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**EXPLICIT  $\bar{\partial}$ -PRIMITIVES OF HENKIN-LEITERER KERNELS  
ON STEIN MANIFOLDS**

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**In this paper we construct explicitly  $\bar{\partial}$ -primitives and use them to obtain a representation formula for holomorphic functions and a theorem on extendability of CR-functions.**

**1. Introduction.** Let  $X$  be a Stein manifold of dimension  $n$ ,  $h : X \rightarrow \mathbb{C}^p$  ( $p \leq n - 1$ ) a holomorphic map and let  $Z(h) = \{\zeta \in X : h(\zeta) = 0\}$ . If  $K(\zeta, z) = K^{(s^*, \nu)}(\zeta, z)$  is a Henkin-Leiterer type kernel on  $X$  (see §2 for notation) then  $K(\zeta, z)$  is a  $\bar{\partial}$ -closed  $(n, n - 1)$ -form in  $\zeta$ , for a fixed  $z$ , i.e.,  $\bar{\partial}_\zeta K(\zeta, z) = 0$ , whose singularity occurs at  $\zeta = z$ . On the other hand, since  $X - Z(h)$  is  $(n - 2)$ -complete (see Sorani and Villani [8, p. 435]), it follows that the cohomology group

$$H^{n-1}(X - Z(h), \mathcal{O}^n) \cong H_{\bar{\partial}}^{(n, n-1)}(X - Z(h))$$

vanishes (see Andreotti and Grauert [1, p. 250]). Therefore, for a fixed  $z \in Z(h)$ , there exists an  $(n, n - 2)$ -form  $\eta(\zeta, z)$ , in  $X - Z(h)$ , so that

$$\bar{\partial}_\zeta \eta(\zeta, z) = K(\zeta, z).$$

For some problems, however, it is important to have explicit formulas for such  $\bar{\partial}$ -primitives,  $\eta$ , of  $K$ ; the problems we have in mind are related to integral representations (see for example Stout [9] and Hatziafratis [2]) and extendability of CR-functions (see for example Lupaccolu [6], Tomassini [11] and Stout [10]). Since such forms  $\eta(\zeta, z)$  are not unique, their dependence on  $z$ , for example, may be difficult to control with cohomological arguments.

In this paper we construct explicitly such  $\bar{\partial}$ -primitives and use them to obtain a representation formula for holomorphic functions and a theorem on extendability of CR-functions.

The arrangement of the paper is as follows. First in §2 we review the main points of the Henkin-Leiterer construction; with  $X$  and  $h$  as above we consider a domain  $D \subset X$ , a Stein neighborhood  $W$  of  $\bar{D}$  and we briefly discuss what a Leray section  $s^* = s^*(\zeta, z)$  and the associated Henkin-Leiterer kernel  $K(\zeta, z) = K^{(s^*, \nu)}(\zeta, z)$  are.

Then in §3 we carry out the construction of the  $\bar{\partial}$ -primitives  $\eta_h(\zeta, z)$  and in Theorem 3.1 we prove that indeed  $\bar{\partial}_\zeta \eta_h(\zeta, z) = K(\zeta, z)$  for  $\zeta \in W - Z(h - h(z))$ ,  $\zeta, z$  being always so that  $s^*(\zeta, z)$  is defined. (At this point we would like to point out that we were led to consider this construction by the paper of Laurent-Thiebaut [5] in which the case  $p = 1$  is studied.)

Our main application of this construction is a Cauchy type integral representation formula for holomorphic functions. Fix a  $z \in D$ , we consider an open set  $\Gamma \subset \partial D$  (open in  $\partial D$ ) with  $\partial\Gamma$  smooth so that  $\Gamma \supset (\partial D) \cap Z(h - h(z))$  and we prove (Theorem 3.2) that for  $f \in C(\bar{\Gamma} \cup D) \cap \mathcal{O}(D)$  we have

$$f(z) = \int_{\zeta \in \Gamma} f(\zeta) K(\zeta, z) - \int_{\zeta \in \partial\Gamma} f(\zeta) \eta_h(\zeta, z).$$

This integral formula expresses the value of  $f$  at  $z$  in terms of its values on a part of the boundary of  $D$  namely  $\bar{\Gamma}$ . In particular it provides a formula for extending CR-functions from parts of the boundary (if such extensions exist); this is the point of Theorem 4.1 in §4. This theorem gives a necessary and sufficient condition for the extendability of a CR-function  $f$  from a part of the boundary of  $D$  to a holomorphic function in  $D$ ; roughly speaking the condition says that certain integrals involving the CR-function and taken over certain cycles which lie in the domain (on  $\partial D$ ) of  $f$  should agree.

Finally with regards to the Theorem 3.2 we mention the work of Patil [7] where a different method was devised for recovering, in some cases, an  $H^2$ -function from its boundary values on a set of positive measure.

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**2. Henkin-Leiterer type kernels.** In this section we will establish notation and recall the main points of the Henkin-Leiterer construction on Stein manifolds.

Let  $X$  be a Stein manifold of dimension  $n$  and let  $T(X)$  denote its holomorphic tangent bundle with the fiber above  $z$  ( $z \in X$ ) denoted by  $T_z(X)$ . Then, following Henkin and Leiterer [4, Ch. 4], there exists a holomorphic map  $s : X \times X \rightarrow T(X)$  and a holomorphic function  $\varphi : X \times X \rightarrow \mathbb{C}$  so that

- (i)  $s(\zeta, z) \in T_z(X)$  for  $(\zeta, z) \in X \times X$ ,
- (ii)  $s(z, z) = 0$  and  $s(\cdot, z)$  is a biholomorphic map from a neighborhood of  $z \in X$  to a neighborhood of  $0 \in T_z(X) \cong \mathbb{C}^n$ ,

(iii)  $\varphi(z, z) = 1$  and there exists a positive integer  $\nu_0$  so that  $\varphi^{\nu_0}(\zeta, z)\|s(\zeta, z)\|^{-2}$  is a  $C^2$ -function on  $X \times X - \Delta = X \times X - \{(z, z) : z \in X\}$ , for any norm  $\|\cdot\|$  on  $T(X)$ ; in particular  $\varphi^\nu\|s\|^{-2}$  is of class  $C^r$  on  $X \times X - \Delta$  provided that  $\nu \geq \nu_1(r)$  for some integer  $\nu_1(r)$ .

Now fix  $D \subset X$ , a relatively compact domain in  $X$  with smooth boundary. Recall that a Leray section for  $(D, s, \varphi)$  is a  $C^1$ -map  $s^* = s^*(\zeta, z)$  defined for  $z \in D$  and for  $\zeta$  in a neighborhood of  $\partial D$ , denoted by  $\text{Dom}(s^*(\cdot, z))$  and depending on  $z$ , with values in  $T^*(X)$ , the holomorphic cotangent bundle of  $X$ , so that:

- (i)  $s^*(\zeta, z) \in T_z^*(X)$  ( $T_z^*(X)$  denotes the fiber of  $T^*(X)$  above  $z$ ),
- (ii)  $\langle s^*(\zeta, z), s(\zeta, z) \rangle \neq 0$  whenever  $\varphi(\zeta, z) \neq 0$  and
- (iii) there is an integer  $\nu^*$  so that the function

$$\varphi^{\nu^*}(\zeta, z)(\langle s^*(\zeta, z), s(\zeta, z) \rangle)^{-1}$$

is of class  $C^1$  for  $(\zeta, z) \in V \times L$ , for each compact subset  $L$  of  $D$  and where  $V$  is a neighborhood of  $\partial D$ , depending on  $L$ . Here  $\langle \cdot, \cdot \rangle$  denotes the pairing of cotangent vectors with tangent vectors.

For examples of Leray sections, which always exist in the above setting, see [4, p. 165].

To a Leray section  $s^*$ , Henkin and Leiterer associate an  $(n, n-1)$ -form in the following way:

$$K^{(s^*, \nu)}(\zeta, z) = \varphi^\nu(\zeta, z) \frac{\omega'_\zeta(s^*(\zeta, z)) \wedge \omega_\zeta(s(\zeta, z))}{\langle s^*(\zeta, z), s(\zeta, z) \rangle^n}$$

where  $\nu$  is assumed to be large enough so that  $K^{(s^*, \nu)}(\zeta, z)$  is continuous in each  $V \times L$  ( $\nu \geq n\nu^*$  is enough); the differential forms  $\omega'_\zeta(s^*(\zeta, z))$  are defined in terms of local coordinates  $(U, \chi)$  at  $z$ ; let  $(s_1, \dots, s_n)$  and  $(s_1^*, \dots, s_n^*)$  be the expressions of  $s$  and  $s^*$  in terms of the local coordinate system  $(U, \chi)$ , i.e.,

$$s(\zeta, z) = \sum_{j=1}^n s_j(\zeta, z) \left( \frac{\partial}{\partial \chi_j} \right)_z \quad \text{and} \quad s^*(\zeta, z) = \sum_{j=1}^n s_j^*(\zeta, z) (d\chi_j)_z;$$

here  $\{(\partial/\partial \chi_j)_z\}_{j=1}^n$  is the usual basis of  $T_z(X)$  with respect to  $(U, \chi)$  and  $\{(d\chi_j)_z\}_{j=1}^n$  is the corresponding basis for  $T_z^*(X)$ .

Then

$$\omega_\zeta(s(\zeta, z)) = d_\zeta s_1(\zeta, z) \wedge \dots \wedge d_\zeta s_n(\zeta, z)$$

and

$$\omega'_\zeta(s(\zeta, z)) = c_n \sum_{j=1}^n (-1)^{j-1} s_j^*(\zeta, z) \bigwedge_{k \neq j} d_\zeta s_k^*(\zeta, z)$$

where  $c_n = (-1)^{n(n-1)/2} (n-1)! / (2\pi i)^n$ .

Of course by the way  $\omega_\zeta(s(\zeta, z))$  and  $\omega'_\zeta(s^*(\zeta, z))$  are defined, they depend on the choice of the local coordinates  $(U, \chi)$ . It turns out, however, that their wedge product and therefore  $K^{(s^*, \nu)}(\zeta, z)$  are independent of the choice of local coordinates, i.e.,  $K^{(s^*, \nu)}(\zeta, z)$  is a globally defined  $(n, n-1)$ -form, see [4, p. 166].

**REMARK.** The discussion, given in §1, in which we justify by a cohomological argument the existence of  $\bar{\partial}$ -primitives,  $\eta(\zeta, z)$ , of  $K(\zeta, z) = K^{(s^*, \nu)}(\zeta, z)$ , applies for a particular class of Leray sections, the ones which are defined for  $(\zeta, z) \in X \times X$ , i.e.,  $D = X$  and  $\text{Dom}(s^*(\cdot, z)) = X$ ; the point here is that, in the general case,  $\text{Dom}(s^*(\cdot, z)) - Z(h)$  is not  $(n-2)$ -complete; however it is possible to give a cohomological argument to prove existence of the  $\bar{\partial}$ -primitives in the general case too; this argument amounts to modifying, in a way,  $s^*(\zeta, z)$  so that the argument given in §1 applies (see also the remark following the proof of Theorem 3.1 below).

**3. Construction of the  $\bar{\partial}$ -primitives.** With the notation of §2, let us consider a holomorphic map  $h : W \rightarrow \mathbb{C}^p$ ,  $p \leq n-1$ , where  $W$  is a Stein neighborhood of  $\bar{D}$ ; let  $Z(h - h(z))$  denote the zero-set of  $h - h(z)$ , i.e.,

$$Z(h - h(z)) = \{\zeta \in W : h(\zeta) = h(z)\}.$$

In this section we will construct a  $\bar{\partial}$ -primitive of  $K^{(s^*, \nu)}(\zeta, z)$  in  $W \cap \text{Dom}(s^*(\cdot, z)) - Z(h - h(z))$ ; in this construction,  $z$  is a fixed point of  $D$ ; the dependence of the construction on  $z$ , however, will be immediately clear, because of the explicit way the construction is carried out.

According to [4, Lemma 4.7.2] there exist holomorphic maps  $h_i^* : W \times W \rightarrow T^*(X)$ ,  $i = 1, \dots, p$ , so that  $h_i^*(\zeta, z) \in T_z^*(X)$  and

$$\langle h_i^*(\zeta, z), s(\zeta, z) \rangle = \varphi(\zeta, z) \cdot (h_i(\zeta) - h_i(z))$$

for  $(\zeta, z) \in W \times W$  and  $i = 1, \dots, p$ . Using such holomorphic maps  $h_i^*$  we now define a  $C^\infty$ -map  $t^* : W \times W \rightarrow T^*(X)$  in the following way:

$$t^*(\zeta, z) = \sum_{i=1}^p (\bar{h}_i(\zeta) - \bar{h}_i(z)) h_i^*(\zeta, z);$$

then it is clear that  $t^*$  is a well-defined  $C^\infty$ -map with  $t^*(\zeta, z) \in T_z^*(X)$ .

Also notice that

$$(I) \quad \langle t^*(\zeta, z), s(\zeta, z) \rangle = \varphi(\zeta, z) \sum_{i=1}^p |h_i(\zeta) - h_i(z)|^2.$$

Let

$$\begin{aligned} & \eta_h^{(s^*, \nu)}(\zeta, z) \\ &= -c'_n \varphi^{\nu-n+1} \sum_{l=0}^{n-2} \varphi^l \frac{\det[s_j^*, t_j^*, \overbrace{\partial_\zeta s_j^*}^l, \overbrace{\partial_\zeta t_j^*}^{n-l-2}] \wedge \omega_\zeta(s(\zeta, z))}{(\langle s^*, s \rangle)^{l+1} (\sum_{i=1}^p |h_i(\zeta) - h_i(z)|^2)^{n-l-1}} \end{aligned}$$

where  $c'_n = (-1)^{n(n-1)/2} (2\pi i)^{-n}$ ;  $(s_1^*, \dots, s_n^*)$  and  $(t_1^*, \dots, t_n^*)$  are the expressions of  $s^*(\zeta, z)$  and  $t^*(\zeta, z)$ , respectively, with respect to the local coordinates  $(U, \chi)$  considered in §2; let us point out that  $\omega_\zeta(s(\zeta, z))$ , in the definition of  $\eta_h^{(s^*, \nu)}(\zeta, z)$  above, is computed with respect to the same coordinates  $(U, \chi)$ ; thus if  $(s_1, \dots, s_n)$  are the expressions of  $s(\zeta, z)$  with respect to  $(U, \chi)$  then  $\omega_\zeta(s(\zeta, z)) = \partial_\zeta s_1 \wedge \dots \wedge \partial_\zeta s_n$ . In the determinants which appear in the definition of  $\eta_h^{(s^*, \nu)}$ ,  $j$  runs from  $j = 1$  to  $j = n$  forming the  $n$  rows of them.

Although the differential form  $\eta_h^{(s^*, \nu)}(\zeta, z)$  is introduced locally, it turns out that it is invariantly defined since we have

LEMMA 3.1.  $\eta_h^{(s^*, \nu)}(\zeta, z)$  is a globally defined  $(n, n-1)$ -form, i.e., it is independent of the choice of local coordinates, with  $\zeta \in W \cap \text{Dom}(s^*(\cdot, z)) - Z(h - h(z))$  and a fixed  $z \in D$ .

*Proof.* Let  $(\tilde{U}, \tilde{\chi})$  be another coordinate system at  $z$ ; let  $(\tilde{s}_1^*, \dots, \tilde{s}_n^*)$ ,  $(\tilde{t}_1^*, \dots, \tilde{t}_n^*)$  and  $(\tilde{s}_1, \dots, \tilde{s}_n)$  be the expressions of  $s^*$ ,  $t^*$  and  $s$ , respectively, with respect to  $(\tilde{U}, \tilde{\chi})$ . Then

$$\begin{aligned} (\tilde{s}_j) &= G \cdot (s_j), \\ (\tilde{s}_j^*) &= (G')^{-1} \cdot (s_j^*), \\ (\tilde{t}_j^*) &= (G')^{-1} \cdot (t_j^*), \end{aligned}$$

where  $G = G(z)$  is the transition matrix from  $(U, \chi)$  to  $(\tilde{U}, \tilde{\chi})$  for the holomorphic vector bundle  $T(X)$ , in which case  $(G')^{-1}$ , the inverse of the transpose of  $G$ , is the transition matrix from  $(U, \chi)$  to  $(\tilde{U}, \tilde{\chi})$  for the bundle  $T^*(X)$ ; of course  $G = G(z)$  depends only on  $z$ ; here  $(s_j)$  denotes the transpose of  $(s_1, \dots, s_n)$  and similarly

for the others; the dot denotes matrix multiplication. Therefore,

$$\begin{aligned}(\partial_\zeta \tilde{s}_j) &= G \cdot (\partial_\zeta s_j), \\(\bar{\partial}_\zeta \tilde{s}_j^*) &= (G')^{-1} \cdot (\bar{\partial}_\zeta s_j^*), \\(\bar{\partial}_\zeta \tilde{t}_j^*) &= (G')^{-1} \cdot (\bar{\partial}_\zeta t_j^*).\end{aligned}$$

It follows from the above relations and properties of determinants with entries differential forms (see [3, p. 94]) that

$$\det[\tilde{s}_j^*, \tilde{t}_j^*, \overbrace{\bar{\partial}_\zeta \tilde{s}_j^*}^l, \overbrace{\bar{\partial}_\zeta \tilde{t}_j^*}^{n-l-2}] = \det[(G')^{-1}] \det[s_j^*, t_j^*, \overbrace{\bar{\partial}_\zeta s_j^*}^l, \overbrace{\bar{\partial}_\zeta t_j^*}^{n-l-2}]$$

and

$$\partial_\zeta \tilde{s}_1 \wedge \cdots \wedge \partial_\zeta \tilde{s}_n = \det(G) \partial_\zeta s_1 \wedge \cdots \wedge \partial_\zeta s_n.$$

Since  $\det[(G')^{-1}] = [\det(G)]^{-1}$ , it follows that  $\eta_h^{(s^*, \nu)}(\zeta, z)$  is, indeed, independent of local coordinates. This completes the proof of the lemma.

**REMARK.** The holomorphic maps  $h_i^*$  ( $i = 1, \dots, p$ ) are by no means unique; thus the differential form  $\eta_h^{(s^*, \nu)}$  depends on the choice of  $h_i^*$ . We will come back to this point later.

**LEMMA 3.2.** *Let  $\sigma^*$  and  $\tau^*$  be defined, for  $(\zeta, z)$  with  $\varphi(\zeta, z) \neq 0$  and  $\zeta \in W \cap \text{Dom}(s^*(\cdot, z)) - Z(h - h(z))$ , as follows:*

$$\begin{aligned}\sigma^*(\zeta, z) &= (\langle s^*(\zeta, z), s(\zeta, z) \rangle)^{-1} \cdot s^*(\zeta, z) \quad \text{and} \\ \tau^*(\zeta, z) &= (\langle t^*(\zeta, z), s(\zeta, z) \rangle)^{-1} \cdot t^*(\zeta, z).\end{aligned}$$

Then

$$\eta_h^{(s^*, \nu)}(\zeta, z) = -c'_n \cdot \varphi^\nu \cdot \sum_{l=0}^{n-2} \det[\sigma_j^*, \tau_j^*, \overbrace{\bar{\partial}_\zeta \sigma_j^*}^l, \overbrace{\bar{\partial}_\zeta \tau_j^*}^{n-l-2}] \wedge \omega_\zeta(s)$$

where  $\sigma_j^*$  and  $\tau_j^*$  are the expressions of  $\sigma^*$  and  $\tau^*$  with respect to the local coordinates  $(U, \chi)$  and  $\omega_\zeta(s) = \omega_\zeta(s(\zeta, z))$  is the differential form as in the definition of  $\eta_h^{(s^*, \nu)}$  with respect to the same coordinates  $(U, \chi)$ .

*Proof.* First notice that  $\sigma^*$  and  $\tau^*$  are well-defined since  $\varphi(\zeta, z) \neq 0$  implies  $\langle s^*(\zeta, z), s(\zeta, z) \rangle \neq 0$  and together with  $\zeta \notin Z(h - h(z))$ , they imply also that  $\langle t^*(\zeta, z), s(\zeta, z) \rangle \neq 0$ ; this is because of (I). It follows from the definition of  $\sigma^*$  and  $\tau^*$  that

$$\begin{aligned}\bar{\partial}_\zeta \sigma_j^* &= (\langle s^*, s \rangle)^{-1} \bar{\partial}_\zeta s_j^* + s_j^* \bar{\partial}_\zeta [(\langle s^*, s \rangle)^{-1}] \quad \text{and} \\ \bar{\partial}_\zeta \tau_j^* &= (\langle t^*, s \rangle)^{-1} \bar{\partial}_\zeta t_j^* + t_j^* \bar{\partial}_\zeta [(\langle t^*, s \rangle)^{-1}].\end{aligned}$$

Now the lemma follows from the above equations, from (I) and properties of determinants.

We are ready now to prove that  $\eta_h^{(s^*, \nu)}$  is a  $\bar{\partial}_\zeta$ -primitive of  $K^{(s^*, \nu)}$ . More precisely we have

**THEOREM 3.1.** *Let  $D$  be a domain on the Stein manifold  $X$ ,  $\dim_{\mathbb{C}} X = n$ , and  $h : W \rightarrow \mathbb{C}^p$  a holomorphic map,  $p \leq n - 1$ , where  $W$  is a Stein neighborhood of  $\bar{D}$ . Let  $s^* = s^*(\zeta, z)$  and  $K^{(s^*, \nu)}$  be as in § 2 and let  $\eta_h^{(s^*, \nu)}$  be the above constructed differential form. Then, for a fixed  $z \in D$ , we have*

$$\bar{\partial}_\zeta \eta_h^{(s^*, \nu)}(\zeta, z) = d_\zeta \eta_h^{(s^*, \nu)}(\zeta, z) = K^{(s^*, \nu)}(\zeta, z)$$

for  $\zeta \in W \cap \text{Dom}(s^*(\cdot, z)) - Z(h - h(z))$ .

*Proof.* Let us consider first  $(\zeta, z)$  with  $\varphi(\zeta, z) \neq 0$ . Then, by the definition of  $\sigma^*$  and  $\tau^*$ ,

$$(1) \quad \langle \sigma^*, s \rangle = 1 \quad \text{and} \quad \langle \tau^*, s \rangle = 1.$$

Working always with a fixed coordinate system  $(U, \chi)$  at  $z$ , (1) can be written as

$$(2) \quad \sum_{j=1}^n \sigma_j^* s_j = 1 \quad \text{and} \quad \sum_{j=1}^n \tau_j^* s_j = 1.$$

It follows from (2) that  $s_j \neq 0$  for at least one  $j \in \{1, \dots, n\}$ . We may assume, without loss of generality, that  $s_1 \neq 0$ . Then, by Lemma 3.2,

$$(3) \quad \eta_h^{(s^*, \nu)} = -\frac{c'_n}{s_1} \varphi^\nu \sum_{l=0}^{n-2} \det \begin{bmatrix} \sigma_1^* s_1 & \tau_1^* s_1 & \overbrace{\partial(\sigma_1^* s_1)}^l & \overbrace{\partial(\tau_1^* s_1)}^{n-l-2} \\ \sigma_j^* & \tau_j^* & \overline{\partial} \sigma_j^* & \overline{\partial} \tau_j^* \end{bmatrix} \wedge \omega_\zeta(s);$$

in the determinants in (3)  $j$  runs from  $j = 2$  to  $j = n$  forming the 2nd up to the  $n$ th row of them. In obtaining (3) we also used the fact that  $s_1 = s_1(\zeta, z)$  is holomorphic in  $\zeta$  (throughout this proof  $\bar{\partial} = \bar{\partial}_\zeta$ ). Next, multiplying the  $j$ th-rows of each determinant in (3) ( $2 \leq j \leq n$ ) by  $s_j$  and adding them to the first row of it we obtain, in view of (2),

$$(4) \quad \eta_h^{(s^*, \nu)} = -\frac{c'_n}{s_1} \varphi^\nu \sum_{l=0}^{n-2} \det \begin{bmatrix} 1 & 1 & \overbrace{0}^l & \overbrace{0}^{n-l-2} \\ \sigma_j^* & \tau_j^* & \overline{\partial} \sigma_j^* & \overline{\partial} \tau_j^* \end{bmatrix} \wedge \omega_\zeta(s).$$

Applying  $\bar{\partial} = \bar{\partial}_\zeta$  to both sides of (4) and using the fact that  $\varphi$  is holomorphic in  $\zeta$ , we obtain

$$\begin{aligned} \bar{\partial}\eta_h^{(s^*, \nu)} = & -\frac{c'_n}{s_1}\varphi^\nu \sum_{l=0}^{n-2} \left( \det \begin{bmatrix} 0 & 1 & \overbrace{0}^l & \overbrace{0}^{n-l-2} \\ \bar{\partial}\sigma_j^* & \tau_j^* & \bar{\partial}\sigma_j^* & \bar{\partial}\tau_j^* \end{bmatrix} \wedge \omega_\zeta(s) \right. \\ & \left. + \det \begin{bmatrix} 1 & 0 & \overbrace{0}^l & \overbrace{0}^{n-l-2} \\ \sigma_j^* & \bar{\partial}\tau_j^* & \bar{\partial}\sigma_j^* & \bar{\partial}\tau_j^* \end{bmatrix} \wedge \omega_\zeta(s) \right) \end{aligned}$$

or, after a computation,

$$\begin{aligned} (5) \quad \bar{\partial}\eta_h^{(s^*, \nu)} = & -\frac{c'_n}{s_1}\varphi^\nu \sum_{l=0}^{n-2} (\det[\overbrace{\bar{\partial}\sigma_j^*}^l, \overbrace{\bar{\partial}\tau_j^*}^{n-l-1}] - \det[\overbrace{\bar{\partial}\sigma_j^*}^{l+1}, \overbrace{\bar{\partial}\tau_j^*}^{n-l-2}]) \wedge \omega_\zeta(s) \\ = & \frac{c'_n}{s_1}\varphi^\nu \det[\overbrace{\bar{\partial}\sigma_j^*}^{n-1}] \wedge \omega_\zeta(s) - \frac{c'_n}{s_1}\varphi^\nu \det[\overbrace{\bar{\partial}\tau_j^*}^{n-1}] \wedge \omega_\zeta(s); \end{aligned}$$

all the determinants in (5) are  $(n-1) \times (n-1)$  and  $j$  runs from  $j=2$  to  $j=n$  forming their  $(n-1)$  rows. Now we claim that

$$(6) \quad \frac{c'_n}{s_1}\varphi^\nu \det \left( \left[ \overbrace{\bar{\partial}\sigma_j^*}^{n-1} \right]_{j=2}^n \right) \wedge \omega_\zeta(s) = K^{(s^*, \nu)}$$

and

$$(7) \quad \frac{1}{s_1} \det \left[ \overbrace{\bar{\partial}\tau_j^*}^{n-1} \right]_{j=2}^n = 0.$$

First let us prove (6). It follows from the definition of  $K^{(s^*, \nu)}$  and the relations between  $s_j^*$  and  $\sigma_j^*$  (exactly as in the proof of Lemma 2.2) that

$$K^{(s^*, \nu)} = \frac{c'_n\varphi^\nu}{s_1} \det \left( \left[ \begin{array}{c} \sigma_1^* s_1 \\ \sigma_j^* \end{array} \quad \overbrace{\bar{\partial}(\sigma_1^* s_1)}^{n-1} \right]_{j=2}^n \right) \wedge \omega_\zeta(s).$$

Therefore, in view of (2),

$$K^{(s^*, \nu)} = \frac{c'_n\varphi^\nu}{s_1} \det \left( \left[ \begin{array}{c} 1 \\ \sigma_j^* \end{array} \quad \overbrace{0}^{n-1} \right]_{j=2}^n \right) \wedge \omega_\zeta(s)$$

which immediately implies (6).

Similarly, to prove (7) we write its left-hand side (in view of the relation between  $\tau^*$  and  $t^*$ ) as follows:

$$(8) \quad \frac{1}{s_1} \det[\overbrace{\bar{\partial}\tau_j^*}^{n-1}] = (\langle t^*, s \rangle)^{-n} \det([t_j^*, \overbrace{\bar{\partial}t_j^*}^{n-1}]_{j=1}^n).$$

Let  $h_{ij}^*$  ( $1 \leq i \leq p$ ,  $1 \leq j \leq n$ ) be the expressions of  $h_i^*$  with respect to the local coordinates  $(U, \chi)$ , i.e.,

$$h_i^*(\zeta, z) = \sum_{j=1}^n h_{ij}^*(\zeta, z)(d\chi_j)_z.$$

Recalling that  $t^* = \sum_{i=1}^p (\bar{h}_i - \bar{h}_i(z))h_i^*$  we obtain

$$(9) \quad t_j^* = \sum_{i=1}^p (\bar{h}_i - \bar{h}_i(z))h_{ij}^* \quad \text{and} \quad \bar{\partial}t_j^* = \sum_{i=1}^p h_{ij}^* \bar{\partial}\bar{h}_i,$$

since  $h_{ij}^*$  are holomorphic in  $\zeta$ . Now to prove (7) we distinguish two cases:

1st case:  $p \leq n - 2$ ; in this case

$$(10) \quad \bar{\partial}t_{j_1} \wedge \cdots \wedge \bar{\partial}t_{j_{n-1}} = 0$$

for  $1 \leq j_1 < \cdots < j_{n-1} \leq n$ ; this follows from (9); but (10) and (8) imply (7) in this case.

2nd case:  $p = n - 1$ ; in this case, substituting (9) into the right-hand side of (8), we obtain

$$(11) \quad \det[t_j^*, \overbrace{\bar{\partial}t_j^*}^{n-1}]_{j=1}^n = p! \det \left( \left[ \sum_{i=1}^p (h_i - h_i(z))h_{ij}^*, h_{1j}^*, \dots, h_{pj}^* \right]_{j=1}^n \right) \times \overline{\partial h_1} \wedge \cdots \wedge \overline{\partial h_p} = 0;$$

since (11) and (8) imply (7), the proof of (7) is complete. Finally (7), (6) and (5) imply the formula of the theorem in the case  $\varphi(\zeta, z) \neq 0$  and, since the set  $\{\varphi(\zeta, z) \neq 0\}$  is dense, this completes the proof of the theorem.

REMARK. As we pointed out before,  $\eta_h^{(s^*, \nu)}$  depends on the choice of  $\{h_i^*\}_{i=1}^p$ ; in the case  $p \leq n - 2$ , however, this dependence is not essential in a sense which we will make precise now.

Let  $[\eta_h^{(s^*, \nu)}]$  denote the cohomology class of  $\eta_h^{(s^*, \nu)}$  in the Dolbault cohomology group  $H_{\bar{\partial}}^{(n, n-2)}(V_z - Z(h - h(z)))$  where  $V_z$  is an open neighborhood of  $\partial D$  with  $\bar{V}_z \subset W \cap \text{Dom}(s^*(\cdot, z))$  (here  $z$  is fixed, as usual, and  $\zeta$  is the variable).

Let  $(h_i^*)' : W \times W \rightarrow T^*(X)$ ,  $i = 1, \dots, p$ , be holomorphic maps, with  $(h_i^*)'(\zeta, z) \in T_z^*(X)$  and  $\langle (h_i^*)', s \rangle = \varphi \cdot (h_i - h_i(z))$ , i.e., another choice for  $h_i^*$  and let  $(\eta_h^{(s^*, \nu)})'$  denote the  $\bar{\partial}$ -primitive of  $K^{(s^*, \nu)}$  in  $W \cap \text{Dom}(s^*(\cdot, z)) - Z(h - h(z))$  associated to  $(h_i^*)'$ . We claim that

$$[\eta_h^{(s^*, \nu)}] = [(\eta_h^{(s^*, \nu)})'];$$

in other words, the cohomology class  $[\eta_h^{(s^*, \nu)}]$  is independent of the choice of  $h_i^*$ . To prove this we argue as follows. Let  $\psi(\zeta, z)$  be a  $C^\infty$  function with  $0 \leq \psi(\zeta, z) \leq 1$ , having compact support contained in  $W \cap \text{Dom}(s^*(\cdot, z))$ , which is identically one in a neighborhood of  $\bar{V}_z$ . Let  $\bar{s}(\zeta, z)$  denote a Leray section for  $(D, s, \varphi)$  with  $\text{Dom}(\bar{s}(\cdot, z)) = W$  and defined for  $z \in W$ ; such a Leray section always exists (see [4, p. 164]; let us point out that  $\bar{s}(\zeta, z)$  is not the complex conjugate of  $s(\zeta, z)$ ). Define

$$\lambda^*(\zeta, z) = \varphi^{\nu_1}(\zeta, z) \left[ \frac{\psi(\zeta, z)}{\langle s^*(\zeta, z), s(\zeta, z) \rangle} s^*(\zeta, z) + \frac{1 - \psi(\zeta, z)}{\langle \bar{s}(\zeta, z), s(\zeta, z) \rangle} \bar{s}(\zeta, z) \right]$$

where  $\nu_1 = \max(\nu_0, \nu^*)$ . Since

$$\langle \lambda^*(\zeta, z), s(\zeta, z) \rangle = \varphi^{\nu_1}(\zeta, z),$$

it follows that  $\lambda^*$  is a Leray section for  $(D, s, \varphi)$ ; thus we may associate to  $\lambda^*$  the  $\bar{\partial}$ -primitives  $\eta_h^{(\lambda^*, \nu)}$  and  $(\eta_h^{(\lambda^*, \nu)})'$  of  $K^{(\lambda^*, \nu)}$ , in  $W - Z(h - h(z))$ , corresponding to  $h_i^*$  and  $(h_i^*)'$ . It follows from Theorem 3.1 that

$$\bar{\partial}(\eta_h^{(\lambda^*, \nu)} - (\eta_h^{(\lambda^*, \nu)})') = K^{(\lambda^*, \nu)} - K^{(\lambda^*, \nu)} = 0 \quad \text{in } W - Z(h - h(z));$$

but  $W - Z(h - h(z))$  is  $(n - 3)$ -complete (here  $p \leq n - 2$ ; see [7, p. 435]) whence  $H^{n-2}(W - Z(h - h(z)), \mathcal{O}^n) = 0$  (see [1, p. 250]); therefore, from Dolbault's theorem, there exists an  $(n, n - 3)$ -form  $\theta$  in  $W - Z(h - h(z))$  with

$$\eta_h^{(\lambda^*, \nu)} - (\eta_h^{(\lambda^*, \nu)})' = \bar{\partial}\theta.$$

Since  $\psi \equiv 1$  in a neighborhood of  $\bar{V}_z$ , it follows from the proof of Lemma 3.2 that

$$\begin{aligned} \eta_h^{(\lambda^*, \nu)} &= \eta_h^{(s^*, \nu)} \quad \text{and} \\ (\eta_h^{(\lambda^*, \nu)})' &= (\eta_h^{(s^*, \nu)})' \quad \text{in } V_z - Z(h - h(z)) \end{aligned}$$

whence

$$\eta_h^{(s^*, \nu)} - (\eta_h^{(s^*, \nu)})' = \bar{\partial}\theta \quad \text{in } V_z - Z(h - h(z)).$$

This proves the claim that the cohomology class  $[\eta_h^{(s^*, \nu)}]$  does not depend on the choice of  $h_i^*$ . Notice also that if  $\Gamma$  is an open subset of  $\partial D$  with  $\bar{\Gamma} \subset \partial D - Z(h - h(z))$  and  $f$  is a smooth CR-function on  $\Gamma$  then

$$f\eta_h^{(s^*, \nu)} - f(\eta_h^{(s^*, \nu)})' = \bar{\partial}(f\theta) = d(f\theta)$$

whence we obtain

$$\int_c f\eta_h^{(s^*, \nu)}(\cdot, z) = \int_c f(\eta_h^{(s^*, \nu)})'(\cdot, z)$$

for every  $(2n - 2)$ -dimensional cycle  $c$  in  $\Gamma$ .

The following theorem is a generalization of the Henkin-Leiterer version of the Cauchy-Fantappiè formula; its proof is similar to the proof of Proposition 2.4 in [6, p. 185].

**THEOREM 3.2.** *Let  $D$  be a domain on the Stein manifold  $X$  and let  $h, W, K^{(s^*, \nu)}$  and  $\eta_h^{(s^*, \nu)}$  be as in Theorem 3.1. Let  $z \in D$  and let  $\Gamma \subset \partial D$  be an open subset of  $\partial D$  with  $\bar{\partial}\Gamma$  smooth and so that  $\Gamma \supset (\partial D) \cap Z(h - h(z))$ . Then for  $f \in C(\bar{\Gamma} \cup D) \cap \mathcal{O}(D)$ , i.e., continuous on  $\bar{\Gamma} \cup D$  and holomorphic in  $D$ , we have the following representation formula:*

$$f(z) = \int_{\zeta \in \Gamma} f(\zeta)K^{(s^*, \nu)}(\zeta, z) - \int_{\zeta \in \partial\Gamma} f(\zeta)\eta_h^{(s^*, \nu)}(\zeta, z).$$

*Proof.* Let  $G \subset D$  be an open subset of  $D$  so that  $\partial G \cap \partial D = \Gamma$  and  $D \cap Z(h - h(z)) \subset G$ . We also assume that  $\partial G = \Gamma \cup \Gamma_0$  where  $\Gamma_0 = \partial G \cap \bar{D} \subset W$ . Then, by [4, Theorem 4.3.4], we have

$$\begin{aligned} (1) \quad f(z) &= \int_{\partial G} fK^{(s^*, \nu)}(\cdot, z) \\ &= \int_{\Gamma} fK^{(s^*, \nu)}(\cdot, z) + \int_{\Gamma_0} fK^{(s^*, \nu)}(\cdot, z). \end{aligned}$$

Since  $\Gamma_0 \subset W - Z(h - h(z))$  it follows from Theorem 3.1, Stokes's theorem and the fact that  $f$  is holomorphic in  $D$  that

$$(2) \quad \int_{\Gamma_0} fK^{(s^*, \nu)}(\cdot, z) = \int_{\partial\Gamma_0} f\eta_h^{(s^*, \nu)}(\cdot, z) \\ = - \int_{\partial\Gamma} f\eta_h^{(s^*, \nu)}(\cdot, z).$$

Now the formula of the theorem follows from (1) and (2).

**REMARK.** If  $\Gamma = \partial D$  then  $\partial\Gamma = \emptyset$  and the formula of Theorem 3.2 reduces to that of [4, Theorem 4.3.4].

**4. Extending CR-functions.** Let  $(D, s, \varphi)$ ,  $W$  and  $s^*$  be as in §3; we assume furthermore that  $s^*(\zeta, z)$  is defined, as a Leray section, for all  $(\zeta, z) \in W \times W$ . Let  $E$  be a closed subset of  $\partial D$  so that each connected component of  $\partial D - E$  contains a peak point for  $\mathcal{O}(\bar{D})$ , i.e., a point  $\zeta_0$  for which there exists a  $g \in \mathcal{O}(\bar{D})$  with  $|g(\zeta_0)| > |g(\zeta)|$  for  $\zeta \in \bar{D} - \{\zeta_0\}$ . For each  $z \in W - E$  let

$$\mathcal{P}_z = \{h : W \rightarrow \mathbb{C}^{n-2} : h \text{ holomorphic, } z \in Z(h) \text{ and } Z(h) \cap E = \emptyset\}.$$

We can now state a criterion for extendability of CR-functions defined on  $\partial D - E$ ; a version of it in  $\mathbb{C}^n$ , with the Bochner-Martinelli kernel in place of the Henkin-Leitner type kernel, is in [2]; its proof is based on ideas from [6] and [5].

**THEOREM 4.1.** *With notation as above, suppose that  $\mathcal{P}_z \neq \emptyset$  for each  $z \in W - E$  and let  $f$  be a smooth CR-function on  $\partial D - E$ . Then a necessary and sufficient condition that  $f$  extends to a holomorphic function in  $D$  is*

$$(1) \quad \int_{\zeta \in \partial\Gamma} f(\zeta)\eta_h^{(s^*, \nu)}(\zeta, z) = \int_{\zeta \in \partial\Gamma} f(\zeta)\eta_g^{(s^*, \nu)}(\zeta, z)$$

for  $h, g \in \mathcal{P}_z$ ,  $\Gamma \supset (\partial D) \cap (Z(h) \cup Z(g))$  open (in  $\partial D$ ) with  $\bar{\Gamma} \subset \partial D - E$  and  $\partial\Gamma$  smooth and  $z \in W - E$ .

**REMARKS.** (i) Of course  $\eta_h^{(s^*, \nu)}$  and  $\eta_g^{(s^*, \nu)}$  are  $\bar{\partial}$ -primitives of  $K^{(s^*, \nu)}$  in  $W \cap \text{Dom}(s^*(\cdot, z)) - Z(h - h(z))$  and  $W \cap \text{Dom}(s^*(\cdot, z)) - Z(g - g(z))$  respectively. As we pointed out before, in the remark following the proof of Theorem 3.1, given  $\Gamma$ , the value of the integral

$$\int_{\zeta \in \partial\Gamma} f(\zeta)\eta_h^{(s^*, \nu)}(\zeta, z)$$

is uniquely determined by  $h$ , i.e., it is independent of the choice of  $h_i^*$ .

(ii) Observe that if  $\Gamma'$  has the properties required for  $\Gamma$  then, by Theorem 3.1 and Stokes's theorem,

$$\begin{aligned} & \int_{\zeta \in \partial \Gamma'} f(\zeta) \eta_h^{(s^*, \nu)}(\zeta, z) - \int_{\zeta \in \partial \Gamma} f(\zeta) \eta_h^{(s^*, \nu)}(\zeta, z) \\ &= \int_{\zeta \in (\Gamma' - \Gamma) \cup (\Gamma - \Gamma')} f(\zeta) K^{(s^*, \nu)}(\zeta, z) \end{aligned}$$

with the various parts of  $(\Gamma' - \Gamma) \cup (\Gamma - \Gamma')$  appropriately oriented; therefore if (1) holds for  $\Gamma$  it will also hold for  $\Gamma'$ .

*Proof of Theorem 4.1.* First the necessity of (1) follows immediately from Theorem 3.2.

Now we prove sufficiency of (1), i.e., we assume that (1) holds and we prove that  $f$  extends to a holomorphic function in  $D$ . To this end let  $z \in W - \partial D$  and let  $h \in \mathcal{P}_z$ ; choose  $\Gamma$  and define

$$(2) \quad F(z) = \int_{\zeta \in \Gamma} f(\zeta) K^{(s^*, \nu)}(\zeta, z) - \int_{\zeta \in \partial \Gamma} f(\zeta) \eta_h^{(s^*, \nu)}(\zeta, z).$$

Condition (1) now guarantees that  $F(z)$  is well-defined, i.e., it is independent of the various choices (basically of the choice of  $h$ , in view of the previous remarks). Next we prove that  $F$  is holomorphic; for this we compute  $\bar{\partial}_z F$ .

$$(3) \quad \bar{\partial}_z F(z) = \int_{\zeta \in \Gamma} f(\zeta) \bar{\partial}_z K^{(s^*, \nu)}(\zeta, z) - \int_{\zeta \in \partial \Gamma} f(\zeta) \bar{\partial}_z \eta_h^{(s^*, \nu)}(\zeta, z).$$

This computation is justified, in part, by the explicit formula for  $\eta_h^{(s^*, \nu)}$ ; notice that if  $h \in \mathcal{P}_z$  then  $h - h(z') \in \mathcal{P}_{z'}$  for  $z'$  close to  $z$ ; thus, in (3),

$$\bar{\partial}_z \eta_h^{(s^*, \nu)}(\zeta, z) = \bar{\partial}_{z'} \eta_h^{(s^*, \nu)}(\zeta, z')|_{z'=z};$$

the point here is that  $h$ , too, depends on  $z$ . But

$$(4) \quad \bar{\partial}_z K^{(s^*, \nu)}(\zeta, z) = \bar{\partial}_\zeta \tilde{K}(\zeta, z)$$

where

$$\tilde{K}(\zeta, z) = -(n-1)c'_n \varphi^\nu \frac{\det[s_j^*, \bar{\partial}_z s_j^*, \overbrace{\bar{\partial}_\zeta s_j^*}^{n-2}]}{(s^*, s)^n} \wedge \omega_\zeta(s)$$

(with the notation of §3; in particular we make use of a local coordinate system  $(U, \chi)$  as in §3; the independence of  $\tilde{K}(\zeta, z)$  of the

choice of  $(U, \chi)$  is proved exactly as Lemma 3.1); this is proved in [3, p. 107];  $\tilde{K}(\zeta, z)$  is defined for  $z \in W$  and  $\zeta \in W - \{z\}$ .

But, by Theorem 3.1,

$$(5) \quad \bar{\partial}_\zeta(\bar{\partial}_z \eta_h^{(s^*, \nu)}) = \bar{\partial}_z K^{(s^*, \nu)}(\zeta, z).$$

It follows from (4) and (5) that

$$\bar{\partial}_\zeta[\bar{\partial}_z \eta_h^{(s^*, \nu)} - \tilde{K}] = 0 \quad \text{in } W - Z(h).$$

Since  $W - Z(h)$  is  $(n - 3)$ -complete, it follows that there exists an  $(n, n - 3)$ -form  $\mu(\zeta, z)$  in  $\zeta$ , whose coefficients are  $(0, 1)$ -forms in  $z$  (locally in  $(U, \chi)$ ), so that

$$(6) \quad \bar{\partial}_z \eta_h^{(s^*, \nu)} - \tilde{K} = \bar{\partial}_\zeta \mu \quad \text{in } W - Z(h)$$

(this argument is similar to the remark following the proof of Theorem 3.1).

But (3), in view of (4) and (6), becomes:

$$\bar{\partial}_z F = \int_\Gamma f \bar{\partial}_\zeta \tilde{K} - \int_{\partial\Gamma} f \tilde{K} - \int_{\partial\Gamma} f \bar{\partial}_\zeta \mu,$$

from which, by Stokes's theorem and the fact that  $f$  is a CR-function we obtain  $\bar{\partial}_z F = 0$ ; thus  $F$  is holomorphic in  $W - \partial D$ . An argument similar to that in [6, pp. 188–190] proves that  $F = 0$  in  $W - \bar{D}$  and that  $F|_D$  is indeed a holomorphic extension of  $f$ . For the Plemelj type formula in the setting of Stein manifolds, which is required here, see [5].

This completes the proof of Theorem 4.1.

*Comments.* (i) The point of using the differential form  $\tilde{K}$  in the proof of Theorem 4.1 is that, although  $\eta_h^{(s^*, \nu)}$  is not defined on  $Z(h)$ ,  $\bar{\partial}_z \eta_h^{(s^*, \nu)}$  is  $\bar{\partial}_\zeta$ -cohomologous to  $\tilde{K}$  in a neighborhood of  $\partial\Gamma$ , and  $\tilde{K}$  is defined in  $W - \{z\}$ .

(ii) A point which may be investigated further is to find geometric conditions under which equality (1) holds; for example, if  $\dim_{\mathbb{C}}(Z(h) \cap Z(g)) \geq 1$ , does it follow that (1) holds?

(iii) If  $h, q \in \mathcal{P}_z$  and  $h_1 = \cdots = h_{n-2}$  and  $g_1 = \cdots = g_{n-2}$  ( $n \geq 3$ ) then the difference  $\eta_h^{(s^*, \nu)} - \eta_g^{(s^*, \nu)}$  is  $\bar{\partial}$ -exact in  $W - (Z(h) \cup Z(g))$  (this is proved in [5]), which implies that (1) holds in this case.

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