

# Pacific Journal of Mathematics

**RATIONAL FORMAL GROUP LAWS**

ROBERT COLEMAN AND FRANCIS OISIN MCGUINNESS

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ROBERT F. COLEMAN AND FRANCIS OISIN MCGUINNESS

**In this paper we determine the rational formal groups defined over a field of characteristic zero. This answers a question originally posed by Robert MacPherson.**

While one can answer this question using Weil's theorem which asserts that every birational group is birationally isomorphic to an actual algebraic group [W1], below we give an elementary argument using methods similar to those used in [C].

**THEOREM.** *Every rational formal group law over an algebraically closed field  $K$  of characteristic zero is of the form*

$$L^{-1}G(L(x), L(y))$$

where  $G(x, y)$  is either  $x+y$  or  $x+y+xy$  and  $L$  is a linear functional transformation over  $K$  such that  $L(0) = 0$ .

One deduces easily from this that

**COROLLARY.** *The rational formal group laws over a field  $K$  of characteristic zero are the rational functions*

$$(x + y + cxy)/(1 - dxy)$$

where  $c$  and  $d$  are elements of  $K$ . Moreover, this formal group is rationally isomorphic to  $x+y$  over  $K$  if  $c^2 - 4d = 0$  and to  $x+y+xy$  over  $K(\sqrt{(c^2 - 4d)})$  otherwise.

*Proof of theorem.* Recall that now  $K$  is algebraically closed. Suppose  $F(x, y)$  is a rational formal group.

Let  $\omega = dx/F_2(x, 0)$  and  $g(x) = F(x, x)$  (the rational function giving multiplication by 2 on  $F$ ). Then  $\omega$  and  $g$  satisfy the hypothesis of the following proposition:

**PROPOSITION.** *Suppose  $\omega \in K(x)dx$  and  $g \in K(x)$ ,  $\omega \neq 0$ ,  $\text{ord}_0 \omega = 0$ ,  $g(0) = 0$  and  $g^*\omega = 2\omega$ . Then  $\omega = L^*(dx)$  or  $L^*(c dx/(x+1))$*

where  $L$  is a linear fractional transformation defined over  $K$  such that  $L(0) = 0$  and  $c \in K^*$ .

*Proof.* Let  $Y$  denote the set of poles and  $Z$  the set of zeros of  $\omega$ . It follows from the hypothesis that  $g^{-1}Y = Y$  and  $g^{-1}Z = Z$ .

The equation  $g^*\omega = 2\omega$  implies that

$$\sum \text{ord}_Q g^*\omega = \sum \text{ord}_Q \omega$$

where the sums run over  $Q \in Y = g^{-1}Y$ . Suppose  $Q \in \mathbb{P}^1(K)$ . Then we also have the formula

$$\sum \text{ord}_P g^*\omega = \deg(g) \cdot \text{ord}_Q \omega + (\deg(g) - \#g^{-1}(Q))$$

where the sum runs over  $P \in g^{-1}(Q)$ . Suppose now  $Q$  is a pole of  $\omega$ . The right-hand side of this formula is less than or equal to  $\text{ord}_Q \omega$ . Hence the last two formulas imply that

$$\deg(g) \cdot \text{ord}_Q \omega + (\deg(g) - \#g^{-1}(Q)) = \text{ord}_Q \omega$$

for all  $Q \in Y$ . This occurs for a given  $Q \in Y$  iff  $\deg(g) = 1$  or  $\text{ord}_Q \omega = -1$  (in which case  $\#g^{-1}(Q) = 1$ ).

Suppose first that  $\deg(g) = 1$  and  $\omega$  has a pole of order greater than one. Since  $g^*\omega = 2\omega$ , no iterate of  $g$  is the identity. As  $g(0) = 0$  it follows that there exists exactly one non-zero point fixed by some iterate of  $g$ . Since  $g^{-1}Y = Y$ ,  $g^{-1}Z = Z$  and  $\text{ord}_0 \omega = 0$ , we see that  $\omega$  has only one pole and no zeros. It follows that  $\omega = L^*(dx)$  for some linear fractional transformation  $L$  which we may assume vanishes at the origin.

Suppose now that  $\omega$  has only simple poles. If  $Q$  is a pole of  $\omega$  we know that  $g^{-1}(Q)$  consists of exactly one point,  $P$  say, and we have the formula

$$\text{Res}_P g^*\omega = \deg(g) \text{Res}_Q \omega$$

by a local computation. Since  $g^*(\omega) = 2\omega$ , this becomes

$$\text{Res}_P \omega = (\deg(g)/2) \text{Res}_Q \omega.$$

Now we know that  $g^{-1}Y = Y$ . Hence, there exists a  $Q$  in  $Y$  and a positive integer  $n$  such that  $\{Q\} = g^{-n}(Q)$ . By iterating the previous equation we deduce that

$$(\deg(g)/2)^n \text{Res}_Q \omega = \text{Res}_Q \omega.$$

Hence, as  $\text{Res}_Q \omega \neq 0$  and  $\deg(g) \in \mathbb{Z}_{>0}$ ,  $\deg(g) = 2$ .

The facts that  $g^{-1}Y = Y$  and  $g^{-1}Z = Z$  imply that the zeros and poles of  $\omega$  lie among the branch points of  $g: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ . Since  $g$  has degree 2 it has only two branch points. Since  $\omega$  is not equal to zero, has only simple poles and its residues sum to zero it must have exactly two poles and no zeros. Hence  $\omega = L^*(c dx/(x+1))$  for some linear fractional transformation  $L$  and some constant  $c \in K^*$ . Since  $\text{ord}_0 \omega = 0$ , we may assume  $L(0) = 0$ . This proves the proposition.  $\square$

The theorem follows from the proposition noting that  $F(x, y) = L^{-1}G(L(x), L(y))$  where  $G(x, y) = x + y$  if  $\omega = L^*(dx)$  and  $G(x, y) = x + y + xy$  if  $\omega = L^*(c dx/(x+1))$ .  $\square$

REMARK. The only place in the above argument where the algebraic closedness of  $K$  was used in a serious manner was in the last step which required finding a linear fractional transformation which moved one pole of  $\omega$  to 0 and the other to  $\infty$ .

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