Pacific Journal of Mathematics

AUTOMATIC CONTINUITY OF *-MORPHISMS BETWEEN NONNORMED TOPOLOGICAL *-ALGEBRAS

MARIA FRAGOULOPOULOU

Vol. 147, No. 1 January 1991

PACIFIC JOURNAL OF MATHEMATICS
Vol. 147, No. 1, 1991

AUTOMATIC CONTINUITY OF *-MORPHISMS BETWEEN NON-NORMED TOPOLOGICAL *-ALGEBRAS

MARIA FRAGOULOPOULOU

Every *-morphism of a Q locally m-convex (lmc) *-algebra E, in an lmc C^*-algebra F, is continuous. The same is also true if E is taken to be a Fréchet locally convex *-algebra. Thus, the topology of a Fréchet locally convex C^*-algebra ($\Leftrightarrow$ Fréchet lmc C^*-algebra) is uniquely determined. Each lmc C^*-algebra has a continuous involution. In the general case, one has that the involution of a barrelled Pták (e.g. Fréchet) locally convex algebra E is continuous iff the real locally convex space $H(E)$ of its self-adjoint elements, is a closed subspace. In particular, every algebra E as before, which admits a continuous faithful *-representation, has a continuous involution. Furthermore (without assuming continuity of the involution), we obtain that every *-representation of an involutive Fréchet Q lmc algebra E, is continuous, while if E has moreover a bounded approximate identity, the same holds also true for each positive linear form of E.

1. Introduction. The continuity of (homo)morphisms between non-normed topological algebras has been considered by several authors. An extensive exposition of what is known about this problem, particularly for multiplicative linear forms, has been given by T. Husain in [**14**]. In this regard, a relevant result is provided by Corollary 3.7 of this paper (cf. also comments following Corollary 3.7). After the recent increasing applications of the theory of (non-normed) topological *-algebras in other fields of mathematics, as for instance, quantum mechanics (cf. [**17**] as well as comments in [**8**; p. 115]), the continuity of *-morphisms between this sort of algebras receives considerable attention. The known results on automatic continuity of *-morphisms between topological *-algebras concern mainly (Hilbert space) *-representations (cf., for instance, [**3**; Lemma 3.1], [**15**; Theorem 3], [**17**; Theorem 4.1]).

This paper provides, in the context of locally convex algebras, new generalizations and extensions of previously known Banach algebra results. Namely, §2 presents the background material. Section 3 deals with several cases of automatic continuity of *-morphisms between locally m-convex (lmc) *-algebras (see, for example, Theorems 3.1, 3.3, 3.11). As a byproduct one gets sufficient conditions under which lmc C^*-algebras become classical C^*-algebras (cf. Proposition 3.8, Corollary 3.12). In this regard, note that the only complete Q lmc C^*-algebras are C^*-algebras (cf. [**11**; Theorem 4.3]). Section 4 gives necessary and sufficient conditions under which the involution of a locally convex algebra becomes continuous (Propositions 4.1, 4.2). In §5 no-continuity of the involution is assumed for the algebras involved. Thus, (Hilbert-space) *-representations of involutive Fréchet Q lmc algebras are continuous (Theorem 5.2), while the same also holds for the positive linear forms if the preceding algebras possess a bounded approximate identity (bai) (Theorem 5.7). In this regard, every positive linear form of $C^\infty(X, E)$ is continuous, whenever X is an n-dimensional compact metrizable C^∞-manifold and E is, for instance, an involutive commutative Banach algebra with a bai (Corollary 5.9).

2. Notation and definitions. All the algebras we deal with are complex, while the topological spaces involved are assumed to be Hausdorff.

A *Q-algebra* is a topological algebra whose set of quasi-invertible elements is open [**19**, **18**]. A *locally m-convex* (lmc) *algebra* is a topological algebra whose topology is defined by a directed family of submultiplicative seminorms [**1**, **19**, **18**]. Such an algebra will be usually denoted by $(E, (p_\alpha))$, $\alpha \in A$. In case E is endowed with an involution * such that $p_\alpha(x^*) = p_\alpha(x)$, for any $x \in E$, $\alpha \in A$, we will speak of an *lmc *-algebra.* When no-continuity of the involution is involved the term *involutive topological algebra* will be used. Denote by $H(E)$ the set of the self-adjoint elements of an involutive algebra E. Every C^*-seminorm on E is then automatically submultiplicative and *-preserving [**22**]. Thus, one defines an *lmc C^*-algebra* as an involutive topological algebra whose topology is defined by a directed family of C^*-seminorms. Complete lmc C^*-algebras are called locally C^*-algebras in [**16**] and pro-C^*-algebras in [**20**]. For a given lmc algebra $(E, (p_\alpha))$, $\alpha \in A$, one has

$$E \underset{\rightarrow}{\subseteq} \varprojlim_{\alpha} E_\alpha, \tag{2.1}$$

up to a not necessarily surjective topological algebraic isomorphism, where E_α is the completion of the normed algebra $(E/N_\alpha, \|\cdot\|_\alpha)$, $\alpha \in \mathrm{A}$, with $N_\alpha \equiv \ker(p_\alpha)$ and $\|x_\alpha\|_\alpha := p_\alpha(x)$, $x_\alpha \equiv x + N_\alpha \in E/N_\alpha$, $\alpha \in \mathrm{A}$ [**1**, **18**, **19**]. Equality in (2.1) occurs when E is moreover complete. Now, a *bounded approximate identity* (bai) of E, is a net

(e_j), $j \in J$, in E with $p_\alpha(e_j) \leq 1$, for all $\alpha \in A$, $j \in J$ and $\lim_j(e_j x) = x = \lim_j(x e_j)$, for each $x \in E$. Every complete lmc C^*-algebra has a bai [**16**; Theorem 2.6]. Example of an lmc algebra with a bai can be found in [**4**; p. 610]. For a given algebra E, denote by $\mathrm{sp}_E(x)$, $r_E(x)$ the *spectrum*, respectively the *spectral radius* of $x \in E$. If E is an *advertibly complete* lmc algebra [**18**; Definition I, 6.4] one has

$$(2.2) \qquad r_E(x) = \sup_\alpha r_{E_\alpha}(x_\alpha), \qquad \forall x \in E,$$

[ibid., Theorem III, 6.1].

Furthermore, if E is an involutive topological algebra by a **-representation* of E we mean a *-morphism φ of E in the *-algebra $\mathscr{L}(H_\varphi)$ of all bounded linear operators on some Hilbert space H_φ. Denote by $P(E)$ (resp. $\mathscr{P}(E)$) the set of all *positive* (resp. *continuous positive*) *linear forms* of E. Then, $\mathscr{P}(E) = P(E)$ whenever E is a Fréchet locally convex *-algebra with a left bai [6; Theorem 4.3]. In this respect, E is called *P-commutative* if $xy - yx \in \bigcap\{L_f : f \in P(E)\}$, for all $x, y \in E$, where $L_f = \{x \in E : f(x^*x) = 0\}$ (cf. [**23**, **9**]). Finally an involutive algebra E is called *symmetric* if $-x^*x$ is quasi-invertible for all $x \in E$ [**19**, **21**].

3. Continuity of *-morphisms between lmc*-algebras. The next theorem has a special bearing on [**5**; 1.3.7. Proposition].

3.1. THEOREM. *Let $(E, (p_\alpha))$, $\alpha \in \mathrm{A}$, be a Q lmc *-algebra and $(F, (q_\beta))$, $\beta \in \mathrm{B}$, an lmc C^*-algebra. Then, every *-morphisjm φ of E in F is continuous. In particular, for each $\beta \in \mathrm{B}$, there is $\alpha_0 \in \mathrm{A}$, such that*

$$q_\beta(\varphi(x)) \leq p_{\alpha_0}(x), \qquad \forall x \in E.$$

Proof. Without loss of generality we assume F to be complete. Then, by [**21**; Lemma (4.8.1), (ii)] one has

$$r_{F_\beta}(z_\beta^* z_\beta) = \|z_\beta^* z_\beta\|_\beta = q_\beta(z^*z) = q_\beta(z)^2, \qquad \forall z \in F.$$

On the other hand, using (2.2), [**18**; Proposition II, 1.1] and the fact that E is a Q-algebra (see [**19**; Proposition 13.5] and/or [**24**; Corollary 1]), we conclude that for every $\beta \in \mathrm{B}$, there is $\alpha_0 \in \mathrm{A}$ with

$$q_\beta(\varphi(x))^2 \leq r_E(x^*x) \leq p_{\alpha_0}(x^*x) \leq p_{\alpha_0}(x)^2, \qquad \forall x \in E.$$

3.2. COROLLARY. *Every *-representation φ of a Q lmc *-algebra $(E, (p_\alpha))$, $\alpha \in \mathrm{A}$, is continuous. In particular, there is $\alpha_0 \in \mathrm{A}$, such* *that*

$$\|\varphi(x)\| \leq p_{\alpha_0}(x), \qquad \forall x \in E.$$

The continuity of a *-representation of a sequentially complete Q lmc *-algebra has been proved by T. Husain-R. Rigelhof [**15**; Theorem 3] by means of another technique. For other alternatives of Corollary 3.2, see Theorem 5.2 below, as well as [**3**; Lemma 3.1] and [**17**; Theorem 4.1]. In [**12**; Corollary 4.7] there is an improvement of [**3**; Lemma 3.1], according to which *every *-representation of a Fréchet locally convex *-algebra is continuous.*. Now, an alternative of Theorem 3.1 is the next Theorem 3.3, which besides provides a slight extension of [**20**; Theorem 5.2]. In fact, N. C. Phillips proves Theorem 3.3 in case E is a Fréchet lmc C^*-algebra and F a complete lmc C^*-algebra.

3.3. THEOREM. *Let $(E, (p_n))$, $n \in \mathbb{N}$, be a Fréchet locally convex *-algebra and $(F, (q_\beta))$, $\beta \in \mathrm{B}$, a locally convex C^*-algebra ($\Leftrightarrow$ lmc C^*-algebra). Then, every *-morphism φ of E in F, is continuous.*

Proof. Using (2.1) and the fact that every F_β, $\beta \in \mathrm{B}$, is a C^*-algebra, the continuity of φ is reduced to that of a *-representation of E, which is valid according to the preceding comments. □

3.4. COROLLARY. *Let E, F be Fréchet locally convex C^*-algebras and φ a 1-1 *-morphism of E in F, with closed image. Then, φ is a topological isomorphism.* □

3.5. COROLLARY. *The topology of a Fréchet locally convex C^*-algebra ($\Leftrightarrow$ Fréchet lmc C^*-algebra) is uniquely determined. That is, any other Fréchet locally convex C^*-topology on E is equivalent to the given one.* □

3.6. COROLLARY. *Let E be a Fréchet locally convex *-algebra. Then, every multiplicative linear form f of E, which is hermitian $(: f(x^*) = \overline{f(x)},\ x \in E)$, is continuous.* □

According to [**19**; Lemma 6.4, b)] every multiplicative linear form of a symmetric algebra is hermitian, so that Corollary 3.6 is now stated as follows.

3.7. COROLLARY. *Every multiplicative linear form of a symmetric Fréchet locally convex *-algebra E is continuous.* □

Corollary 3.7 generalizes a previous result of E. A. Michael [**19**; Theorem 12.6] (cf. also [**14**; Theorem 2.28]) stated for commutative symmetric Fréchet lmc *-algebras. In fact, E. A. Michael [**19**] posed in 1952 the question of whether each multiplicative linear form of a commutative Fréchet lmc algebra E is continuous; he himself proved that this is true if moreover E is a symmetric lmc *-algebra (ibid.). Thus, *Corollary* 3.7 *gives an affirmative answer to the previous question for the wider class of locally convex algebras, although still based on the additional structure of the continuous involution.*

Of course, since E in Corollary 3.7 is not commutative we may have $\mathfrak{M}(E) = \varnothing$, where $\mathfrak{M}(E)$ denotes the *spectrum* (Gel'fand space) of E. But assuming for E a weaker concept of commutativity, the so-called P-commutativity [**9**, **23**] (cf. also §2) we get that $\mathfrak{M}(E) \neq \varnothing$ *for every unital P-commutative symmetric Fréchet lmc *-algebra* E [**9**; Theorem 6.1] (in the commutative case this is true for unital lmc algebras). Hence, *Corollary* 3.7 *in either aspect, commutative or P-commutative, contains the result of E. A. Michael.*

The next proposition provides a sufficient condition for an lmc C^*-algebra to be a C^*-algebra.

3.8. PROPOSITION. *The image, under a *-morphism* φ, *of a Pták* Q *lmc *-algebra* E *onto a barrelled lmc* C^**-algebra* F *is, up to a topological algebraic isomorphism, a* C^**-algebra.*

Proof. By Theorem 3.1 φ is continuous, hence $\ker(\varphi)$ is closed. Thus, $E/\ker(\varphi)$ endowed with the quotient topology is a Hausdorff Pták Q lmc *-algebra (cf., for instance, [**13**; p. 300, Proposition 5] and [**19**; Proposition 13.5]). Now, the canonical algebraic isomorphism

$$E/\ker(\varphi) \to F : x + \ker(\varphi) \mapsto \varphi(x),$$

is clearly continuous, hence (open mapping theorem) a topological isomorphism. Property Q for E implies now the assertion (cf. [**11**; Theorem 4.3]). □

Proposition 3.8 remains also true if E is replaced by a Fréchet locally convex *-algebra and F is moreover ssb (*strong spectrally bounded*; i.e., if (q_β), $\beta \in \mathrm{B}$, is a defining family of seminorms for F, then $\sup_\beta q_\beta(y) < \infty$, for all $y \in F$ [**10**]). This follows by Theorem 3.3 and [**10**; Theorem 2.3]. On the other hand, it is clear from Proposition 3.8 that *if* F *is a Fréchet lmc* C^**-algebra and* φ *a *-morphism of* E *in* F *with closed image, then* $\mathrm{Im}(\varphi)$ *is* (up to a topological isomorphism) *a* C^**-algebra.*

The next theorem generalizes a standard Banach *-algebra result (see [**5**; 1.8.1 Proposition]) to the case of lmc *-algebras. The author is indebted to Professor A. Mallios for his comments, which made hypotheses for E in Theorem 3.9, more reasonable and contributed to the respective part of the proof.

3.9. THEOREM. *Let $(E, (p_\alpha))$, $\alpha \in \mathrm{A}$, be a complete lmc C^*-algebra, whose every self-adjoint element has a compact spectrum. Let also $(F, (q_\beta))$, $\beta \in \mathrm{B}$, be an lmc *-algebra and φ a* 1-1 **-morphism of E in F such that $\overline{\mathrm{Im}(\varphi)}$ is a Q-*-subalgebra of F. Then, $\varphi^{-1}|\mathrm{Im}(\varphi)$ is continuous. In particular, for every $\alpha \in \mathrm{A}$, there is $\beta_0 \in \mathrm{B}$, with*

$$p_\alpha(x) \leq q_{\beta_0}(\varphi(x)), \qquad \forall x \in E. \tag{3.1}$$

Proof. Without loss of generality we suppose E and F unital. Now, since each p_α is a C^*-seminorm and $q_\beta(\varphi(x^*x)) \leq q_\beta(\varphi(x))^2$, for any $\beta \in \mathrm{B}$, $x \in E$, it suffices to prove (3.1) for every $x \in H(E)$. Thus, let $x \in H(E)$ and E_0 be the complete lmc C^*-subalgebra of E generated by x and the unit of E. Then, $E_0 = \mathscr{C}(\mathrm{sp}_E(x))$ up to a topological algebraic *-isomorphism (see also, for instance, [**5**; 1.5.1, Theorem]), so that E_0 is a unital commutative C^*-algebra. Thus, we may suppose E to be such an algebra. On the other hand, since in this case $\mathrm{Im}(\varphi)$ is also commutative and $\mathfrak{M}(\mathrm{Im}(\varphi)) = \mathfrak{M}(\overline{\mathrm{Im}(\varphi)})$, up to a homeomorphism [**18**; Corollary V, 2.1], we may think of F as a unital commutative complete Q lmc *-algebra. So the map

$${}^t\varphi: \mathfrak{M}(F) \to \mathfrak{M}(E): f \mapsto f \circ \varphi,$$

is now continuous; hence $\mathrm{Im}({}^t\varphi)$ is closed (see [**18**; Lemma VI, 1.3]). Following now the reasoning of [**5**; 1.8.1. Proposition] we conclude that ${}^t\varphi$ is surjective. Thus, (see also [**18**; Corollary III, 6.4]) one gets that $\mathrm{sp}_F(\varphi(x)) = \mathrm{sp}_E(x)$, $x \in E$, which by a similar argument to that in the proof of Theorem 3.1 implies (3.1). □

It would be interesting to have any strengthening of Theorem 3.9.

3.10. COROLLARY. *Let E be as in Theorem* 3.9 *and φ a faithful *-representation of E. Then, $\varphi^{-1}|\mathrm{Im}(\varphi)$ is continuous.* □

For the existence of faithful *-representations of an lmc C^*-algebra and/or an lmc *-algebra see [**10**; Corollary 1.2, Theorem 2, 3] as well as [**11**; Proposition 4.11].

3.11. THEOREM. *Let* $(E, (p_n))$, $n \in \mathbb{N}$, *be a Fréchet Q lmc algebra,* $(F, (q_\beta))$, $\beta \in \mathrm{B}$, *an lmc* C^**-algebra and* φ *a* 1-1 *morphism of* E *in* F *with self-adjoint image* $(: \varphi(x)^* \in \mathrm{Im}(\varphi)$, *for every* $x \in E)$. *Then,* φ *is continuous.*

Proof. The map $\varphi^{-1} \circ (*|_{\mathrm{Im}(\varphi)}) \circ \varphi$, is clearly an involution of E, which by a closed graph argument [**13**] is continuous. In fact, let (x_n) be a sequence in E with $x_n \to x \in E$ and $x_n^* \to y \in E$. Arguing as in Theorem 3.1, we obtain that for every $\beta \in \mathrm{B}$, there is $n_0 \in N$ such that

$$\begin{aligned} q_\beta(\varphi(x) - \varphi(x_n))^2 &\le p_{n_0}(x_n^* - x^*) p_{n_0}(x_n - x) \\ &\le (p_{n_0}(x_n^*) + p_{n_0}(x^*)) p_{n_0}(x_n - x), \end{aligned}$$

where $p_{n_0}(x_n - x) \to 0$ and $(p_{n_0}(x_n^*))$ is bounded [**13**; p. 135]. Hence, $\varphi(x_n) \to \varphi(x)$ and similarly $\varphi(x_n^*) \to \varphi(y)$, where moreover $\varphi(x_n)^* \to \varphi(x)^*$. Thus, $x^* = y$, and consequently the assertion now follows from either of the Theorems 3.1, 3.3. □

3.12. COROLLARY. *Let* E, F *be as in Theorem* 3.11 *and* φ *a bijective morphism between E and F. Then, whenever F is barrelled, both* E *and* F *become* C^**-algebras up to topological algebraic isomorphisms.*

Proof. Apply Theorem 3.11 and use open mapping theorem together with [**11**; Theorem 4.3]. □

3.13. *Scholium.* Based on a geometric version of [**19**; Proposition 13.5] given by Y. Tsertos [**24**; Theorem 1], actually for any topological algebra, we get the following improved form of [**9**; Corollary 7.4] (see also Remark 7.1 of the same reference).

THEOREM. *Let* E *be a symmetric Fréchet Q lmc* **-algebra with a bai. Then, E is P-commutative iff* $r_E(x^*x) \le r_E(x)^2$, *for all* $x \in E$.

Proof. Since E is Fréchet every $f \in P(E)$ is continuous [**6**; Theorem 4.3], therefore uniquely extended to a (continuous) positive linear form f_1 on the unitization E_1 of E, with

$$|f(x)| \le f_1(0, 1) r_E(x), \qquad \forall x \in E$$

(cf. [**7**; Proposition 3.4, (i)] and [**9**; Lemma 7.1]). On the other hand, since E is Q there is $n_0 \in \mathbb{N}$ with $r_E(x) \le p_{n_0}(x)$, for all $x \in E$ (see [**24**; Corollary 1]). P-commutativity of E results now from an application of [**9**; Theorem 7.1] for the bilinear map $h(x, y) := f(yx)$, $x, y \in E$ and from [**9**; Scholium 4.1, (vi)]. For the converse, see Theorem 6.2, (ii) of the last reference. □

4. Continuity of the involution of a topological algebra. In this section we consider necessary and sufficient conditions under which the involution of a locally convex algebra is continuous. For the Banach algebra analogues of these results, see [**2**; p. 190, §36]. In the same reference there are also examples of discontinuous involution. Here, we present another example of this kind; namely, we construct a Fréchet lmc algebra with a discontinuous involution.

Let now E be an involutive algebra and E^* the algebraic dual of E. Then, the map $E^* \to E^*: f \mapsto f^*$ with $f^*(x) := \overline{f(x^*)}$, $x \in E$, is a linear involution of E^* in the sense of [**2**] (see also (ibid., p. 187)). In this respect, one easily has the following propositions.

4.1. PROPOSITION. *For an involutive locally convex algebra E, whose topological dual is E', consider the next statements*:

(i) *The involution of E is continuous.*
(ii) *E' is invariant* (*under the linear involution of E^**).*

Then, (i) $\Rightarrow$ (ii), *while* (ii) $\Rightarrow$ (i) *in case E is moreover barrelled and Pták* (*e.g., Fréchet*). □

4.2. PROPOSITION. *For an involutive locally convex algebra E, consider the following statements*:

(i) *The involution of E is continuous.*
(ii) *$H(E')$ separates the points of E.*
(iii) *$H(E)$ is a closed subspace of E.*

Then, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), *while* (iii) $\Rightarrow$ (i) *in case E is moreover barrelled and Pták.* □

4.3. COROLLARY. *Let E be an involutive barrelled Pták locally convex algebra and φ a faithful *-representation of E. Then, the involution of E is continuous.*

Proof. Let (x_δ) be a net in $H(E)$ with $\lim_\delta x_\delta = x \in E$. Then, for any $\xi, \eta \in H_\varphi$ we have $\langle \varphi(x)(\xi), \eta\rangle = \langle \xi, \varphi(x)(\eta)\rangle$, which implies $x \in H(E)$. The assertion now follows by Proposition 4.2. □

4.4. *Example of a discontinuous involution.* Let (E_n), $n \in \mathbb{N}$, be a sequence of Banach algebras with respect to two norms $\|\cdot\|_n$, $\|\cdot\|'_n$,

$n \in \mathbb{N}$, which are not equivalent. Suppose also that each E_n, $n \in \mathbb{N}$, has an isometric involution with respect to $\|\cdot\|_n$, $n \in \mathbb{N}$. Then, $E \equiv \prod_n E_n$ is a Fréchet lmc algebra with respect to two topologies τ, τ' induced from $\|\cdot\|_n$, $\|\cdot\|'_n$, $n \in \mathbb{N}$, respectively. In particular (E, τ) is an lmc *-algebra, while τ, τ' are not equivalent. Now let $F \equiv (E, \tau)\oplus(E, \tau')$ with algebraic operations defined coordinatewise and involution by $(x, y)^* := (y^*, x^*)$, $(x, y) \in F$. Then, F is an involutive Fréchet lmc algebra under the respective direct sum topology. In particular, we can find a sequence (y_n) in E with $y_n \underset{\tau}{\longrightarrow} 0$, but $y_n \underset{\tau'}{\not\to} 0$; so that there is a sequence (z_n) in F such that $z_n \to 0$, but $z_n^* \not\to 0$.

5. Continuity of *-representations on topological algebras without continuous involution. Every *-representation of an involutive Banach algebra E is continuous, while if E has moreover a bai, then every positive linear form of E is also continuous (N. Th. Varopoulos [**25**]) (see also [**2**; pp. 196, 201 Theorems 3, 15 resp.]). Concerning the first result we have already mentioned in §3 its analogues when E is a (non-normed) topological algebra with continuous involution. Regarding positive linear forms in the same case, see [**15**; Theorem 2] and [**6**; Theorem 4.3]. Theorems 5.2, 5.7 of this section give conditions under which a *-representation, as well as a positive linear form of an involutive lmc algebra, is continuous, without any assumption of continuity for the involution.

For a given algebra E denote by J_E the *Jacobson radical* of E [**2**; p. 124, Definition 13]. E will be called *semisimple* if $J_E = \{0\}$ (ibid.).

5.1. LEMMA. *Let φ be a morphism of a Q lmc algebra $(E, (p_\alpha))$, $\alpha \in \mathrm{A}$, onto a semisimple lmc algebra F. Then,* $\ker(\varphi)$ *is closed and $J_E \subset \ker(\varphi)$.*

Proof. Let $x \in \overline{M}$, $M = \ker(\varphi)$. Then, $yx \in \overline{M}$ for all $y \in E$. So for every $\alpha \in \mathrm{A}$, there is $z \in M$ with $p_\alpha(yx - z) < 1$. On the other hand, since E is Q, $r_E \le p_{\alpha_0}$, for some $\alpha_0 \in \mathrm{A}$ (cf. [**24**; Corollary 1]). Consequently, there exists $z \in M$ such that $r_E(yx - z) < 1$, which by [**18**; Proposition III, 6.1 and Theorem I, 6.4] yields quasi-invertibility for $yx - z$. Thus, there is $w \in E$ with

$$(yx - z)w = w(yx - z) = (yx - z) + w.$$

Now, since $\varphi(z) = 0$, we get that $\varphi(yx) = \varphi(y)\varphi(x)$ is quasi-invertible in $F = \mathrm{Im}(\varphi)$. Hence, (cf. [**2**; Proposition 16]) $\varphi(x) \in J_F = \{0\}$. The rest of the proof goes now exactly as in [**2**; 131, Proposition 10]. $\square$

The lack of property Q from the first version of Lemma 5.1 was pointed out to the author by Professor A. Mallios.

5.2. THEOREM. *Let E be an involutive Fréchet Q lmc algebra. Then, every *-representation φ of E is continuous.*

Proof. The self-adjoint 2-sided ideal $M \equiv \ker(\varphi)$ of E is closed by [**2**; p. 195, Lemma 2] and Lemma 5.1. Hence E/M endowed with the respective quotient topology (q_n), $n \in \mathbb{N}$, becomes an involutive Fréchet Q lmc algebra. A closed graph argument shows continuity of the involution of E/M. In fact, let $(\dot{x}_n \equiv x_n + M)$ be a sequence in E/M with $\dot{x}_n \to \dot{x}_1$ and $\dot{x}_n^* \to \dot{x}_2$. Since $E/M = \mathrm{Im}(\varphi)$ algebraically, E/M becomes also a normed *-algebra with the C^*-property, equipped with the norm $\|\dot{x}\| := \|\varphi(x)\|$, $x \in E$. Thus, [**21**; Lemma (4.8.1), (ii)] and [**18**; Proposition II, 1.1] imply

$$\begin{aligned} \|\dot{x}_n^* - \dot{x}_2\|^2 &= r_{\mathscr{L}(H)}(\varphi(x_n^* - x_2)^*(x_n^* - x_2)) \\ &\leq r_{E/M}((\dot{x}_n^* - \dot{x}_2)^*(\dot{x}_n^* - \dot{x}_2)). \end{aligned}$$

Since now E/M is Q one has $r_{E/M} \leq q_m$, for some $m \in \mathbb{N}$ [**24**; Corollary 1], so that

$$\|\dot{x}_n^* - \dot{x}_2\|^2 \leq (q_m(\dot{x}_n) + q_m(\dot{x}_2^*))q_m(\dot{x}_n^* - \dot{x}_2),$$

where $(q_m(\dot{x}_n))$ is bounded [**13**; p. 135], and $q_m(\dot{x}_n^* - \dot{x}_2) \to 0$. Therefore,

$$\|\dot{x}_n^* - \dot{x}_2\| \to 0 \quad \text{and similarly} \quad \|\dot{x}_n^* - \dot{x}_1^*\| \to 0,$$

which implies $\dot{x}_2 = \dot{x}_1^*$. On the other hand, arguing as before, we get

$$\|\dot{x}\|^2 \leq q_m(\dot{x}^*)q_m(\dot{x}), \qquad \forall \dot{x} \in E/M,$$

for some $m \in \mathbb{N}$, and this yields the assertion. $\square$

5.3. COROLLARY. *Let X be a compact metrizable n-dimensional C^∞-manifold, E an involutive Banach algebra and $C^\infty(X, E)$ the involutive Fréchet lmc algebra of all E-valued C^∞-maps on X. Then, every *-representation μ of $C^\infty(X, E)$, which is of the form $\mu = \varphi \otimes \psi$, with φ, ψ *-representations of $C^\infty(X)$, E respectively, is continuous.*

Proof. $C^\infty(X, E) = C^\infty(X) \widehat{\otimes}_\tau E$ up to a topological algebraic isomorphism [**18**; p. 394, (2.8)], where τ is an admissible topology on $C^{\infty}(X) \otimes E$ [**8**; Definition 2.1]. The result follows now by [**18**; p. 134], Theorem 5.2 and [**8**; Lemma 3.2]. □

Clearly E in Corollary 5.3 can be replaced by any involutive Fréchet Q lmc algebra. Applying Theorem 5.2, as well as the reasoning of Theorem 3.3, we now get another automatic continuity result for *-morphisms between topological *-algebras, which does not require continuity of the involution for the domain of the given morphism.

5.4. COROLLARY. *Every *-morphism of an involutive Fréchet Q lmc algebra in an lmc C^*-algebra, is continuous.* □

5.5. PROPOSITION. *Let E be an involutive complete Q lmc algebra and f a positive linear form on E. Then,*

(i) *there is a *-representation φ_f of E, such that*

$$\|\varphi_f(x)\| \leq r_E(x^*x)^{1/2}, \qquad \forall x \in E.$$

(ii) *If E is moreover Fréchet and $y \in E$, the positive linear form f_y, with $f_y(x) := f(y^*xy)$, $x \in E$, is continuous.*

Proof. Apply the reasoning of [**2**; p. 198], together with [**9**; Lemma 7.1] and Theorem 5.2. □

5.6. COROLLARY. *Let E be a unital involutive Fréchet Q lmc algebra. Then, every positive linear form of E is continuous.* □

The next theorem extends to our case a known result of N. Th. Varopoulos [**25**], according to which every positive linear form of an involutive Banach algebra with a bai, is continuous. In case E in Theorem 5.7, below has a continous involution, property Q becomes redundant as this can be seen by an analogue of Varopoulos' result for Fréchet locally convex algebras with "continuous involution" and left bai, proved by P. G. Dixon [**6**; Theorem 4.3]. Continuity of the involution makes a left bai to be sufficient in this case. On the other hand, T. Husain-R. Rigelhof [**15**; Theorem 2] have proved that every positive linear form of a unital sequentially complete Q lmc *-algebra (i.e., continuity of the involution is again assumed) is continuous.

5.7. THEOREM. *Let E be an involutive Fréchet Q lmc algebra with a bai. Then, every positive linear form f of E is continuous.*

Proof. For any $x, y, z \in E$ one has

$$4xyz = (z+x^*)^*y(z+x^*) - (z-x^*)^*y(z-x^*) \\ + i(z+ix^*)^*y(z+ix^*) - i(z-ix^*)^*y(z-ix^*),$$

so that by Proposition 5.5, (ii), for fixed $x, z \in E$ the linear form

$$g: E \to \mathbb{C}: y \mapsto g(y) := f(xyz),$$

is continuous. Let now (y_n) be a null sequence of E. Then, using twice [**4**; p. 608, Theorem] (cf. also [**6**; Corollary 4.2]), we find $x, z \in E$ and a null sequence (w_n) of E such that $y_n = xw_nz$. Hence

$$\lim_n f(y_n) = \lim_n g(w_n) = 0,$$

which proves the assertion. □

5.8. Corollary. *Let E be an involutive Banach algebra (and/or an involutive Fréchet Q lmc algebra) with a bai, and X, $C^\infty(X, E)$ as in Corollary 5.3. Then, every positive linear form f of $C^\infty(X, E)$ given by $f = g \otimes h$ with g, h positive linear forms of $C^\infty(X)$, E respectively, is continuous.*

Proof. The assertion follows by Theorem 5.7, applying the reasoning in the proof of Corollary 5.3 together with [**8**; Lemma 3.1]. □

5.9. Corollary. *Let $E, X, C^\infty(X, E)$ be as in Corollary 5.8, with E being moreover commutative. Then, every positive linear form of $C^\infty(X, E)$ is continuous.*

Proof. According to the proof of Corollary 5.3, we have

$$C^\infty(X, E) = C^\infty(X) \,\widehat{\otimes}_\tau E,$$

where $C^\infty(X)$ is a (commutative) Fréchet Q lmc *-algebra [**18**; p. 134]. Thus, $C^\infty(X, E)$ is an involutive (commutative) Fréchet Q lmc algebra (ibid., Corollary XII, 2.2). Now, since moreover $C^\infty(X, E)$ has a bai (given as the tensor product of the unit of $C^\infty(X)$ with the bai of E), the assertion follows by Theorem 5.7. □

It is clear by Corollary 5.9 that every positive linear form of $C^\infty(X)$, X as before, is continuous. Of course, this result can be also taken as a consequence either of [**6**; Theorem 4.3] or [**15**; Theorem 2] (cf. comments after Corollary 5.6).

Acknowledgments. An initial form of this paper was completed while the author was visiting the Department of Mathematics, University of Maryland, College Park. She is indebted to all those from the staff of that Institution who made her stay there most comfortable and pleasant. In particular, she would like to express her thanks to Professor John Horváth for his kind attention to her research interests and his willingness for advice. Author's thanks are also due to the referee whose comments led to a more compact form of this paper.

REFERENCES

[1] R. R. Arens, *A generalization of normed rings*, Pacific J. Math., **2** (1952), 455–471.

[2] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer-Verlag, Berlin, Heidelberg, New York, 1973.

[3] R. M. Brooks, *On representing F^*-algebras*, Pacific J. Math., **39** (1971), 51–69.

[4] I. G. Craw, *Factorization in Fréchet algebras*, J. London Math. Soc., **44** (1969), 607–611.

[5] J. Dixmier, *C^*-algebras*, North-Holland, Amsterdam, 1977.

[6] P. G. Dixon, *Automatic continuity of positive functionals on topological involution algebras*, Bull. Austr. Math. Soc., **23** (1981), 265–281.

[7] M. Fragoulopoulou, *Spaces of representations and enveloping lmc *-algebras*, Pacific J. Math., **95** (1981), 61–73.

[8] ———, *Representations of tensor product lmc *-algebras*, Manuscripta Math., **42** (1983), 115–145.

[9] ———, *Symmetry and $\mathscr{P}$-commutativity of topological *-algebras*, Period. Math. Hungar., **17** (1986), 185–209.

[10] ———, *On strong spectrally bounded lmc *-algebras*, Math. Nachr., **134** (1987), 309–315.

[11] ———, *Symmetric topological *-algebras*, Period. Math. Hungar., **19** (1988), 181–208.

[12] ———, *An introduction to the representation theory of topological *-algebras*, Schriftenreihe, Univ. Münster, **48** (1988), 1–81.

[13] J. Horváth, *Topological Vector Spaces and Distributions*, vol. I, Addison-Wesley Publ. Co., Reading, Massachusetts, 1966.

[14] T. Husain, *Multiplicative Functionals on Topological Algebras*, Pitman Advanced Publ. Program, Boston, London, Melbourne, 1983.

[15] T. Husain and R. Rigelhof, *Representations of MQ^*-algebras*, Math. Ann., **180** (1969), 297–306.

[16] A. Inoue, *Locally C^*-algebras*, Mem. Faculty Sci. Kyushu Univ. (Ser. A), **25** (1971), 197–235.

[17] G. Lassner, *Topological algebras of operators*, Reports on Math. Physics, **3** (1972), 279–293.

[18] A. Mallios, *Topological Algebras. Selected Topics*, North-Holland, Amsterdam, 1986.

[19] E. A. Michael, *Locally multiplicatively-convex topological algebras*, Memoirs Amer. Math. Soc., No. 11, (1952).

[20] N. C. Phillips, *Inverse of C^*-algebras*, J. Operator Theory, **19** (1988), 159–195.

[21] C. E. Rickart, *Banach Algebras*, R. E. Krieger, Huntington, New York, 1974.
[22] Z. Sebestyén, *Every C^*-seminorm is automatically submultiplicative*, Period. Math. Hungar., **10** (1979), 1–8.
[23] W. Tiller, *P-commutative Banach *-algebras*, Trans. Amer. Math. Soc., **180** (1973), 327–336.
[24] Y. Tsertos, *Spectral radius and Q property in topological algebras*, (to appear).
[25] N. Th. Varopoulos, *Sur les formes positives d'une algèbre de Banach*, C. R. Acad. Sci. Paris, **253** (1964), 2465–2467.

Received January 27, 1989.

MATHEMATICAL INSTITUTE
UNIVERSITY OF ATHENS
PANEPISTIMIOPOLIS
ATHENS 157 84, GREECE

Pacific Journal of Mathematics
Vol. 147, No. 1 January, 1991