightarrow 0$ as $\gamma \rightarrow 0$ for $P \in \mathcal{P}$ (note that $\alpha_\gamma(A) = \exp(i\gamma[l(A) - r(a)])A$ for reduced words A). For general X in \mathcal{L} , the convergence $\|\alpha_\gamma(X) - X\|_0 \rightarrow 0$ as $\gamma \rightarrow 0$ follows from the uniform density of \mathcal{P} in \mathcal{L} . Summing up, we have:

LEMMA 3.2. *Let \mathcal{L} be the C^* -algebra obtained as the completion of \mathcal{P} in the norm $\| \cdot \|_0$. Then there exists a unique strongly continuous one-parameter group $\alpha = \{\alpha_\gamma : \gamma \in \mathbb{R}\}$ of $*$ -automorphisms on \mathcal{L} defined by $\alpha_\gamma(U_t) = e^{i\gamma t}U_t$ and $\alpha_\gamma(S_t) = e^{i\gamma t}S_t$.*

THEOREM 3.3. *For any $X \in \mathcal{L}$, $\lim_{T \rightarrow \infty} (2T)^{-1} \int_{-T}^T \alpha_\gamma(X) d\gamma$ converges uniformly to an element $\Phi_0(X) \in \mathcal{L}_0$. The linear mapping $\Phi_0 : \mathcal{L} \rightarrow \mathcal{L}_0$ is a conditional expectation from \mathcal{L} to \mathcal{L}_0 .*

Proof. If A is an even reduced word then $\alpha_\gamma(A) = A$, so $\Phi_0(A) = A$. If A is uneven, $\alpha_\gamma(A) = \exp(i\gamma[l(A) - r(A)])A$, so $\Phi_0(A) = 0$. Hence $\Phi_0(P)$ exists for $P \in \mathcal{P}$, $\Phi_0(P)$ is the sum of the even terms comprising P , so $\Phi_0(P) \in \mathcal{P}_0$. Since \mathcal{P} is uniformly dense in \mathcal{L} it

is clear that $\Phi_0(X)$ exists for all $X \in \mathcal{L}$, and moreover,

$$\begin{aligned} \|\Phi_0(X)\|_0 &= \lim_{T \rightarrow \infty} (2T)^{-1} \left\| \int_{-T}^T \alpha_\gamma(X) d\gamma \right\|_0 \\ &\leq \lim_{T \rightarrow \infty} (2T)^{-1} \int_{-T}^T \|\alpha_\gamma(X)\| d\gamma = \|X\|_0. \end{aligned}$$

Clearly Φ_0 preserves positivity.

Now suppose (P_n) is a sequence of polynomials converging uniformly to X . Then $\|\Phi_0(X) - \Phi_0(P_n)\|_0 \leq \|X - P_n\|_0$, so $\Phi_0(X)$ is the uniform limit of even polynomials of \mathcal{P} . Hence $\Phi_0(X) \in \mathcal{L}_0$. Conversely, if $X \in \mathcal{L}_0$, then since $X = \lim_{n \rightarrow \infty} P_n$ for a sequence of even polynomials, $\Phi_0(X) = \lim_{n \rightarrow \infty} \Phi_0(P_n) = \lim_{n \rightarrow \infty} P_n = X$, so that Φ_0 is surjective and $\Phi_0 \circ \Phi_0 = \Phi_0$. Hence Φ_0 is a conditional expectation on γ . \square

Using some elementary results on almost periodic functions we show (see also [12]) that the mapping Φ_0 is one-to-one on the positive elements. We shall assume \mathcal{L} to be unitaly embedded in $\mathfrak{B}(\mathfrak{H}')$ for some Hilbert space \mathfrak{H}' . If $P \in \mathcal{P}$ is written as a linear combination of reduced words, $P = \sum_{j=1}^q c_j A_j$, then from the expression $\alpha_\gamma(P) = \sum_{j=1}^q c_j e^{i\gamma \xi_j} A_j$, where $\xi_j = l(A_j) - r(A_j)$, it is clear that the mapping $\gamma \rightarrow (\alpha_\gamma(P)f, g)$ is an almost periodic function of γ , for any $f, g \in \mathfrak{H}'$. For $X \in \mathcal{L}$, consider the function $\varphi(\gamma) = (\alpha_\gamma(X)f, g)$; and define $\varphi_m(\gamma) = (\alpha_\gamma(P_m)f, g)$ for some sequence of polynomials $\{P_m\}$ converging uniformly to X . Then for $\gamma \in \mathbb{R}$,

$$\begin{aligned} |\varphi(\gamma) - \varphi_m(\gamma)| &= |(\alpha_\gamma(X)f, g) - (\alpha_\gamma(P_m)f, g)| \\ &\leq \|\alpha_\gamma(X - P_m)\|_0 \|f\| \|g\|, \end{aligned}$$

so that φ is the uniform limit of a sequence of almost periodic functions. Hence φ is itself almost periodic, [5, Theorem 49.V]. Now if X is a non-zero positive element of \mathcal{L} we may choose a vector $f = g$ in \mathfrak{H}' such that $\varphi(0) = (Xf, f) > 0$. But then $\varphi(\gamma)$ is a non-negative, almost periodic function which is not identically equal to 0, so that its mean, $\mathfrak{M}(\varphi)$, is strictly positive, [5, Theorem 72]. But

$$\begin{aligned} \mathfrak{M}(\varphi) &= \lim_{T \rightarrow \infty} (2T)^{-1} \int_{-T}^T \varphi(\gamma) d\gamma \\ &= \lim_{T \rightarrow \infty} (2T)^{-1} \int_{-T}^T (\alpha_\gamma(X)f, f) d\gamma = (\Phi_0(X)f, f), \end{aligned}$$

so that $\Phi_0(X)$ is a non-zero positive element of \mathcal{L}_0 . Hence we have established the following (cf. [12, Proposition 2]).

PROPOSITION 3.4. *The condition expectation $\Phi_0: \mathcal{L} \rightarrow \mathcal{L}_0$ is one-to-one on the positive elements of \mathcal{L} .*

As in the previous section let \mathcal{U} and \mathcal{S} be a pair of semigroups of isometries acting on the Hilbert space \mathfrak{H} , and let \mathfrak{A} be the C^* -algebraic completion of \mathcal{P} in $\mathfrak{B}(\mathfrak{H})$. We shall show that the completion of \mathcal{P}_0 in $\mathfrak{B}(\mathfrak{H})$ is isometrically $*$ -isomorphic to the completion \mathcal{L}_0 of \mathcal{P}_0 in \mathcal{L} . To begin this, suppose $P = \sum_{j=1}^q d_j A_j$ is the unique decomposition of an even polynomial P in \mathfrak{A} into a sum of (even) terms in reduced form. Let $L = \max\{l(A_j): 1 \leq j \leq q\}$ ($= \max\{r(A_j): 1 \leq j \leq q\}$). For each j , if A_j has the form (3), then let R_j be the partition of $[0, L]$ formed as the union of the partitions $\{0, L - (l_1 + \dots + l_{2\mu-1}), L - (l_1 + \dots + l_{2\mu-2}), \dots, L - l_1, L\}$ and $\{0, L - (r_1 + \dots + r_{2\nu-1}), L - (r_1 + \dots + r_{2\nu-2}), \dots, L - r_1, L\}$.

Let R be the union of all of the partitions R_j , $1 \leq j \leq q$. Then there are positive real numbers c_1, c_2, \dots, c_n , for some n , such that

$R = \{0, L - (c_1 + \dots + c_{n-1}), L - (c_1 + \dots + c_{n-2}), \dots, L - c_1, L\}$ (and $0 = L - (c_1 + \dots + c_n)$). Then clearly any A_j may be written in the form

$$A_j = W_{c_1, a_1} \cdots W_{c_{k_j}, a_{k_j}} W_{c_{k_j}, b_{k_j}}^* \cdots W_{c_1, b_1}^*$$

where $a_i, b_i \in \{1, 2\}$ depend on A_j , for $1 \leq i \leq k_j$, where $k_j \leq n$ satisfies $\sum_{i=1}^{k_j} c_i = l(A_j)$ ($= r(A_j)$), and as above, $W_{t,1} = U_t, W_{t,2} = S_t$. If $k_j < n$, then we may rewrite A_j as

$$A_j = W_{c_1, a_1} \cdots W_{c_{k_j}, a_{k_j}} J_{c_{k_j+1}} W_{c_{k_j}, b_{k_j}}^* \cdots W_{c_1, b_1}^* + W_{c_1, a_1} \cdots W_{c_{k_j}, a_{k_j}} F_{c_{k_j+1}} W_{c_{k_j}, b_{k_j}}^* \cdots W_{c_1, b_1}^*$$

From Definition 2.1, the second term above may be rewritten as a linear combination of four terms, each of the form

$$W_{c_1, a_1} \cdots W_{c_{k_j}, a_{k_j}} W_{c_{k_j+1}, a_{k_j+1}} W_{c_{k_j+1}, b_{k_j+1}}^* W_{c_{k_j}, b_{k_j}}^* \cdots W_{c_1, b_1}^*.$$

If $k_j + 1 = n$ we do nothing; otherwise, we rewrite each of the four terms as the sum of two terms

$$W_{c_1, a_1} \cdots W_{c_{k_j}, a_{k_j}} W_{c_{k_j+1}, a_{k_j+1}} J_{c_{k_j+2}} W_{c_{k_j+1}, b_{k_j+1}}^* W_{c_{k_j}, b_{k_j}}^* \cdots W_{c_1, b_1}^* + W_{c_1, a_1} \cdots W_{c_{k_j}, a_{k_j}} W_{c_{k_j+1}, a_{k_j+1}} F_{c_{k_j+2}} W_{c_{k_j+1}, b_{k_j+1}}^* W_{c_{k_j}, b_{k_j}}^* \cdots W_{c_1, b_1}^*.$$

Continuing this process, we may rewrite P as a linear combination of terms each of which takes one of the following three forms:

$$(7.1) \quad J_{c_1}$$

$$(7.2) \quad W_{c_1, a_1} \cdots W_{c_r, a_r} J_{c_{r+1}} W_{c_r, b_r}^* \cdots W_{c_1, b_1}^*, \quad 0 < r < n,$$

$$(7.3) \quad W_{c_1, a_1} \cdots W_{c_n, a_n} W_{c_n, b_n}^* \cdots W_{c_1, b_1}^*.$$

Using the identities (4), we may further decompose (7.2) and (7.3) so that P may be rewritten as a linear combination of terms, each of which takes one of the following three forms:

$$(8.1) \quad J_{c_1},$$

$$(8.2) \quad T_{c_1, a_1} \cdots T_{c_r, a_r} J_{c_{r+1}} T_{c_r, b_r}^* \cdots T_{c_1, b_1}^*, \quad 0 < r < n,$$

$$(8.3) \quad T_{c_1, a_1} \cdots T_{c_n, a_n} T_{c_n, b_n}^* \cdots T_{c_1, b_1}^*.$$

Note that any two distinct terms above (with either the same or different forms) have product 0; this follows from Lemma 2.2.2. Using the commutation relations and (5) shows that for fixed r , $0 \leq r \leq n$, the 4^r terms in \mathcal{P} having the form (8.1) if $r = 0$, (8.2) if $0 < r < n$, and (8.3) if $r = n$, are matrix units for a $2^r \times 2^r$ matrix subalgebra \mathfrak{M}_r of \mathcal{P} . Since $BC = 0$ for any elements $B \in \mathfrak{M}_r$ and $C \in \mathfrak{M}_{r_0}$, for $r \neq r_0$, the totality of terms of the form in (8) are matrix units for a finite-dimensional C^* -subalgebra of \mathcal{P} . Since P lies in this algebra, we may reassemble P as a sum of polynomials $\sum_{r=0}^n P_r$, where $P_r \in \mathfrak{M}_r$. Since the subalgebras \mathfrak{M}_r are mutually orthogonal, it is now clear that $\|P\| = \max\{\|P_r\| : 0 \leq r \leq n\}$. Hence we have:

PROPOSITION 3.5. *Let \mathcal{U} and \mathcal{S} be a pair of one-parameter semigroups of isometries on a Hilbert space \mathfrak{H} , satisfying (1). Let \mathcal{P} be the algebra of polynomials in these isometries. If $P \in \mathcal{P}_0$ there is a finite-dimensional C^* -subalgebra of \mathcal{P} containing P .*

Using the decomposition of P above we see that for any even polynomial P , $\|P\|$ is the same in any representation of the semigroups \mathcal{U} and \mathcal{S} , by the uniqueness of the C^* -algebraic norm on finite-dimensional matrix algebras. In particular, if \mathfrak{A} is the C^* -algebraic completion of \mathcal{U} and \mathcal{S} in $\mathfrak{B}(\mathfrak{H})$, as above, with norm $\|\cdot\|$, then for all $P \in \mathcal{P}_0$, $\|P\| = \|P\|_0$ (cf. [11, 1.9]). This yields the following result.

THEOREM 3.6. *Let \mathcal{U} and \mathcal{S} be a pair of one-parameter semigroups of isometries on $\mathfrak{B}(\mathfrak{H})$ satisfying the commutation relations (1), and let*

\mathfrak{A} be the C^* -algebra obtained as the uniform closure of the polynomial algebra \mathcal{P} in the isometries $U_t, S_t, t \geq 0$. Let \mathfrak{A}_0 be the C^* -subalgebra of \mathfrak{A} obtained as the completion of the even polynomials \mathcal{P}_0 in the norm. Then there exists a $*$ -isometric isomorphism from \mathfrak{A}_0 to \mathcal{L}_0 .

THEOREM 3.7. *Let \mathcal{U}, \mathcal{S} , and \mathfrak{A} be as above. For any element $P \in \mathcal{P} (\subset \mathfrak{A})$ there exists a projection $Q \in \mathcal{P}_0$, depending on P , such that $QPQ \in \mathcal{P}_0$ and $\|QPQ\| = \|\Phi_0(P)\|$.*

Proof. Let $P = \sum_{j=1}^q d_j A_j$ be a decomposition of P into a linear combination of words in reduced form. If $\Phi_0(P) = 0$, then we may choose $Q = 0$. Hence, we may assume $P \neq 0$ and that there are even reduced words A_j in the decomposition of P . Let $L \geq 0$ be the maximum length ($L = l(A) = r(A)$) among all of the even words. Note that if $\Phi_0(P)$ is just a scalar multiple of I , then $L = 0$. First suppose $L > 0$. For each reduced word (even or uneven) A_j , form a partition R_j of $[0, L]$ as follows: if A_j has the form (3), let $n_j + 1$ be the first index such that $\sum_{i=1}^{n_j+1} l_i \geq L$, let $m_j + 1$ be the first index such that $\sum_{i=1}^{m_j+1} r_i \geq L$, and set R_j to be the union of the partitions $\{0, L - (l_1 + \dots + l_{n_j}), \dots, L - l_1, L\}$ and $\{0, L - (r_1 + \dots + r_{m_j}), \dots, L - r_1, L\}$. Let $R = \{0, L - (c_1 + \dots + c_{n-1}), \dots, L - c_1, L\}$ be the union of these partitions, and let $c_n = L - (c_1 + \dots + c_{n-1})$. As in the proof of Proposition 3.5, each of the even terms may be decomposed into a linear combination of terms of the form (7), which in turn may be rewritten as a linear combination of the terms appearing in (8).

Suppose $A = A_j$ is an uneven term in the decomposition of P . If $l(A) \geq L$ and $r(A) \geq L$, A may be rewritten in the form

$$(9.1) \quad W_{c_1, a_1} \cdots W_{c_n, a_n} W V^* W_{c_n, b_n}^* \cdots W_{c_1, b_1}^*$$

where W and V are words in reduced form such that $l(W) > 0$ or $l(V) > 0$, and $r(W) = r(V) = 0$. If $l(A) < L$ (respectively, $r(A) < L$), $l(A) = \sum_{i=1}^{k_j} c_i$ (resp., $r(A) = \sum_{i=1}^{k_j} c_i$) for some $k_j < N$, then by using a procedure similar to that used in the proof of Theorem 3.6, we may decompose A into a linear combination of terms taking one of the forms below (where W is a reduced word with $l(W) > 0$

and $r(W) = 0$)

$$(9.2) \quad J_{c_1} W^*, \quad \text{if } l(A) = 0,$$

$$(9.2') \quad W J_{c_1}, \quad \text{if } r(A) = 0,$$

$$(9.3) \quad W_{c_1, a_1} \cdots W_{c_r, a_r} J_{c_{r+1}} W^* W_{c_r, b_r}^* \cdots W_{c_1, b_1}^*, \quad 0 < r < n,$$

$$(9.3') \quad W_{c_1, a_1} \cdots W_{c_r, a_r} W J_{c_{r+1}} W_{c_r, b_r}^* \cdots W_{c_1, b_1}^*, \quad 0 < r < n.$$

From the proof of Proposition 3.5, $\Phi_0(P)$ decomposes into a sum $\sum_{r=0}^n P_r$ of even polynomials, where each P_r is in turn a linear combination of terms each of which has the form of one of the elements in (8). Also we have shown that $\|\Phi_0(P)\| = \max \|P_r\|$. Choose r such that $\|\Phi_0(P)\| = \|P_r\|$. If $r = 0$, set $Q = Q_0 = J_{c_1}$. If $0 < r < n$, set

$$Q = Q_r = \sum_{a_1, \dots, a_1=1} T_{c_1, a_1} \cdots T_{c_r, a_r} J_{c_{r+1}} T_{c_r, a_r}^* \cdots T_{c_1, a_1}^*.$$

Then it is clear, using the relations (5), that Q_r is a projection. It is also straightforward to show, appealing to Lemma 2.2.2 (and recalling that $T_{l, 1}$ is a linear combination of U_l and S_l) that if B is any term in (9) arising from the decomposition of an uneven reduced term in the expression for P , that $QBQ = 0$. Hence $Q A_j Q = 0$ for all uneven terms A_j . Using the argument establishing that $P_r P_{r_0}$ for $r \neq r_0$ in the proof of the proposition above, we also conclude that $Q_r P_{r_0} Q_r = 0$ for $r \neq r_0$. Finally, if B is any term in the decomposition of P_r , then it is easy to see, using (5), that $Q_r B Q_r = B$, whence $Q_r P_r Q_r = P_r$. Assembling these equations we obtain $Q_r P Q_r = P_r$.

Now suppose $r = n$. Then we modify an argument in [11] to show that there is a projection $Q_n \in \mathcal{P}_0$ such that $\|Q_n P Q_n\| = \|P_n\|$. Consider the matrix units (8.3) constructed in the proof of the proposition for the $2^n \times 2^n$ matrix algebra \mathfrak{M}_n . For any $\varepsilon > 0$ it is straightforward to verify that if

$$Q = \sum_{e_1, \dots, e_n=1}^2 T_{c_1, e_1} \cdots T_{c_n, e_n} J_\varepsilon T_{c_n, e_n}^* \cdots T_{c_1, e_1}^*,$$

then Q is a projection in \mathcal{P}_0 , and the mapping $D \rightarrow Q D Q$ on \mathfrak{M}_n is an isomorphism from \mathfrak{M}_n to another matrix subalgebra, $Q \mathfrak{M}_n Q$, of \mathcal{P} . It is also easy to verify that if B is any even term of the form in (8.1) or (8.2), then $QBQ = 0$ by using Lemma 2.2.2. Now suppose B is one of the terms of the form in (9) arising from the decomposition of an uneven term A_j of P . It is clear, again from Lemma 2.2.2, that for any term B of the form in (9.2), (9.3), (9.2'),

real numbers c_1, c_2, \dots, c_r, c , such that QPQ is a self-adjoint operator in the $2^r \times 2^r$ matrix algebra \mathfrak{M} generated by matrix units of the form $T_{c_1, a_1}, \dots, T_{c_r, a_r} J_c T_{c_r, b_r}^* \dots T_{c_1, b_1}^*$. Let $\sum_{k=1}^q \gamma_k E_k$ be the spectral decomposition of QPQ in \mathfrak{M} , where the E_k are rank one orthogonal projections in \mathfrak{M} and $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_q$. From the inequalities above, $\gamma_1 > 1 - \varepsilon$, and $\|QPQ\| = \gamma_1$. Let $V \in \mathfrak{M}$ be a partial isometry such that

$$VV^* = E_1 \quad \text{and} \quad V^*V = E'_1 = T_{c_1, 1} \dots T_{c_r, 1} J_c T_{c_r, 1}^* \dots T_{c_1, 1}^*.$$

Setting $W = T_{c_1, 1} \dots T_{c_r, 1}$, we have $W^*V^*QPQVW = \gamma_1 W^*E'_1W = \gamma_1 J_c$. Finally define $Y_i = Z_{c/2, i} / \sqrt{1 - e^{-\lambda c}}$, $i = 1, 2$, where $Z_{t, i}$ is defined as in Lemma 3.1. Then Y_1 and Y_2 are isometries satisfying $Y_2^*Y_1^*F_cY_1Y_2 = 0$, so setting $Y = Y_1Y_2$, $Y^*J_cY = I$. Hence $Y^*W^*QPQVW = \gamma_1 I$. Let $D = QVW$. Then $\|D\|_0 \leq 1$, so

$$\begin{aligned} \|D^*XD - I\|_0 &\leq \|D^*XD - D^*PD\|_0 + \|D^*PD - I\|_0 \\ &\leq \|X - P\|_0 + \|\gamma_1 I - I\|_0 < 2\varepsilon, \end{aligned}$$

so D^*XD is invertible, and we are done. □

COROLLARY 3.10. *\mathcal{L} is a simple C^* -algebra.*

We may now prove the following uniqueness result.

COROLLARY 3.11. *Let \mathcal{U} and \mathcal{S} be a pair of one-parameter semigroups of isometries acting on a separable Hilbert space \mathfrak{H} and satisfying the commutation relations (1). Let $\mathfrak{A} \subset \mathfrak{B}(\mathfrak{H})$ be the C^* -algebraic completion of the polynomial $*$ -algebra \mathcal{P} in the operators $U_t, S_t, t \geq 0$. Then \mathcal{L} and \mathfrak{A} are isomorphic.*

Proof. From the definition of \mathcal{L} it follows that \mathfrak{A} must be a quotient of \mathcal{L} , i.e., $\mathfrak{A} = \pi(\mathcal{L}) \cong \mathcal{L} / \ker(\pi)$, for some representation π . But $\ker(\pi) = 0$. □

Suppose Γ is a subgroup of \mathbb{R} , and $\mathcal{U}_\Gamma = \{U_t : t \in \Gamma^+\}$, $\mathcal{S}_\Gamma = \{S_t : t \in \Gamma^+\}$ are semigroups of isometries on a Hilbert spaces \mathfrak{H} which satisfy the commutation relations

$$S_t^*U_t = e^{-\lambda t}I, \quad t \in \Gamma^+.$$

Then we may consider the polynomial $*$ -algebra \mathcal{P}_Γ generated by the operators $U_t, S_t, t \in \Gamma^+$, and we define \mathfrak{A}_Γ to be the C^* -algebraic

completion of \mathcal{P}_Γ in the norm on $\mathfrak{B}(\mathcal{H})$. It is easy to see that the techniques used to prove the results above for the case $\Gamma = \mathbb{R}$ may be applied virtually without change to show that \mathfrak{A}_Γ is a simple C^* -algebra, if Γ is dense in \mathbb{R} . Combining Theorem 2.4 with these observations, we arrive at the following extension of the results above.

THEOREM 3.12. *Let Γ be a subgroup of \mathbb{R} with corresponding C^* -algebra \mathfrak{A}_Γ . If Γ is discrete, \mathfrak{A}_Γ contains a maximal closed two-sided ideal isomorphic to the C^* -algebra of compact operators \mathcal{K} , and $\mathfrak{A}_\Gamma/\mathcal{K}$ is isomorphic to the Cuntz algebra O_2 . If Γ is dense in \mathbb{R} , then \mathfrak{A}_Γ is a simple C^* -algebra, and the C^* -algebra generated by pairs of semigroups of isometries $\mathcal{U}_\Gamma, \mathcal{S}_\Gamma$ acting on a Hilbert space is canonically unique.*

It would be interesting to obtain necessary and sufficient conditions on a pair of dense semigroups Γ^+, Γ_0^+ of \mathbb{R}^+ for the corresponding C^* -algebras $\mathfrak{A}_\Gamma, \mathfrak{A}_{\Gamma_0}$ to be isomorphic. In the situation where $\mathfrak{B}_\Gamma, \mathfrak{B}_{\Gamma_0}$ are the C^* -algebras generated by single one-parameter semigroups $\mathcal{U}_\Gamma, \mathcal{U}_{\Gamma_0}$ of isometries, R. G. Douglas has shown in [12] that \mathfrak{B}_Γ and \mathfrak{B}_{Γ_0} are isomorphic if and only if Γ and Γ_0 are order isomorphic. We suspect that the isomorphism classes of algebras \mathfrak{A}_Γ are also determined by order isomorphism classes of semigroups.

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