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Let X be a compact PL manifold and Q denote the Hilbert cube I^ω. In this paper, we show that the following subgroups of the homeomorphism group $H(X \times Q)$ of $X \times Q$ are manifolds:

$$H^{\mathrm{fd}}(X \times Q) = \{h \times \mathrm{id} \mid h \in H(X \times I^n) \text{ for some } n \in \mathbb{N}\},$$
$$H^{\mathrm{PL}}(X \times Q) = \{h \times \mathrm{id} \in H^{\mathrm{fd}}(X \times Q) \mid h \text{ is } PL\} \text{ and}$$
$$H^{\mathrm{LIP}}(X \times Q) = \text{ all Lipschitz homomorphisms of } X \times Q \text{ under some suitably chosen metric}.$$

In fact, let $\widetilde{H}^*(X \times Q)$ denote the subspace consisting of those homeomorphisms which are isotopic to a member of $H^*(X \times Q)$, where $* = \mathrm{fd}$, PL or LIP respectively. Then it is shown that

(1) $(\widetilde{H}^{\mathrm{PL}}(X \times Q), H^{\mathrm{PL}}(X \times Q))$ is an (l_2, l_2^f)-manifold pair,

(2) $(\widetilde{H}^{\mathrm{LIP}}(X \times Q), H^{\mathrm{LIP}}(X \times Q))$ is an (l_2, l_2^Q)-manifold pair and

(3) $H^{\mathrm{fd}}(X \times Q)$ is an $(l_2 \times l_2^f)$-manifold and dense in $\widetilde{H}^{\mathrm{fd}}(X \times Q)$,

where l_2 is the separable Hilbert space, $l_2^f = \{(x_i) \in l_2 \mid x_i = 0$ except for finitely many $i\}$ and $l_2^Q = \{(x_i) \in l_2 \mid \sup |i \cdot x_i| < \infty\}$.

0. Introduction. By $H(X)$, we denote the homeomorphism group of a compactum X onto itself with the compact-open topology. Let $Q = I^\omega$ be the Hilbert cube and l_2 the separable Hilbert space. A separable manifold modeled on Q or l_2 is called a *Q-manifold* or *l_2-manifold*, respectively. By the combined works of **[Ge$_1$]**, **[To$_1$]** and **[Fe]** or **[To$_2$]**, it was shown that $H(M)$ is an l_2-manifold for a compact Q-manifold M. In this paper, we concern ourselves with subgroups of $H(M)$ which are manifolds.

Let l_2^f and l_2^Q be the linear spans of the natural orthonormal basis of l_2 and the Hilbert cube $\prod_{n\in\mathbb{N}}[-1/n, 1/n]$ in l_2, respectively, that is

$$l_2^f = \{x \in l_2 \mid x(n) = 0 \text{ except for finitely many } n\},$$
$$l_2^Q = \left\{x \in l_2 \mid \sup_{n\in\mathbb{N}} |n \cdot x(n)| < \infty\right\},$$

where $x(n)$ denotes the nth coordinate of x. A separable manifold modeled on l_2^f or l_2^Q is called an *l_2^f-manifold* or *l_2^Q-manifold*, respectively. A pair (M, N) is called an *(l_2, l_2^f)-manifold pair* or *(l_2, l_2^Q)-manifold pair* if (M, N) is locally homeomorphic to the pair (l_2, l_2^f) or (l_2, l_2^Q), that is, there exist an open cover $\mathscr{U}$ of M and open embeddings $\varphi_U : U \to l_2$, $U \in \mathscr{U}$, such that $\varphi_U(U) \cap l_2^f = \varphi_U(U \cap N)$ or $\varphi_U(U) \cap l_2^Q = \varphi_U(U \cap N)$, respectively. It is well known that a pair (M, N) is an (l_2, l_2^f)-manifold pair or (l_2, l_2^Q)-manifold pair if and only if M is an l_2-manifold and N is an fd-cap set or cap set for M, respectively. For the definition and results of (fd-)cap sets, we refer to **[Ch$_1$]**.

Let X be a compact Euclidean polyhedron with $\dim X > 0$ (i.e., $X \subset \mathbb{R}^n$ for some $n \in \mathbb{N}$ and X inherits the Euclidean metric). By $H^{\mathrm{PL}}(X)$ and $H^{\mathrm{LIP}}(X)$, we denote the subspaces of $H(X)$ consisting of all PL homeomorphisms and all Lipschitz homeomorphisms, respectively. Then $H^{\mathrm{PL}}(X) \subset H^{\mathrm{LIP}}(X)$ (see **[LV]**). In case $\dim X = 1$ or 2, $(H(X), H^{\mathrm{PL}}(X))$ is an (l_2, l_2^f)-manifold pair and $(H(X), H^{\mathrm{LIP}}(X))$ is an (l_2, l_2^f)-manifold pair (**[GH]** and **[SW$_1$]**). In general, $H^{\mathrm{PL}}(X)$ is not dense in $H(X)$ as mentioned in the Introduction of **[GH]**. It is not known whether $H(X)$ is an l_2-manifold when $\dim X \geq 3$ even if X is a manifold. (This is referred to as the Homeomorphism Group Problem.) However, by the combined works of **[Ge$_2$]**, **[Ga]**, **[Ha]**, **[To$_1$]** and **[KW]** (cf. **[SW$_1$]**), $H^{\mathrm{PL}}(X)$ is an l_2^f-manifold. Let $\widetilde{H}^{\mathrm{PL}}(X)$ denote the subspace of $H(X)$ consisting of those homeomorphisms which are isotopic to PL homeomorphisms. Given a compact PL manifold $M \subset \mathbb{R}^n$ with $\dim M \neq 4$ which inherits the Euclidean metric and we assume $\partial M = \varnothing$ in case $\dim M = 5$. Suppose that $H(M)$ is an l_2-manifold, then $(\widetilde{H}^{\mathrm{PL}}(M), H^{\mathrm{PL}}(M))$ is an (l_2, l_2^f)-manifold pair **[GH]** and $(H(M), H^{\mathrm{LIP}}(M))$ is an (l_2, l_2^Q)-manifold pair **[SW$_1$]**.

For any compact polyhedron X, $X \times Q$ is a compact Q-manifold. Conversely, each compact Q-manifold M is homeomorphic ($\cong$) to such a product, where X can be a compact PL manifold. For further properties on Q-manifolds, we refer to **[Ch$_3$]** or **[Mi]**. For each $n \in \mathbb{N}$, we write $Q = I^n \times Q_{n+1}$. For a compact Euclidean polyhedron X, we regard $H(X \times I^n) \subset H(X \times Q)$ by identifying $h \in H(X \times I^n)$ with $h \times \mathrm{id} \in H(X \times Q)$. Let

$$H^{\mathrm{fd}}(X \times Q) = \bigcup_{n \in \mathbb{N}} (X \times I^n) \subset H(X \times Q)$$

and

$$H^{\mathrm{PL}}(X \times Q) = \bigcup_{n \in \mathbb{N}} (X \times I^n) \subset H^{\mathrm{fd}}(X \times Q).$$

Let d_X be a metric for X inherited from the Euclidean space $\mathbb{R}^n$ which contains X. For any $p \in \mathbb{N} \cup \{\infty\}$, by choosing a suitable sequence $a_i > 0$, $i \in \mathbb{N}$, we have a natural affine homeomorphism of Q onto $\prod_{i \in \mathbb{N}} [0, a_i] \subset l_p$. Let d_Q be a metric for Q induced by such an embedding and the norm of l_p. Let d be a metric for $X \times Q$ which is Lipschitz equivalent to a metric

$$d_{X \times Q}((x, y), (x', y')) = \max\{d_X(x, x'), d_Q(y, y')\}.$$

Let $H^{\mathrm{LIP}}(X \times Q)$ be the space of Lipschitz homeomorphisms of $X \times Q$ with respect to such a metric. It is natural to conjecture that these subgroups of $H(X \times Q)$ are manifolds.

By $\widetilde{H}^*(X \times Q)$, we denote the subspace of $H(X \times Q)$ consisting of those homeomorphisms which are isotopic to a member of $H^*(X \times Q)$, where $* = \mathrm{fd}$, PL or LIP. Then each $\widetilde{H}^*(X \times Q)$ is an l_2-manifold. In case X is a PL manifold,

$$\widetilde{H}^{\mathrm{PL}}(X \times Q) \subset \widetilde{H}^{\mathrm{fd}}(X \times Q) \subset \widetilde{H}^{\mathrm{LIP}}(X \times Q).$$

Indeed, the second inclusion follows from [**Su**$_2$, Corollary 3]. However in general, we do not know about equality and whether $\widetilde{H}^*(X \times Q) = H(X \times Q)$. It may be true that $\widetilde{H}^{\mathrm{PL}}(X \times Q) = \widetilde{H}^{\mathrm{fd}}(X \times Q)$ even if $\bigcup_{n \in \mathbb{N}} \widetilde{H}^{\mathrm{PL}}(X \times I^n) \neq H^{\mathrm{fd}}(X \times Q)$. In fact, by the result of Kirby and Siebenmann [**KS**], there is an $h \in H(S^2 \times S^3)$ such that $h \times \mathrm{id} \notin \widetilde{H}^{\mathrm{PL}}(S^2 \times S^3 \times I^n)$ for any I^n (cf. [**Ki**]). Therefore

$$\bigcup_{n \in \mathbb{N}} \widetilde{H}^{\mathrm{PL}}(S^2 \times S^3 \times I^n) \neq H^{\mathrm{fd}}(S^2 \times S^3 \times Q).$$

However by the results of [**Su**$_1$] and [**Ch**$_2$], $\widetilde{H}^{\mathrm{PL}}(S^2 \times S^3 \times Q) = \widetilde{H}^{\mathrm{fd}}(S^2 \times S^3 \times Q)$.

Here are statements of our main results:

THEOREM I. *For a compact polyhedron X, there is a homeomorphism*

$$\varphi : H(X \times Q) \times l_2 \to H(X \times Q)$$

such that

$$\varphi(H^{\mathrm{fd}}(X \times Q) \times l_2) = H^{\mathrm{fd}}(X \times Q),$$
$$\varphi(H^{\mathrm{PL}}(X \times Q) \times l_2^f) = H^{\mathrm{PL}}(X \times Q) \quad \text{and}$$
$$\varphi(H^{\mathrm{LIP}}(X \times Q) \times l_2^Q) = H^{\mathrm{LIP}}(X \times Q).$$

THEOREM II. *For each compact polyhedron X, $H^{\mathrm{PL}}(X \times Q)$ is an l_2^f-manifold. Moreover if X is a* PL *manifold, then the pair $(\widetilde{H}^{\mathrm{PL}}(X \times Q), H^{\mathrm{PL}}(X \times Q))$ is an (l_2, l_2^f)-manifold pair.*

THEOREM III. *For each compact* PL *manifold $X \subset \mathbb{R}^n$, the pair $(\widetilde{H}^{\mathrm{LIP}}(X \times Q), H^{\mathrm{LIP}}(X \times Q))$ is an (l_2, l_2^Q)-manifold pair. Hence $H^{\mathrm{LIP}}(X \times Q)$ is an l_2^Q-manifold.*

THEOREM IV. *For each compact* PL *manifold X, $H^{\mathrm{fd}}(X \times Q)$ is an $(l_2 \times l_2^f)$-manifold and dense in $\widetilde{H}^{\mathrm{fd}}(X \times Q)$.*

We should remark that $H^{\mathrm{PL}}(X\times Q)$ and $H^{\mathrm{fd}}(X\times Q)$ are not dense in $\widetilde{H}^{\mathrm{PL}}(X\times Q)$ and $\widetilde{H}^{\mathrm{fd}}(X\times Q)$ respectively for a compact polyhedron X; hence the second half of Theorem II is not true for a polyhedron. In fact, let T be the simple triad, that is,

$$T = [-1, 1] \times \{0\} \cup \{0\} \times [0, 1] \subset \mathbb{R}^2.$$

Since $T\times Q \cong Q$ and $H(Q)$ is contractible, $\widetilde{H}^{\mathrm{PL}}(T\times Q) = H(T\times Q)$. Each $h \in H^{\mathrm{fd}}(T \times Q)$ must leave the set $\{(0, 0)\} \times Q$ invariant. By homogeneity of $T \times Q$, there is a $g \in H(T \times Q)$ such that $g(0, 0, 0, \dots) = (1, 0, 0, \dots)$. Clearly g cannot be approximated by any $h \in H^{\mathrm{fd}}(X \times Q)$.

As one of incomplete infinite-dimensional manifolds, $(l_2 \times l_2^f)$-manifolds were studied by Bestvina and Mogilski **[BM]**. Theorem IV provides a natural setting for such manifolds.

1. Proof of Theorem I. In this section we shall prove Theorem I. The following proof is a modification of a construction in **[SW$_1$]**. First we note that the metric d defined in §0 is compatible with the piecewise linear structure on $X \times Q$. Thus Lemma 1.3 in **[SW$_1$]** is valid for $X \times Q$. The rest of the argument is parallel to the finite-dimensional case of **[SW$_1$]**. So we only outline the proof as follows:

Let $s = (-1, 1)^\omega$, $\Sigma = \{z \in s \mid \sup|z(i)| < 1\}$ and $\sigma = \{z \in s \mid z(i) = 0$ except for finitely many $i\}$. Let A denote the space of arcs in $X \times Q$ parametrized by the interval $[-1, 1]$ (i.e., embeddings of

$[-1, 1]$ into $X \times Q$). Since Lemma 1.3 in [$\mathbf{SW}_1$] is valid for $X \times Q$ as mentioned above, we have a map $\tau : A \to (-1, 1)$ satisfying

(1) $\tau(f) = t$ if and only if $\mu(f|[-1, t]) = \mu(f|[t, 1])$, and

(2) $\tau(f) = 0$ if f is linear,

where $\mu(f)$ denotes the Morse's μ-length of an arc (or a path) f with respect to the metric d defined in §0 (cf. [**Mo**] or [$\mathbf{Ge}_1$]). Let B be a rectilinear convex cell in X with $\dim B = \dim X$. By Lemma 1.5 in [$\mathbf{SW}_1$], we have a map

$$\xi : (s, \Sigma, \sigma) \to (H(B), H^{\mathrm{LIP}}(B), H^{\mathrm{PL}}(B))$$

such that for each $z \in s$ and $i \in \mathbb{N}$,

(3) $\xi(z) = \mathrm{id}$ on a neighborhood of the boundary of B,

(4) $\xi(z)(\alpha_i([-1, 0])) = \alpha_i([-1, z_i])$ and

(5) $\xi(z)(\alpha_i([0, 1])) = \alpha_i([z_i, 1])$,

where $\alpha_i \in A(B)$, $i \in \mathbb{N}$, are suitable linear arcs with pairwise disjoint images. We identify $X = X \times \{0\} \subset X \times Q$ and regard $H(X) \subset H(X \times Q)$ by identifying $h = h \times \mathrm{id}$ for each $h \in H(X)$. As in the proof of Theorem 1.1 in [$\mathbf{SW}_1$], let $F : s \to H(X) \subset H(X \times Q)$ be the map defined by $F(z)|B = \xi(z)$ and $F(z)|X \setminus B = \mathrm{id}$. Then $F(\sigma) \subset H^{\mathrm{PL}}(X) \subset H^{\mathrm{PL}}(X \times Q)$ and $F(\Sigma) \subset H^{\mathrm{LIP}}(X) \subset H^{\mathrm{LIP}}(X \times Q)$. We define a map $\widetilde{T} : H(X \times Q) \to s$ by $\widetilde{T}(h) = (\tau(h \circ \alpha_i))_{i \in \mathbb{N}}$, which is a natural extension of T in [$\mathbf{SW}_1$]. Using the same argument as [$\mathbf{SW}_1$], we can show that $\widetilde{T}(H^{\mathrm{PL}}(X \times Q)) \subset \sigma$ and $\widetilde{T}(H^{\mathrm{LIP}}(X \times Q)) \subset \Sigma$. Let

$$H_0(X \times Q) = \widetilde{T}^{-1}(0), \quad H_0^{\mathrm{fd}}(X \times Q) = H^{\mathrm{fd}}(X \times Q) \cap \widetilde{T}^{-1}(0),$$
$$H_0^{\mathrm{PL}}(X \times Q) = H^{\mathrm{PL}}(X \times Q) \cap \widetilde{T}^{-1}(0) \quad \text{and}$$
$$H_0^{\mathrm{LIP}}(X \times Q) = H^{\mathrm{LIP}}(X \times Q) \cap \widetilde{T}^{-1}(0).$$

Then as shown in [$\mathbf{SW}_1$], $h \circ F\widetilde{T}(h) \in H_0(X \times Q)$ for each $h \in H(X \times Q)$. Thus we can define a map $G : H(X \times Q) \to H_0(X \times Q)$ by $G(h) = h \circ F\widetilde{T}(h)$. Similarly as in [$\mathbf{SW}_1$], we can show that the map

$$P : H(X \times Q) \to H_0(X \times Q) \times s$$

defined by $P(h) = (G(h), \widetilde{T}(h))$ is a homeomorphism and its inverse is given by $P^{-1}(h, z) = h \circ F(z)^{-1}$. Then

$$P(H^{\mathrm{fd}}(X \times Q)) = H_0^{\mathrm{fd}}(X \times Q) \times s.$$

In fact, $G(h) = h \circ F\widetilde{T}(h) \in H_0^{\mathrm{fd}}(X \times Q)$ for each $h \in H^{\mathrm{fd}}(X \times Q)$.

Hence $P(H^{\mathrm{fd}}(X \times Q)) \subset H_0^{\mathrm{fd}}(X \times Q) \times s$. Since $F(z) \in H(X) \subset H^{\mathrm{fd}}(X \times Q)$ for each $z \in s$, $P^{-1}(H_0^{\mathrm{fd}}(X \times Q) \times s) \subset H^{\mathrm{fd}}(X \times Q)$. Moreover we have

$$G(H^{\mathrm{PL}}(X \times Q)) = H_0^{\mathrm{PL}}(X \times Q) \quad \text{and}$$
$$G(H^{\mathrm{LIP}}(X \times Q)) = H_0^{\mathrm{LIP}}(X \times Q).$$

In fact, $\widetilde{T}(h) \in \sigma$ if $h \in H^{\mathrm{PL}}(X \times Q)$ and $\widetilde{T}(h) \in \Sigma$ if $h \in H^{\mathrm{LIP}}(X \times Q)$. It follows that $G(h) = h \circ F\widetilde{T}(h) \in H_0^{\mathrm{PL}}(X \times Q)$ for each $h \in H^{\mathrm{PL}}(X \times Q)$ and $G(h) = h \circ F\widetilde{T}(h) \in H_0^{\mathrm{LIP}}(X \times Q)$ for each $h \in H^{\mathrm{LIP}}(X \times Q)$. Hence we have

$$P(H^{\mathrm{PL}}(X \times Q)) \subset H_0^{\mathrm{PL}}(X \times Q) \times \sigma \quad \text{and}$$
$$P(H^{\mathrm{LIP}}(X \times Q)) \subset H_0^{\mathrm{LIP}}(X \times Q) \times \Sigma.$$

Since $F(z) \in H^{\mathrm{PL}}(X) \subset H^{\mathrm{PL}}(X \times Q)$ for each $z \in \sigma$ and $F(z) \in H^{\mathrm{LIP}}(X) \subset H^{\mathrm{LIP}}(X \times Q)$ for each $z \in \Sigma$, we have

$$P^{-1}(H_0^{\mathrm{PL}}(X \times Q) \times \sigma) \subset H^{\mathrm{PL}}(X \times Q) \quad \text{and}$$
$$P^{-1}(H_0^{\mathrm{LIP}}(X \times Q) \times \sigma) \subset H^{\mathrm{LIP}}(X \times Q).$$

Thus P is a homeomorphism of the triple

$$(H(X \times Q), H^{\mathrm{LIP}}(X \times Q), H^{\mathrm{PL}}(X \times Q))$$

onto the triple

$$(H_0(X \times Q) \times s, H_0^{\mathrm{LIP}}(X \times Q) \times \Sigma, H_0^{\mathrm{PL}}(X \times Q) \times \sigma).$$

Since $(s \times s, \Sigma \times \Sigma, \sigma \times \sigma) \cong (s, \Sigma, \sigma) \cong (l_2, l_2^Q, l_2^f)$ ([**SW**$_2$]), we can obtain the desired homeomorphism φ as required. □

2. Local contractibility of $H^{\mathrm{fd}}(X \times Q)$ and $H^{\mathrm{PL}}(X \times Q)$. In this section, we shall prove the following

2.1. THEOREM. *For any compact polyhedron X, there is a neighborhood $\mathscr{N}$ of* id *in $H(X \times Q)$ and a homotopy $\varphi : \mathscr{N} \times I \to H(X \times Q)$ such that*

(i) $\varphi(h) = h$ *and* $\varphi(h, 1) = \mathrm{id}$ *for each* $h \in \mathscr{N}$,
(ii) $\varphi(\mathrm{id}, t) = \mathrm{id}$ *for each* $t \in I$,
(iii) $\varphi((\mathscr{N} \cap H^{\mathrm{fd}}(X \times Q)) \times I) \subset H^{\mathrm{fd}}(X \times Q)$ *and*
(iv) $\varphi((\mathscr{N} \cap H^{\mathrm{PL}}(X \times Q)) \times I) \subset H^{\mathrm{PL}}(X \times Q)$.

2.2. COROLLARY. *For any compact polyhedron X, $H(X \times Q)$, $H^{\mathrm{fd}}(X \times Q)$ and $H^{\mathrm{PL}}(X \times Q)$ are locally contractible.* □

Since our proof is an adaptation to known techniques, we will provide only an outline here. The proof is based on the following two lemmas. In Lemma 2.3, we contract the Q-coordinates of $X \times Q$ by using the coordinate switching technique in **[Wo₁]**, and in Lemma 2.4, we contract the X-coordinates by imitating Gauld's proof of the local contractibility of $H(X)$ as in **[Ga]**.

Let $p: X \times Q \to X$ and $q: X \times Q \to Q$ denote the projections. For each $n \in \mathbb{N}$, we denote the projections of $Q = I^n \times Q_{n+1}$ onto I^n and Q_{n+1} by p_n and q_{n+1}, respectively. For each $n \in \mathbb{N} \cup \{0\}$ and $\varepsilon > 0$, let

$$H_{n,\varepsilon} = \{h \in H(X \times Q) | p_n \circ q \circ h = p_n \circ q,\ d_X(p, p \circ h) < \varepsilon\},$$
$$H^{\mathrm{fd}}_{n,\varepsilon} = H_{n,\varepsilon} \cap H^{\mathrm{fd}}(X \times Q) \quad \text{and} \quad H^{\mathrm{PL}}_{n,\varepsilon} = H_{n,\varepsilon} \cap H^{\mathrm{PL}}(X \times Q),$$

where $p_0 : Q \to \{0\}$ is the constant map. Then $H_{0,\varepsilon}$ is a neighborhood of id in $H(X \times Q)$.

2.3. LEMMA. *Given any $n \in \mathbb{N} \cup \{0\}$ and $\varepsilon > 0$, there is a homotopy $\varphi : H_{n,\varepsilon} \times I \to H_{n,\varepsilon}$ such that*

(i) *$\varphi(h, 0) = h$ and $\varphi(h, 1) \in H_{n+1,\varepsilon}$ for each $h \in H_{n,\varepsilon}$,*
(ii) *$\varphi_t(\mathrm{id}) = \mathrm{id}$ for each $t \in I$,*
(iii) *$\varphi(H^{\mathrm{fd}}_{n,\varepsilon} \times I) \subset H^{\mathrm{fd}}_{n,\varepsilon}$ and $\varphi(H^{\mathrm{PL}}_{n,\varepsilon} \times I) \subset H^{\mathrm{PL}}_{n,\varepsilon}$.*

Proof. We can define a pseudo-isotopy $\theta: X \times Q \times I \times I \to X \times Q$ by $\theta_0(x, y, s) = (x, y)$ and by connecting the following homeomorphisms using the coordinate switching isotopies of **[Wo]**:

$$\theta_1(x, y, s) = (x, y(1), \dots, y(n), s, -y(n+1), -y(n+2), \dots),$$
$$\theta_{1/2}(x, y, s) = (x, y(1), \dots, y(n), y(n+1), s, -y(n+2), -y(n+3), \dots),$$
$$\theta_{1/3}(x, y, s) = (x, y(1), \dots, y(n), y(n+1), y(n+2), s, -y(n+3), -y(n+4), \dots)$$
$$\dots .$$

Then for each $t > 0$, θ_t is a homeomorphism and θ_t^{-1} is continuous with respect to t. Now we define $\varphi: H(X \times Q) \times I \to H(X \times Q)$ as follows:

$$\varphi(h, t) = \begin{cases} \theta_t \circ (h \times \mathrm{id}_I) \circ \theta_t^{-1} & \text{if } t > 0, \\ h & \text{if } t = 0. \end{cases}$$

It is easy to see that φ is the desired map. □

In **[Ga]**, the wrapping and unwrapping procedures of Edwards (cf. **[EK]**) are employed to show that $H(X)$ is locally contractible for any compact polyhedron X. The process preserves the piecewise linearity of homeomorphisms, so it also provides another proof of local contractibility of $H^{\mathrm{PL}}(X)$. By crossing everything with Q, the same proof can be routinely modified to obtain the following modification of the first step in the proof of **[Ga**, Theorem 1]. The details are left to the reader.

2.4. LEMMA. *Let* $\dim X = m$. *Then for each* $n \in \mathbb{N} \cup \{0\}$ *and sufficiently small* $\varepsilon > 0$, *there is a map* $\psi\colon H_{n,\varepsilon} \times I \to H_{n,6^m\varepsilon}$ *such that*

(i) $\psi(h, 0) = h$ *and* $\psi(h, 1) \in H_{n,\varepsilon/2}$ *for each* $h \in H_{n,\varepsilon}$,
(ii) $\psi_t(\mathrm{id}) = \mathrm{id}$ *for each* $t \in I$,
(iii) $\psi(H^{\mathrm{fd}}_{n,\varepsilon} \times I) \subset H^{\mathrm{fd}}_{n,6^m\varepsilon}$ *and* $\psi(H^{\mathrm{PL}}_{n,\varepsilon} \times I) \subset H^{\mathrm{PL}}_{n,6^m\varepsilon}$. □

Proof of Theorem 2.1. By using Lemmas 2.3 and 2.4 in that order, we can construct a homotopy $\varphi\colon H_{0,\varepsilon} \times I \to H(X \times Q)$ for some sufficiently small $\varepsilon > 0$ so that for each $h \in H_{0,\varepsilon}$,

$$\begin{aligned}
&\varphi_0(h) = h,\\
&\varphi_{1/2}(h) \in H_{1,\varepsilon}, \quad \varphi_{2/3}(h) \in H_{1,\varepsilon/2},\\
&\varphi_{3/4}(h) \in H_{2,\varepsilon/2}, \quad \varphi_{4/5}(h) \in H_{2,\varepsilon/3},\\
&\dots\\
&\varphi_1(h) = \mathrm{id}.
\end{aligned}$$

Then φ is the desired homotopy. □

3. Density of $H^{\mathrm{fd}}(X \times Q)$ **in** $\widetilde{H}^{\mathrm{fd}}(X \times Q)$. This section contains the main idea of the paper. Lemma 3.6 is the key lemma which shows that $H^{\mathrm{fd}}(Q)$ is dense in $H(Q)$. As demonstrated in §0, $H^{\mathrm{fd}}(X \times Q)$ need not be dense in $\widetilde{H}^{\mathrm{fd}}(X \times Q)$ for a compact polyhedron X even if $X \times Q \cong Q$. But it is true (Theorem 3.8) that $H^{\mathrm{fd}}(X \times Q)$ is dense in $\widetilde{H}^{\mathrm{fd}}(X \times Q)$ for any finite-dimensional compact manifold X. This is our main result.

Before we start, we recall the definition of Z-sets. A closed set A in a space M is called a Z-*set* in M if for each open cover $\mathscr{U}$, there is a map $f\colon M \to M \setminus A$ which is $\mathscr{U}$-close to id, that is, each pair of points $x \in M$ and $f(x)$ are contained in some $U \in \mathscr{U}$. In case M is an ANR, a closed set A in M is a Z-set if each map $f\colon Q \to M$ is approximated by maps $g\colon Q \to M \setminus A$. For basic results of Z-sets, we refer to **[Ch₃]**. In the following, let $\overline{H}^{\mathrm{fd}}(X \times Q)$ denote the closure of $H^{\mathrm{fd}}(X \times Q)$ in $H(X \times Q)$.

3.1. LEMMA. *Each homeomorphism $h: A \to B$ between Z-sets in Q extends to an $\tilde{h} \in \overline{H}^{\mathrm{fd}}(Q)$.*

Proof. The proof is a modification of the standard proof of the Homeomorphism Extension Theorem (the Z-sets Unknotting Theorem). In particular, we refer to Chapter II of **[Ch$_3$]** for details. First we give several elementary results. Their proofs are rather straightforward and will be omitted.

(a) If $f, g \in \overline{H}^{\mathrm{fd}}(Q)$, so are f^{-1} and $g \circ f$.
(b) If $f_n \in \overline{H}^{\mathrm{fd}}(Q)$, $n \in \mathbb{N}$, and $f = \lim_{n\to\infty} f_n \circ \cdots \circ f_1 \in H(Q)$, then $f \in \overline{H}^{\mathrm{fd}}(Q)$.
(c) Given $Q = \prod_{n\in\mathbb{N}} Q_n$, where each Q_n is a copy of Q, and $f_n \in \overline{H}^{\mathrm{fd}}(Q_n)$, $n \in \mathbb{N}$, then $f = \prod_{n\in\mathbb{N}} f_n \in \overline{H}^{\mathrm{fd}}(Q)$.
(d) Given $Q = Q_1 \times Q_2$, where Q_1 and Q_2 are copies of Q, and $h \in H(Q) = H(Q_1 \times Q_2)$ defined by

$$h(x, y)(n) = \begin{cases} (x, (1 + k(x)(n)) \cdot y(n) + k(x)(n)) & \text{if } -1 \le y(n) \le 0, \\ (x(1 - k(x)(n)) \cdot y(n) + k(x)(n)) & \text{if } 0 \le y(n) \le 1, \end{cases}$$

where $k: Q_1 \to Q_2$ is a map, then $h \in \overline{H}^{\mathrm{fd}}(Q)$.

We observe that all the homeomorphisms employed in proving the Homeomorphism Extension Theorem in **[Ch$_3$]** can be chosen in $\overline{H}^{\mathrm{fd}}(Q)$ as follows: First we easily see that $h \in H(Q)$ obtained in **[Ch$_3$**, 6.1**]** belongs to $\overline{H}^{\mathrm{fd}}(Q)$ by (b) and then so does $h \in H(Q)$ in **[Ch$_3$**, 6.2**]** by (c). Thus in **[Ch$_3$**, 7.1**]**, h can be extended to a homeomorphism of $\overline{H}^{\mathrm{fd}}(Q)$ by (a) and (d). It follows from (a) that $h \in H(Q)$ in **[Ch$_3$**, 9.1**]** belongs to $\overline{H}^{\mathrm{fd}}(Q)$. Finally we can see that $h \in H(Q)$ obtained in **[Ch$_3$**, 10.1**]** belongs to $\overline{H}^{\mathrm{fd}}(Q)$ by (b) and then so does $h \in H(Q)$ in **[Ch$_3$**, 10.2**]**. Thus Lemma 3.1 follows. □

Let $A \subset X$ and $a, b: X \to \mathbb{R}$ be maps of X to $\mathbb{R}$ such that $a < b$, that is, $a(x) < b(x)$ for all $x \in X$. We denote

$$[a, b]_A = \{(t, x) \in \mathbb{R} \times A | a(x) \le t \le b(x)\},$$
$$[a, b)_A = \{(t, x) \in \mathbb{R} \times A | a(x) \le t < b(x)\}, \text{ etc.}$$

In case a and b are constant,

$$[a, b]_A = [a, b] \times A, \quad [a, b)_A = [a, b) \times A, \text{ etc.}$$

3.2. LEMMA. *Let $a, c: Q \to I$ be maps, A a closed set in Q and V an open set in $I \times Q$ such that $a < c$ and $[0, a]_Q \cup [0, c]_A \subset V$. Then there is a map $b: Q \to I$ such that $a < b$ and*

$$[0, c]_A \subset [0, b)_Q \subset [0, b]_Q \subset V.$$

Proof. This is an elementary consequence of a Urysohn map. □

In the following lemmas, let $q: Q \times I \to Q$ denote the projection.

3.3. LEMMA. *Let A be a Z-set in Q, $g \in H(I \times Q)$, $\varepsilon > 0$ and let $a, b, c, d: Q \to I$ be maps such that $a < b < d$ and $a < c < d$. Then there exists an $f \in H(I \times Q)$ such that*

$$gfg^{-1} \in \overline{H}^{\mathrm{fd}}(I \times Q), \quad f(I \times A) = I \times A, \quad f([0, b)_Q) \supset [0, c]_A, \\ f|[0, a]_Q \cup [d, 1]_Q = \mathrm{id} \quad \textit{and} \quad d(qf, q) < \varepsilon.$$

Proof. Let $a', b', c', d': Q \to I$ be maps such that $a < a' < b' < b$ and $c < c' < d' < d$. Let

$$\delta = \min\left\{\varepsilon/2, \inf_{x \in Q}(a'(x) - a(x)), \inf_{x \in Q}(d(x) - d'(x))\right\} > 0.$$

Since $I \times A$ is a Z-set in $I \times Q$, we have $g' \in \overline{H}^{\mathrm{fd}}(I \times Q)$ such that $g'|I \times A = g|I \times A$ by Lemma 3.1. There are neighborhoods $V \subset U$ of A in Q such that $d(g|I \times U, g'|I \times U) < \delta$, $g'^{-1}g(I \times V) \subset I \times U$ and $g^{-1}g'(I \times Q) \subset I \times Q$. Let $b'': Q \to I$ be a map such that $b''|A = c'$ and $b''|Q \setminus V = b'$. Then we have $f' \in H(I \times Q)$ such that $f'|[0, a']_Q \cup [d', 1]_Q = \mathrm{id}$ and f' maps each $[a', b']_{\{x\}}$ and $[b', d']_{\{x\}}$ onto $[a', b'']_{\{x\}}$ and $[b'', d']_{\{x\}}$ linearly. Then f' is Q-preserving, that is, $qf' = q$. It is straightforward to see that $f' \in \overline{H}^{\mathrm{fd}}(I \times Q)$. Observe that $f'|I \times (Q \setminus V) = \mathrm{id}$. The composition

$$f = g^{-1}g'f'g'^{-1}g \in H(I \times Q)$$

is the desired homeomorphism. In fact, $gfg^{-1} = g'f'g'^{-1} \in \overline{H}^{\mathrm{fd}}(I \times Q)$. Obviously $f(I \times A) = I \times A$ and

$$f([0, b)_Q) \supset g^{-1}g'g'g'^{-1}g([0, b']_A) = f'([0, b']_A) \\ = [0, b'']_A = [0, c']_A \supset [0, c)_A.$$

Since $f'|I \times (Q \setminus V) = \mathrm{id}$ and

$$g'^{-1}g(I \times (Q \setminus U)) \subset I \times (Q \setminus V),$$

we have $f|I \times (Q \setminus U) = \mathrm{id}$ and also $f'g'^{-1}g(I \times U) = g'^{-1}g(I \times U)$. It follows that

$$\begin{aligned}
&d(qf|I \times U, q|I \times U) \\
&\quad = d(qg^{-1}g'f'g'^{-1}g|I \times U, q|I \times U) \\
&\quad \leq d(qg^{-1}g'|g'^{-1}g(I \times U), q|g'^{-1}g(I \times U)) \\
&\qquad + d(qf'|g'^{-1}g(I \times U), q|g'^{-1}g(I \times U)) \\
&\qquad + d(qg'^{-1}g|I \times U, q|I \times U) \\
&\quad \leq 2 \cdot d(g'^{-1}g|I \times U, \mathrm{id}) < 2\delta < \delta .
\end{aligned}$$

Therefore $d(qf, q) < \varepsilon$. It follows that

$$g'^{-1}g([0, a]_U \cup [d, 1]_U) \subset [0, a']_Q \cup [d', 1]_Q,$$

which implies $f|[0, a]_U \cup [d, 1]_U = \mathrm{id}$. Hence $f|[0, a]_Q \cup [d, 1]_Q = \mathrm{id}$. This completes the proof of the lemma. □

3.4. LEMMA. *Let $g \in H(I \times Q)$, $\varepsilon > 0$ and let $a, b, c, d: Q \to I$ be maps such that $a < b < d$ and $a < c < d$. Then there exists an $f \in H(I \times Q)$ such that*

$$gfg^{-1} \in \overline{H}^{\mathrm{fd}}(I \times Q), \quad f([0, b)_Q) \supset [0, c]_Q,$$
$$f|[0, a]_Q \cup [d, 1]_Q = \mathrm{id} \quad \textit{and} \quad d(qf, q) < \varepsilon .$$

Proof. Let $a', c': Q \to I$ be maps such that $a < a' < b$ and $c < c' < d$. By Lemma 3.3, we have $f_1 \in H(I \times Q)$ such that

$$gf_1g^{-1} \in \overline{H}^{\mathrm{fd}}(I \times Q), \quad f_1([0, b)_Q) \supset [0, c']_{I \times \{0\}},$$
$$f_1|[0, a']_Q \cup [d, 1]_Q = \mathrm{id} \quad \text{and} \quad d(qf_1, q) < 2^{-2}\varepsilon .$$

By Lemma 3.2, we have maps $a_1, b_1: Q \to I$ such that

$$a' < a_1 < b_1 < d \quad \text{and}$$
$$[0, c']_{I \times \{0\}} \subset [0, a_1)_Q \subset [0, b_1]_Q \subset g^{-1}f_1g([0, b)_Q).$$

Let U_1 be an open neighborhood of $I \times \{0\}$ in Q such that

$$[0, c']_{I \times \{0\}} \subset [0, c']_{U_1} \subset [0, a_2)_Q .$$

Again by Lemma 3.3, we have $f_2' \in H(I \times Q)$ such that

$$gf_2'g^{-1} \in \overline{H}^{\mathrm{fd}}(I \times Q), \quad f_2'([0, b_1)_Q) \supset [0, c']_{I^2\times\{0\}},$$
$$f_2'|[0, a_1]_Q \cup [d, 1]_Q = \mathrm{id} \quad \text{and} \quad d(qf_2', q) < 2^{-3}\varepsilon.$$

By Lemma 3.2, we have maps $a_1, b_2 \colon Q \to I$ such that

$$a_1 < a_2 < b_2 < d \quad \text{and}$$
$$[0, c']_{I^2\times\{0\}} \subset [0, a_2)_Q \subset [0, b_2]_Q \subset g^{-1}f_2g([0, b_1)_Q).$$

Let $f_2 = f_2'f_1 \in H(I \times Q)$ and let U_2 be an open neighborhood of $I^2 \times \{0\}$ in Q such that

$$[0, c']_{I^2\times\{0\}} \subset [0, c']_{U_2} \subset [0, a_2)_Q.$$

Then it follows

$$gf_2g^{-1} \in \overline{H}^{\mathrm{fd}}(I \times Q), \quad f_2([0, b)_Q) \supset [0, c']_{U_1\cup U_2},$$
$$f_2|[0, a']_Q \cup [d, 1]_Q = \mathrm{id} \quad \text{and} \quad d(qf_2, q) < (2^{-2} + 2^{-3}) \cdot \varepsilon.$$

Thus by using Lemmas 3.2 and 3.3 inductively, we obtain open sets U_n in Q and $f_n \in H(I \times Q)$, $n \in \mathbb{N}$, such that

$$I^n \times \{0\} \subset U_n, \quad gf_ng^{-1} \in \overline{H}^{\mathrm{fd}}(I \times Q),$$
$$f_n([0, b)_Q) \supset [0, c']_{U_1\cup\cdots\cup U_n},$$
$$f_n|[0, a']_Q \cup [d, 1]_Q = \mathrm{id} \quad \text{and}$$
$$d(qf_n, q) < (2^{-2} + \cdots + 2^{-1-n}) \cdot \varepsilon.$$

Then $A = Q \setminus \bigcup_{n\in\mathbb{N}} U_n$ is a Z-set in Q. By Lemma 3.3, we have an $h \in H(I \times Q)$ such that

$$ghg^{-1} \in \overline{H}^{\mathrm{fd}}(I \times Q), \quad h(I \times A) = I \times A,$$
$$h([0, a')_Q) \supset [0, c]_A, \quad h|[0, a]_Q \cup [c, 1]_Q = \mathrm{id} \quad \text{and}$$
$$d(qh, q) < 2^{-1}\varepsilon.$$

Let V be an open neighborhood of A in Q such that

$$[0, c]_A \subset [0, c]_V \subset h([0, a')_Q).$$

Since $h([0, c]_A) = [0, c]_A$ and $h|[c, 1]_Q = \mathrm{id}$,

$$[0, c]_{Q\setminus V} \subset [0, c]_{Q\setminus A} = h([0, c]_{Q\setminus A})$$
$$\subset h\left([0, c')_{\bigcup_{n\in\mathbb{N}} U_n}\right) = \bigcup_{n\in\mathbb{N}} h([0, c')_{U_1\cup\cdots\cup U_n}).$$

From compactness,

$$[0, c]_{Q\setminus V} \subset h([0, c')_{U_1\cup\cdots\cup U_n}) \subset hf_n([0, b)_Q)$$

for some $n \in \mathbb{N}$. Since $f_n|[0, a']_Q = \mathrm{id}$,

$$[0, c]_V \subset h([0, a')_Q) = hf_n([0, a')_Q) \subset hf_n([0, b)_Q).$$

Thus we have $[0, c]_Q \subset hf_n([0, b)_Q)$. We observe that

$$gfg^{-1} = (ghg^{-1}) \circ (gf_m g^{-1}) \in \overline{H}^{\mathrm{fd}}(I \times Q) \quad \text{and}$$
$$d(qf, q) \le d(qh, q) + d(qf_n, q) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore the composition $f = hf_n \in H(I \times Q)$ is the desired homeomorphism. □

By applying a similar argument as in the proof of [**Co**, Theorem 1], the following follows from Lemma 3.4.

3.5. Lemma. *Let $g \in H(I\times Q)$ and $h \in H(I\times Q)$ such that $qh = q$ and $ph \ge p$. Then $ghg^{-1} \in \overline{H}^{\mathrm{fd}}(I \times Q)$.*

Proof. We will provide only an outline. Given any $\varepsilon > 0$, we choose a partition $0 = t_0 < t_1 < \cdots < t_{m+1} = 1$ of I so that $t_{i+1} - t_{i-1} < \varepsilon$ for each $i = 1, \dots, m$. For each $i = 1, \dots, m$, let $u_i: Q \to (0, 1)$ be a map defined by $h^{-1}(t_i, x) = (u_i(x), x)$ (i.e., $h(u_i(x), x) = (t_i, x)$). Note that $0 = u_0 < u_1 < \cdots < u_{m+1} = 1$ and $u_i \le t_i$ for each $i = 1, \dots, m$. By applying Lemma 3.4 repeatedly, we can obtain an $f \in H(I \times Q)$ such that $gfg^{-1} \in \overline{H}^{\mathrm{fd}}(I \times Q)$, $d(qf, q) < \varepsilon$ and $[0, t_i]_Q \subset f([0, u_i)_Q) \subset [0, t_{i+1})_Q$ for each $i = 1, \dots, m$, which implies $d(f, \mathrm{id}) < \varepsilon$. □

For $A \subset M$, let

$$H_A(M) = \{h \in H(M) |\ h|A = \mathrm{id}\}.$$

The following is the key lemma which is a special case of Theorem 3.8.

3.6. Lemma. *$H^{\mathrm{fd}}_{\{1\}\times Q}(I \times Q)$ is dense in $H_{\{1\}\times Q}(I \times Q)$.*

Proof. For simplicity, we denote $F = \{1\} \times Q$. Let

$$H' = \bigcup\{H_U(I \times Q) | U \text{ is a neighborhood of } F \text{ in } Q\} \text{ and}$$
$$H'' = \{h \in H' | \forall g \in H',\ \ g^{-1}hg \in \mathrm{cl}\, H^{\mathrm{fd}}_F(I \times Q)\}.$$

Then H' is dense in $H_F(I \times Q)$, H'' is a normal subgroup of H' and $H'' \subset \mathrm{cl}\, H^{\mathrm{fd}}_F(I \times Q)$. By [**Fi**] (cf. [**Wo**$_2$, Theorem 6]), H' is a simple subgroup of $H_F(I \times Q)$. Since $H'' \neq \{\mathrm{id}\}$ (Lemma 3.5), we have $H'' = H'$, which implies that $H_F^{\mathrm{fd}}(I \times Q)$ is dense in $H_F(I \times Q)$. □

The following lemma is a consequence of the Deformation theorem of **[FV]**. The proof can be carried out by mimicking the first paragraph in the proof of **[EK**, Corollary 1.3].

3.7. LEMMA. *Let $\{U_1, \dots, U_p\}$ be an open cover of a compact Q-manifold M. Then there exists a neighborhood $\mathscr{N}$ of* id *in $H(M)$ and maps $\varphi_i \colon \mathscr{N} \to H_{M \setminus U_i}(M)$, $i = 1, \dots, p$, such that for each $h \in \mathscr{N}$, $h = \varphi_p(h) \circ \cdots \circ \varphi_1(h)$.*

We are now ready to prove the main result in this section.

3.8. THEOREM. *For each compact finite-dimensional manifold X, $\overline{H}^{\mathrm{fd}}(X \times Q) = \widetilde{H}^{\mathrm{fd}}(X \times Q)$, that is, $H^{\mathrm{fd}}(X \times Q)$ is dense in $\widetilde{H}^{\mathrm{fd}}(X \times Q)$.*

Proof. Let $\dim X = n$. Clearly each point of $X \times I$ has a closed neighborhood N such that $(N, \mathrm{bd}\, N) \cong (I^{n+1}, \{1\} \times I^n)$. Then we have an open cover $\{V_1, \dots, V_p\}$ of $X \times I$ such that $(\mathrm{cl}\, V_i, \mathrm{bd}\, V_i) \cong (I^{n+1}, \{1\} \times I^n)$. For each $i = 1, \dots, p$, let $g_i \colon I \times I^n \to \mathrm{cl}\, V_i$ be a homeomorphism such that $\mathrm{bd}\, g_i(I \times I^n) = g_i(\{1\} \times I^n)$. Identifying the Q_{n+1}-factor of $I \times Q = (I \times I^n) \times Q_{n+1}$ with the Q_2-factor of $X \times Q = (X \times I) \times Q_2$, we define a homeomorphism $\tilde{g}_i \colon I \times Q \to \mathrm{cl}\, V_i \times Q_2$ by $\tilde{g}_i = g_i \times \mathrm{id}$. Let

$$F_i = (X \times Q) \setminus (V_i \times Q_2)$$

and define a homeomorphism

$$\psi_i \colon (H_{F_i}(X \times Q), H_{F_i}^{\mathrm{fd}}(X \times Q)) \to (H_{\{1\} \times Q}(I \times Q), H_{\{1\} \times Q}^{\mathrm{fd}}(I \times Q))$$

by $\psi_i(h) = \tilde{g}_i^{-1} h \tilde{g}_i$. To prove the theorem, we can apply the same argument as in the proof of **[GH**, Theorem 1]. By Lemma 3.7, there are maps $\psi_i \colon \mathscr{N} \to H_{F_i}(X \times Q)$ of a neighborhood $\mathscr{N}$ of id in $H(X \times Q)$ such that $h = \psi_p(h) \circ \cdots \circ \psi_1(h)$ for each $h \in \mathscr{N}$. By Lemma 3.6, each $\psi_i \varphi_i(h)$ can be approximated by $f_i \in H_{\{1\} \times Q}^{\mathrm{fd}}(I \times Q)$. Then h can be approximated by $\psi_p(f_p) \circ \cdots \circ \psi_i(f_1) \in H^{\mathrm{fd}}(X \times Q)$. Therefore $H^{\mathrm{fd}}(X \times Q) \cap \mathscr{N}$ is dense in $\mathscr{N}$. By local connectedness, $H^{\mathrm{fd}}(X \times Q)$ is dense in every component of $H(X \times Q)$ containing a member of $H^{\mathrm{fd}}(X \times Q)$. In other words, $H^{\mathrm{fd}}(X \times Q)$ is dense in $\widetilde{H}^{\mathrm{fd}}(X \times Q)$. □

4. Proofs of Theorems II and III. By using Theorems I and 3.8, we can prove Theorems II and III as follows.

Proof of Theorem II. Let X be a compact polyhedron. Then $H^{\mathrm{PL}}(X \times Q) = \bigcup_{n\in\mathbb{N}} H^{\mathrm{PL}}(X \times I^n)$ is a countable union of finite-dimensional compacta by **[Ge$_2$**, Theorem 1.9]. By the result of **[Ha]**, Corollary 1.2 implies that $H^{\mathrm{PL}}(X \times Q)$ is an ANR. By the result of **[To$_1$]** and Theorem I, $H^{\mathrm{PL}}(X \times Q)$ is an l_2^f-manifold.

In case X is a compact PL manifold, we denote

$$H^\sharp = H^{\mathrm{fd}}(X \times Q) \cap \widetilde{H}^{\mathrm{PL}}(X \times Q).$$

Since $\widetilde{H}^{\mathrm{PL}}(X \times Q)$ is open in $H(X \times Q)$, $H^\sharp$ is dense in $\widetilde{H}^{\mathrm{PL}}(X \times Q)$ (Theorem 3.8). Furthermore $H^\sharp$ is uniformly locally connected since it is a locally connected topological group. We conclude easily that $H^\sharp = \bigcup_{n\in\mathbb{N}} \widetilde{H}^{\mathrm{PL}}(X \times I^n)$. Since $H^{\mathrm{PL}}(X \times I^n)$ is dense in $\widetilde{H}^{\mathrm{PL}}(X \times I^n)$ for each $n \geq 5$ by **[GH**, Theorem 1], $H^{\mathrm{PL}}(X \times Q)$ is dense in $H^\sharp$, hence in $\widetilde{H}^{\mathrm{PL}}(X \times Q)$. We can employ the same argument as in the proof of **[GH**, Theorem 2] to show that $H^{\mathrm{PL}}(X \times Q)$ is an fd-cap set for $\widetilde{H}^{\mathrm{PL}}(X \times Q)$, that is, $(\widetilde{H}^{\mathrm{PL}}(X \times Q), H^{\mathrm{PL}}(X \times Q))$ is an (l_2, l_2^f)-manifold pair. □

Proof of Theorem III. Let X be a compact PL manifold. First we observe that $H^{\mathrm{LIP}}(X \times Q)$ is dense in $\widetilde{H}^{\mathrm{LIP}}(X \times Q)$. In fact, when we regard $H(X \times I^n) \subset H(X \times Q)$ by identifying $h \in H(X \times I^n)$ with $h \times \mathrm{id} \in H(X \times Q)$, $H^{\mathrm{LIP}}(X \times I^n) \subset H^{\mathrm{LIP}}(X \times Q)$ with respect to the metric $d_{X\times Q}$ on $X \times Q$ defined in §0 and a product metric of $X \times I^n$ defined by d_X and the Euclidean metric of I^n. Since each $H^{\mathrm{LIP}}(X \times I^n)$ is dense in $H(X \times I^n)$ by **[Su**, Corollary 3],

$$H^{\mathrm{LIP}}(X \times Q) \cap H^{\mathrm{fd}}(X \times Q)$$

is dense in $H^{\mathrm{fd}}(X \times Q)$, hence in $\widetilde{H}^{\mathrm{fd}}(X \times Q)$ by Theorem 3.8. Therefore $H^{\mathrm{LIP}}(X \times Q)$ is dense in $\widetilde{H}^{\mathrm{LIP}}(X \times Q)$.

Since $H^{\mathrm{PL}}(X \times I^n) \subset H^{\mathrm{LIP}}(X \times I^n)$ for each $n \in \mathbb{N}$ by **[LV**, Theorem 2.18], we have $H^{\mathrm{PL}}(X \times Q) \subset H^{\mathrm{LIP}}(X \times Q)$. Now, by using exactly the same argument as **[SW**, Theorem 2.3], we can conclude the result as follows: First by Theorem I, we can find an l_2^Q-manifold M such that

$$H^{\mathrm{PL}}(X \times Q) \subset M \subset H^{\mathrm{LIP}}(X \times Q) \cap \widetilde{H}^{\mathrm{PL}}(X \times Q).$$

Next by Theorem II, we can show that M is a cap set for $\widetilde{H}^{\mathrm{PL}}(X\times Q)$, whence

$$H^{\mathrm{LIP}}(X \times Q) \cap \widetilde{H}^{\mathrm{PL}}(X \times Q)$$

is also a cap set for $\widetilde{H}^{\mathrm{PL}}(X \times Q)$, that is,

$$(\widetilde{H}^{\mathrm{PL}}(X \times Q), H^{\mathrm{LIP}}(X \times Q) \cap \widetilde{H}^{\mathrm{PL}(X\times Q)})$$

is an (l_2, l_2^Q)-manifold pair. Since $H^{\mathrm{LIP}}(X \times Q)$ is dense in $\widetilde{H}^{\mathrm{LIP}}(X \times Q)$ and $\widetilde{H}^{\mathrm{LIP}}(X \times Q)$ is homogeneous,

$$(\widetilde{H}^{\mathrm{LIP}}(X \times Q), \widetilde{H}^{\mathrm{LIP}}(X \times Q))$$

is also an (l_2, l_2^Q)-manifold pair. □

5. Proof of Theorem IV. We obtain Theorem IV as a direct consequence of Theorem 5.5. Before we state the theorem, we need some notions. A subset A of a space M is said to be *locally homotopy negligible* in M **[To₃]** if for each open set U in M the inclusion $U\backslash A \subset U$ is a weak homotopy equivalence. If M is an ANR and A is locally homotopy negligible, then for each open cover $\mathscr{U}$ of M, there is a $\mathscr{U}$-homotopy $f: M \times I \to M$ such that $f_0 = \mathrm{id}$ and $f_t(M) \subset M \setminus A$ for each $t > 0$ **[To₃**, Theorem 2.4]. Hence $M \setminus A$ is also an ANR. Furthermore if A is closed in M then A is a Z-set in M. A closed set A in M is called a *strong Z-set* in M if for each open cover $\mathscr{U}$ of M, there is a map $f: M \to M$ such that f is $\mathscr{U}$-close to id and $A \cap \mathrm{cl}\, f(M) = \varnothing$. A Z-set in a locally compact space is a strong Z-set (cf. **[BBMW]**). Note that each Z-set in an l_2-manifold is a strong Z-set **[He]**. Hence we have the following:

5.1. LEMMA. *If N is contained in an l_2-manifold M such that $M \setminus N$ is locally homotopy negligible in M, then each Z-set in N is a strong Z-set.* □

An embedding $h: A \to M$ is called a *Z-embedding* if $h(A)$ is a Z-set in M. Let $\mathscr{C}$ be an additive topological class hereditary with respect to closed sets. A subset N of an l_2-manifold M is called a *$\mathscr{C}$-absorbing set* in M **[BM]** if $M \setminus N$ is locally homotopy negligible in M, $N = \bigcup_{n\in\mathbb{N}} N_n$, where each N_n is a Z-set in M and $N_n \in \mathscr{C}$ and N has the following property named the *strongly $\mathscr{C}$-universal* *property*:

(*) For each open cover $\mathscr{U}$ of N and each map $f: A \to N$ from a space $A \in \mathscr{C}$ such that $f|B$ is an embedding for a closed set B in A, there is a Z-embedding $g: A \to N$ such that $g|B = f|B$ and g is $\mathscr{U}$-close to f.

By $\mathscr{M}_1$ we denote the class of completely metrizable spaces. Concerning the strong $\mathscr{M}_1$-universal property, we have the following:

5.2. LEMMA. *If N is an ANR, $N \times l_2 \cong N$ and each Z-set is a strong Z-set, then N has the strong $\mathscr{M}_1$-universal property.*

Proof. Let $f: A \to N$ be a map of $A \in \mathscr{M}_1$ such that $f|B$ is a Z-embedding for a closed set B in A and let $\mathscr{U}$ be an open cover of N. By [**BM**, Lemma 1.1], we can assume that $f(A \setminus B) \cap f(B) = \varnothing$, and for each $x \in B$ and each neighborhood U of x in A, there is a neighborhood V of $f(x)$ in M such that $f^{-1}(V) \subset U$. By [**Sa**, Theorem 2.2], the projection $p: N \times l_2 \to N$ is $\mathscr{U}$-close to a map $\varphi: N \times l_2 \to N$ such that $\varphi|f(B) = p$ and φ maps $(N \setminus f(B)) \times l_2$ homeomorphically onto $N \setminus f(B)$. Since $A \in \mathscr{M}_1$, we have a Z-embedding $h: A \to l_2$. We define a map $g : A \to N$ by $g(x) = \varphi(f(x), h(x))$. Then clearly g is $\mathscr{U}$-close to f and $g|B = f|B$. It is easy to see that g is closed and injective, that is, g is a closed embedding. Since $(N \setminus f(B)) \times h(A)$ is a countable union of Z-sets in $M \times l_2$, $g(A \setminus B)$ is a countable union of Z-sets in M; hence so is $g(A) = f(B) \cup g(A \setminus B)$. By [**CDM**, Lemma 2.4], $g(A)$ is a Z-set in N; hence g is a Z-embedding. This completes the proof of the lemma. □

Combining the above two lemmas, we have the following:

5.3. PROPOSITION. *Let N be a countable union of Z-sets in an l_2-manifold M. If $M \setminus N$ is locally homotopy negligible and $N \times l_2 \cong N$, then N is an $\mathscr{M}_1$-absorbing set in M.* □

Recall that we regard $H(X \times I^n) \subset H(X \times Q)$ for each $n \in \mathbb{N}$ by identifying $h \in H(X \times I^n)$ with $h \times \mathrm{id} \in H(X \times Q)$.

5.4. LEMMA. *Let X be a compact polyhedron. Then for each $n \in \mathbb{N}$, $H(X \times I^n)$ is a Z-set in $H(X \times Q)$ (hence also a Z-set in $\widetilde{H}^{\mathrm{fd}}(X \times Q)$).*

Proof. Note that $H(X \times Q)$ is an ANR ([**Fe**], [**To**$_2$]). Clearly $H(X \times I^n)$ is closed in $H(X \times Q)$. We use the metric $d = d_{X\times Q}$ on $X \times Q$ as defined in §0, where

$$d_Q(y, y') = \sup\{2^{-i} \cdot |y(i) - y'(i)| \,|\, i \in \mathbb{N}\}.$$

Let $f: Q \to H(X \times Q)$ be a map and $\varepsilon > 0$. Since $f(Q)$ is equicontinuous by Ascoli's Theorem, there is a $\delta > 0$ such that $d(z, z') < \delta$ implies $d(h(z), h(z')) < \varepsilon/2$ for each $h \in f(Q)$. Choose $m > n$ so that $2^{-m} < \min\{\varepsilon/2, \delta\}$. For each $y \in Q$, we denote

$$y' = (y(1), \dots, y(m-1), y(m+1), y(m+2), \dots) \in Q.$$

We define a homeomorphism $\tau: X \times Q \to X \times Q \times I$ by $\tau(x, y) = (x, y', y(m))$. Choose any $k \in H(I) \setminus \{\mathrm{id}_I\}$ and define a map $\varphi: H(X \times Q) \to H(X \times Q) \setminus H(X \times I^n)$ by

$$\varphi(h) = \tau^{-1} \circ (h \times k) \circ \tau.$$

Then for each $h \in f(Q)$ and $(x, y) \in X \times Q$, $d((x, y), (x, y')) \le 2^{-m} < \delta$ implies

$$\begin{aligned} &d(\varphi(h)(x, y), h(x, y)) \\ &\quad \le d(\tau^{-1}(h(x, y'), k(y(m))), h(x, y')) + d(h(x, y'), h(x, y)) \\ &\quad \le 2^{-m} + \varepsilon/2 < \varepsilon. \end{aligned}$$

Hence $d(\varphi|f(Q), \mathrm{id}) < \varepsilon$. Thus we have a map

$$\varphi \circ f: Q \to H(X \times Q) \setminus H(X \times I^n)$$

which is ε-close to f. Therefore $H(X \times I^n)$ is a Z-set in $H(X \times Q)$. □

Since $l_2 \times l_2^f$ is an $\mathscr{M}_1$-absorbing set in $l_2 \times l_2 \cong l_2$ [**BM**, Proposition 2.6], Theorem IV is a consequence of the following theorem:

5.5. THEOREM. *For each compact* PL *manifold* X, $H^{\mathrm{fd}}(X \times Q)$ *is an* $\mathscr{M}_1$*-absorbing set in* $\widetilde{H}^{\mathrm{fd}}(X \times Q)$.

Proof. Since $H^{\mathrm{PL}}(X \times Q)$ is an fd-cap set for $\widetilde{H}^{\mathrm{PL}}(X \times Q)$,

$$\widetilde{H}^{\mathrm{PL}}(X \times Q) \setminus H^{\mathrm{fd}}(X \times Q)$$

is locally homotopy negligible in $\widetilde{H}^{\mathrm{PL}}(X \times Q)$. Since $\widetilde{H}^{\mathrm{fd}}(X \times Q)$ is homogeneous and $\widetilde{H}^{\mathrm{PL}}(X \times Q)$ is open in $\widetilde{H}^{\mathrm{fd}}(X \times Q)$,

$$\widetilde{H}^{\mathrm{fd}}(X \times Q) \setminus H^{\mathrm{fd}}(X \times Q)$$

is locally homotopy negligible in $\widetilde{H}^{\mathrm{fd}}(X \times Q)$. Recall that

$$H^{\mathrm{fd}}(X \times Q) = \bigcup_{n \in \mathbb{N}} H(X \times I^n).$$

Each $H(X \times I^n)$ is a Z-set in $\widetilde{H}^{\mathrm{fd}}(X \times Q)$ by Lemma 5.4. Then the theorem follows from Theorem I and Proposition 5.3. □

6. Problems. The first problem has been mentioned in §0.

6.1. *Problem.* For a PL manifold X, is $\widetilde{H}^{\mathrm{fd}}(X \times Q) = \widetilde{H}^{\mathrm{PL}}(X \times Q)$ or $\widetilde{H}^{\mathrm{LIP}}(X \times Q) = \widetilde{H}^{\mathrm{fd}}(X \times Q)$?

Concerning Theorems III and IV, we ask the following

6.2. *Problem.* For a PL manifold X, is $H^{\mathrm{LIP}}(X \times Q)$ or $H^{\mathrm{fd}}(X \times Q)$ dense in $H(X \times Q)$? In other words, is $\widetilde{H}^{\mathrm{LIP}}(X \times Q) = H(X \times Q)$ or $\widetilde{H}^{\mathrm{fd}}(X \times Q) = H(X \times Q)$?

For a decreasing sequence $s_n > 0$, $n \in \mathbb{N}$, converging to 0, we define a metric for Q by

$$d_Q(y, y') = \sup\{s_n \cdot |h(n) - y'(n)| \,|\, n \in \mathbb{N}\}.$$

By [Vä] and [Ho], Q is Lipschitz homogeneous if and only if

$$R((s_n)_{n \in \mathbb{N}}) = \sup\{s_n / s_{n+1} | n \in \mathbb{N}\} < \infty.$$

Theorem III is valid for any such metric even if $R((s_n)_{n \in \mathbb{N}}) = \infty$. The following problem seems to be interesting.

6.3. *Problem.* What metric for a compact Q-manifold M does make $H^{\mathrm{LIP}}(M)$ an l_2^Q-manifold or $(\widetilde{H}^{\mathrm{LIP}}(M), H^{\mathrm{LIP}}(M))$ an (l_2, l_2^Q)-manifold pair or $H^{\mathrm{LIP}}(M)$ dense in $H(M)$?

It is not known whether Theorem III is valid for any compact polyhedron X, that is,

6.4. *Problem.* Let X be a compact polyhedron and d a metric for $X \times Q$ which is Lipschitz equivalent to $d_{X \times Q}$ defined in §0. Then, with respect to d, is $H^{\mathrm{LIP}}(X \times Q)$ an l_2^Q-manifold? Is $(\widetilde{H}^{\mathrm{LIP}}(X \times Q), H^{\mathrm{LIP}}(X \times Q))$ an (l_2, l_2^Q)-manifold pair? Is $H^{\mathrm{LIP}}(X \times Q)$ dense in $H(X \times Q)$?

In general, $H^{\mathrm{fd}}(X \times Q)$ need not be dense in $\widetilde{H}^{\mathrm{fd}}(X \times Q)$ as shown in §0, but the following remains open.

6.5. *Problem.* For any compact polyhedron X, is $H^{\mathrm{fd}}(X \times Q)$ an $(l_2 \times l_2^f)$-manifold?

This is reduced to the following

6.6. *Problem.* For any compact polyhedron X, is each $H(X \times I^n)$ a Z-set in $H(X \times Q)$?

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