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POINT SPECTRUM ON A QUASIHOMOGENEOUS TREE

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Dedicated to Professor Sh. Murakami on his 60th birthday

By using the algebraicity of the Green kernel it is shown that a linear operator of nearest neighbour type on a quasi homogeneous tree i.e. a tree admitting of a group of automorphism with finite quotient has no point spectrum on the space of square summable functions, provided the tree has a regular property and that the operator is invariant under the group of automorphism.

0. Introduction. This result is an extension of spectrum theorem on an anisotropic random walk on a homogeneous tree (see [Ao1] and [Fi]). In case of one dimensional lattice relevant results have been obtained in full generality (see [Mo1] and [Mo2]). See [Ko] for a similar problem on a Riemannian manifold. The author is indebted to the referee for various improvements of statements in this note. Among other things, in Theorem 1 the author has originally restricted himself to the graph Γ without loops and multiple edges. The referee has suggested the more complete present form with its proof.

1. Basic properties of the Green kernel. Let T be a connected locally finite tree with the set of vertices $V(T)$ and the set of edges $E(T)$. Let A be a symmetric operator on $l^2(T)$, the space of square summable complex valued functions on $V(T)$:

$$(1.1) \quad Au(x) = \sum_{\langle x, x' \rangle} a_{x, x'} u(x') + a_{x, x} u(x)$$

for $u(x) \in l^2(T)$, with $a_{x, x}$ and $a_{x, x'} = a_{x', x} \in \mathbb{R}$. $\langle x, x' \rangle$ means that two vertices x, x' are adjacent to each other with respect to an edge $\overline{x, x'}$ binding x and x' .

We assume first that A is regular in the following sense:

$$(\mathcal{E}1) \quad a_{x, x'} \neq 0 \quad \text{for all } \langle x, x' \rangle.$$

Suppose further that a discrete group of automorphism G of T acts fix point-freely on T :

$$(1.2) \quad G \times V(T) \ni (g, x) \rightarrow g \cdot x \in V(T)$$

and that the quotient $\Gamma = G \backslash T$ is a finite graph. Recall that $V(\Gamma) = G \backslash V(T)$ and $E(\Gamma) = G \backslash E(T)$, where $G\bar{x}, \bar{y}$ connects the vertices Gx and Gy of Γ . Observe that Γ may have loops and multiple edges. In particular, G must be a finitely generated free group (see [S] or [T]). Γ is locally homeomorphic to T . We call the tree T “quasi homogeneous”. T can be regarded as the set of paths in T from a base point $* \in V(T)$ to points in $V(T)$.

We set up the following condition:

$$(\mathcal{E}2) \quad A \text{ is invariant under the action of } G.$$

Then A becomes a bounded and hence self-adjoint operator with domain $\mathcal{D}(A) = l^2(T)$. So the resolvent $(z - A)^{-1}$ is uniquely defined for $z \in \mathbb{C} - \mathbb{R}$. We denote by $G(x, y|z)$, $x, y \in V(T)$ and $z \in \mathbb{C} - \mathbb{R}$, the Green kernel for A , i.e., the matrix elements of the resolvent: $(e_x, (z - A)^{-1}e_y)$ for $e_x, e_y \in l^2(T)$, where $(,)$ and e_x denote the inner product on $l^2(T)$ and the function on $V(T)$ which is equal to 1 at x and zero elsewhere respectively.

It is obvious that

$$(1.3) \quad G(g \cdot x, g \cdot y|z) = G(x, y|z)$$

for an arbitrary $g \in G$. We denote by $W_x(z)$ the inverse $G(x, x|z)^{-1}$. As a function of x , W_x depends only on the coset $G \cdot x \in V(\Gamma) = G \backslash V(T)$.

We shall frequently use the following lemma which has been proved in our previous paper [Ao2].

LEMMA 1.1. $W_x(z)$, $x \in V(T)$, satisfy the basic equations:

$$(\mathcal{E}) \quad z - a_{x,x} - W_x = \sum_{\substack{\langle x,y \rangle \\ y \in V(T)}} \frac{1}{2}(-W_x + \sqrt{W_x^2 + 4a_{x,y}^2 W_x / W_y})$$

and

$$(1.4) \quad W_x(z) \sim z \text{ for } \text{Im } z \rightarrow \pm\infty.$$

$W_x(z)$ are uniquely determined by (\mathcal{E}) and (1.4).

Since W_x depends only on the coset $G \cdot x \in G \backslash V(T)$, there are N algebraic equations for the unknown W_x , $x \in V(\Gamma)$, where N denotes the number of vertices in $V(\Gamma)$. Hence $W_x(z)$ are all algebraic functions in z .

We shall also write $W_{\bar{x}}(z) = G(\bar{x}, \bar{x}|z)^{-1}$ in place of $W_x(z) = G(x, x|z)^{-1}$ in the case where $G \cdot x = \bar{x} \in G \setminus V(T)$. This will not lead to any confusion. The following is an easy consequence of the spectral representation of the Green kernel (see [Ak or Ca]).

LEMMA 1.2. For each $x \in V(T)$,

$$(1.5) \quad \text{Im } W_x(z) \cdot \text{Im } z > 0 \quad \text{for } z \in \mathbb{C} - \mathbb{R}.$$

The following two were proved in [Ao2]:

LEMMA 1.3. For each adjacent pair $x, y \in V(T)$, the multiplier $\alpha(x|y|z) = G(\omega, y|z)/G(\omega, x|z)$ is expressed as

$$\frac{-W_x + \sqrt{W_x^2 + 4a_{x,y}^2 W_x/W_y}}{2a_{x,y}} = W_x \cdot \frac{-1 + \sqrt{1 + 4a_{x,y}^2/(W_x W_y)}}{2a_{x,y}}$$

provided x lies on the geodesic line from ω to y . We have thereby

$$(1.6) \quad 2a_{x,y} G(x, y|z) = -1 + \sqrt{1 + 4a_{x,y}^2 G(x, x|z) \cdot G(y, y|z)}.$$

LEMMA 1.4. For each adjacent pair $x, y \in V(T)$, the function $\{-W_x + \sqrt{W_x^2 + 4a_{x,y}^2 W_x/W_y}\}$ is holomorphic in z and its imaginary part has the opposite sign to $\text{Im } z$, provided $z \in \mathbb{C} - \mathbb{R}$.

2. Statement of main theorems. We take and fix a real number λ . Since $W_x(z)$, $x \in V(\Gamma)$, are algebraic in z , they are expressed in Puiseux expansions. There exists a minimal exponent ρ_x such that

$$(2.1) \quad \lim_{\substack{z \rightarrow \lambda \\ \text{Im } z > 0}} W_x(z) \cdot (z - \lambda)^{-\rho_x} = c_x \neq 0.$$

Owing to Lemma 1.2, we have $-1 \leq \rho_x \leq 1$ and $\rho_x \in \mathbb{Q}$. Indeed if $|\rho_x| > 1$, then

$$\left| \lim_{z \rightarrow \lambda+0} \arg W_x(z) - \lim_{z \rightarrow \lambda-0} \arg W_x(z) \right| > \pi$$

from (2.1), which contradicts Lemma 1.2, $\rho_x \in \mathbb{Q}$ follows from the algebraicity of the function $W_x(z)$ in z . We denote by $V_\alpha(\Gamma)$ the set of vertices $x \in V(\Gamma)$ such that $\rho_x = \alpha$. λ is an eigenvalue of the operator A if and only if there exists at least one $x \in V(\Gamma)$ such that $\rho_x = 1$, i.e., $V_1(\Gamma) \neq \emptyset$. In fact the spectral kernel $d\theta(x, y|\lambda)$ of the operator A is positive definite:

$$(2.2) \quad d\theta(x, x|\lambda) \cdot d\theta(y, y|\lambda) \geq d\theta(x, y|\lambda)^2$$

and therefore λ is an eigenvalue if and only if

$$\theta(x, x|\lambda + 0) - \theta(x, x|\lambda - 0) \neq 0$$

for some $x \in V(\Gamma)$ (see [Ca] for details). We denote by N_α the number of vertices in $V_\alpha(\Gamma)$ for each $\alpha \in \mathbb{R}$ where $-1 \leq \alpha \leq 1$. Then we have $\sum_{-1 \leq \alpha \leq 1} N_\alpha = N$.

The set of vertices $V_\alpha(\Gamma)$ and edges connecting vertices in $V_\alpha(\Gamma)$ define a finite subgraph Γ_α of Γ such that $V(\Gamma_\alpha) = V_\alpha(\Gamma)$. We denote by N'_α the number of edges and L_α the number of loops in Γ_α . A proper circuit in Γ is a sequence x_0, x_1, \dots, x_k of successive adjacent vertices such that $k \geq 3$, $x_0 = x_k$ and $x_i \neq x_j$ for $0 \leq i < j < k$. Then the main theorem can be stated as follows.

THEOREM 1. (i) Γ_1 has no proper circuit. Thus, Γ_1 is a disjoint union of pseudo-trees, i.e., trees where loops and multiple edges are allowed. (ii) If $-1 < \rho < 1$, then no vertex of Γ_ρ is adjacent to any vertex of Γ_1 . (iii) We have the equality:

$$(2.3) \quad \sum_{x \in V_1(\Gamma)} \frac{1}{c_x} = N_1 - \left(N'_1 - \frac{1}{2} L_1 \right) - N_{-1}.$$

Hence λ is an eigenvalue of A if and only if $N_1 - (N'_1 - \frac{1}{2} L_1) - N_{-1} > 0$.

REMARK 1. The sum in the left-hand side in (2.3) is equal to $\sum_{x \in V(\Gamma)} \text{Res}_{z=\lambda} G(x, x|z)$.

THEOREM 2. If Γ is regular, i.e., if there is an equal number (greater than 1) of edges in Γ emanating from each vertex in $V(\Gamma)$, then the operator A is point spectrum free on $l^2(T)$.

COROLLARY. If T is a Cayley graph of a free group with finitely many generators and G is a subgroup of finite index, then A is point spectrum free.

REMARK 2. In this corollary the operator A has at most N bands of continuous spectra in view of the projection freeness theorem for reduced C^* -algebras of free groups (see [Cu] and references therein). It seems likely that there appear exactly N bands for generic A .

The following question raised by the referee seems very likely.

Question 1. Does the set of exponents $\{\rho_x\}$ consist of only $\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$?

3. Proofs of the theorems. We start to prove

LEMMA 3.1. Γ_1 has no proper circuit.

Proof. Γ_1 is decomposed into connected components $\Gamma_1^{(1)}, \dots, \Gamma_1^{(K)}$. For each $k, 1 \leq k \leq K$, we denote by $T_1^{(k)}$ the set of paths in $\Gamma_1^{(k)}$ starting from a base point $\bar{x} \in V(\Gamma_1^{(k)})$ and ending in points of $V(\Gamma_1^{(k)})$. Then $T_1^{(k)}$ can be regarded as a subtree of T . We take an arbitrary adjacent pair $x, y \in V(T_1^{(k)})$. We have Puiseux expansions at $z = \lambda$:

$$(3.1) \quad W_x(z) = c_x(z - \lambda) + (\text{higher degree terms}),$$

$$(3.2) \quad W_y(z) = c_y(z - \lambda) + (\text{higher degree terms}),$$

where c_x and c_y are both positive as is seen from Lemma 1.2. In the same way we have

$$(3.3) \quad G(x, y|z) = \frac{c_{x,y}}{z - \lambda} + (\text{higher degree terms})$$

for $c_{x,y} \in \mathbb{R}$. Let $\{u_j(x)\}_{1 \leq j \leq M}, 1 \leq M \leq +\infty$, be an orthonormal system of λ -eigenfunctions for A . The matrix $((c_{x,y}))$ defines the projection operator from $l^2(T)$ onto the λ -eigenspace. Since $c_{x,x} = 1/c_x$ and $c_{y,y} = 1/c_y$, we have

$$(3.4) \quad \frac{1}{c_x} = \sum_{j=1}^M u_j(x)^2,$$

$$(3.5) \quad \frac{1}{c_y} = \sum_{j=1}^M u_j(y)^2, \quad \text{and}$$

$$(3.6) \quad c_{x,y} = \sum_{j=1}^M u_j(x) \cdot u_j(y).$$

The relation (1.6) shows that

$$(3.7) \quad \left\{ \sum_{j=1}^M u_j(x) \cdot u_j(y) \right\}^2 = \sum_{j=1}^M u_j(x)^2 \cdot \sum_{j=1}^M u_j(y)^2.$$

As a result of the Schwarz inequality, this implies that the two M dimensional vectors $\{u_j(x)\}_{1 \leq j \leq M}$ and $\{u_j(y)\}_{1 \leq j \leq M}$ are linearly dependent: $u_j(x) = t(x, y)u_j(y)$ for $t(x, y) \in \mathbb{R}$. Let $x \in V(T_1^{(k)})$ be

an arbitrary point such that $G \cdot x = \bar{x} \in V(\Gamma_1^{(k)})$. Applying the above relation successively we have

$$(3.8) \quad u_j(x) = t(x) \cdot u_j(x^*)$$

for an arbitrary $x^* \in V(T_1^{(k)})$ and j such that $u_j(x^*) \neq 0$, where $t(x)$ is a real function on $V(T_1^{(k)})$ such that $\sum_{x \in V(T_1^{(k)})} t(x)^2 < +\infty$. But then

$$(3.9) \quad \sum_{j=1}^M \sum_{x \in V(T_1^{(k)})} u_j(x)^2 = \sum_{j=1}^M u_j(x^*)^2 \cdot \sum_{x \in V(T_1^{(k)})} t(x)^2 < +\infty.$$

Suppose $\Gamma_1^{(k)}$ has a proper circuit. Then this represents a non-trivial action of an element g of G on T and it follows from [T] that g must have infinite order. Indeed, by [T], g has either infinite order or order 2. Now one has to verify that the second case is impossible: if $g^2 = \text{identity}$ and the circuit is even then g fixes a vertex of T , in contradiction with fixed point-freeness. If the circuit is odd, then g inverts an edge \bar{x}, \bar{y} in T whose G -orbit lies on the circuit. But then $G\bar{x}, \bar{y}$ is a loop, in contradiction with properness of the circuit. Since $1/c_x$ is invariant under this action: $1/c_x = 1/c_{g \cdot x}$, we have

$$(3.10) \quad +\infty = \sum_{-\infty}^{+\infty} \frac{1}{c_{g^l \cdot x}} \leq \sum_{j=1}^M \sum_{x \in V(T_1^{(k)})} u_j(x)^2$$

which is finite from (3.9). This is a contradiction. Hence each $\Gamma_1^{(k)}$ has no proper circuit and it is a pseudo-tree. T_1 is itself a disjoint union of pseudo-trees. Lemma 3.1 has thus been proved.

LEMMA 3.2. *Let $x \in V_1(\Gamma)$ and $y \in V_{-1}(\Gamma)$ which are adjacent to each other. We have Puiseux expansions at $z = \lambda$ for $W_x(z)$ as in (3.1) and for $W_y(z)$ as follows:*

$$(3.11) \quad W_y(z) = \frac{c_y}{z - \lambda} + (\text{higher degree terms})$$

where $c_y < 0$ from Lemma 1.2. Then

$$(3.12) \quad c_x \cdot c_y \leq -4a_{x,y}^2 \quad \text{and}$$

$$(3.13) \quad b_{x,y} = -1 \pm \sqrt{1 + \frac{4a_{x,y}^2}{c_x \cdot c_y}} < 0,$$

where we take the positive root $\sqrt{\quad}$ in the right hand side.

Proof. From Lemma 1.4, for $\text{Im } z > 0$,

$$(3.14) \quad \begin{aligned} \text{Im}(-W_x + \sqrt{W_x^2 + 4a_{x,y}^2 W_x/W_y}) \\ = \text{Im}\{c_x(-1 \pm \sqrt{1 + 4a_{x,y}^2/(c_x c_y)})(z - \lambda) \\ + (\text{higher degree terms})\} < 0. \end{aligned}$$

c_x being positive, $-1 \pm \sqrt{1 + 4a_{x,y}^2/(c_x c_y)}$ must be negative, i.e.

$$(3.15) \quad 0 \leq \sqrt{1 + 4a_{x,y}^2/(c_x c_y)} < 1.$$

Lemma 3.2 has thus been proved.

We shall denote by $\varepsilon_{x,y}$ the sign \pm appearing in the right hand side of (3.14). Note that $\varepsilon_{x,y}$ is symmetric in x, y from Lemma 1.3.

In the sequel, whenever we speak of a sum $\sum_{\langle x,y \rangle}$, where $x \in V(\Gamma)$, the sum refers to all edges from x to y , each carrying the weight inherited from T .

LEMMA 3.3. *Suppose $x \in V_1(\Gamma) = V(\Gamma_1)$. Then there is no $y \in V_\rho(\Gamma)$ adjacent to x in Γ for $-1 < \rho < 1$.*

Proof. W_x has a Puiseux expansion (3.1). Suppose that there exists one $y \in V_\rho(\Gamma)$, $-1 < \rho < 1$, which is adjacent to x . Let α be the greatest exponent among these ρ . Then comparing the constant term and the term $(z - \lambda)^{(1-\alpha)/2}$ respectively, we have from (3)

$$(3.16) \quad \lambda - a_{x,x} = \sum_{\substack{\langle x,y \rangle \\ y \in V_1(\Gamma)}} \varepsilon_{x,y} a_{x,y} \sqrt{c_x/c_y},$$

$$(3.17) \quad 0 = \sum_{\substack{\langle x,y \rangle \\ y \in V_\alpha(\Gamma)}} \varepsilon_{x,y} a_{x,y} \sqrt{c_x/c_y} (z - \lambda)^{(1-\alpha)/2}.$$

Here we take $\sqrt{c_x/c_y}$ as positive. Since, as a result of Lemma 1.4, the imaginary part of each term in the right hand side satisfies

$$(3.18) \quad \text{Im}\{\varepsilon_{x,y} a_{x,y} \sqrt{c_x/c_y} (z - \lambda)^{(1-\alpha)/2}\} < 0$$

for $\text{Im } z > 0$, which is a contradiction to (3.17). Hence the set of $y \in V_\alpha(\Gamma)$ which is adjacent to x must be empty. Namely there is no $y \in V_\rho(\Gamma)$, $-1 < \rho < 1$, which is adjacent to x .

Therefore for $x \in V_1(\Gamma)$, the equation (\mathcal{E}) becomes

$$(3.19) \quad z - W_x - a_{x,x} = \frac{1}{2} \sum_{\substack{\langle x,y \rangle \\ y \in V_1(\Gamma) \cup V_{-1}(\Gamma)}} (-W_x + \sqrt{W_x^2 + 4a_{x,y}^2 W_x / W_y}).$$

Comparing the term $z - \lambda$, we have

$$(3.20) \quad 1 - c_x = \frac{1}{2} \sum_{\substack{\langle x,y \rangle \\ y \in V_{-1}(\Gamma)}} (-c_x + \varepsilon_{x,y} c_x \sqrt{1 + 4a_{x,y}^2 / (c_x c_y)}) + \frac{1}{2} \sum_{\substack{\langle x,y \rangle \\ y \in V_1(\Gamma)}} \left\{ -c_x + 2\varepsilon_{x,y} a_{x,y} \sqrt{c_x / c_y} \left(\frac{c'_x}{c_x} - \frac{c'_y}{c_y} \right) \right\},$$

where $c'_x(z - \lambda)^2$ denotes the quadratic term in the Puiseux expansion of $W_x(z)$. (3.20) is reexpressed as

$$(3.21) \quad \frac{1}{c_x} - 1 = \frac{1}{2} \sum_{\substack{\langle x,y \rangle \\ y \in V_{-1}(\Gamma)}} (-1 + \varepsilon_{x,y} \sqrt{1 + 4a_{x,y}^2 / (c_x c_y)}) + \frac{1}{2} \sum_{\substack{\langle x,y \rangle \\ y \in V_1(\Gamma)}} \left\{ -1 + \varepsilon_{x,y} \frac{2a_{x,y}}{\sqrt{c_x c_y}} \left(\frac{c'_x}{c_x} - \frac{c'_y}{c_y} \right) \right\}.$$

Summing up both sides over the vertices of $V_1(\Gamma)$ and seeing that the terms

$$\varepsilon_{x,y} \frac{2a_{x,y}}{\sqrt{c_x c_y}} \left(\frac{c'_x}{c_x} - \frac{c'_y}{c_y} \right)$$

are alternating in x and y for the proper edges $\overline{x, y}$ of Γ_1 (while they are zero for loops), we have that they cancel. We get

$$(3.22) \quad \sum_{x \in V_1(\Gamma)} \frac{1}{c_x} - N_1 = \frac{1}{2} \sum_{\substack{x \in V_1(\Gamma), y \in V_{-1}(\Gamma) \\ \langle x,y \rangle}} b_{x,y} - \left(N'_1 - \frac{1}{2} L_1 \right).$$

To compute the right hand side of (3.22), we observe

LEMMA 3.4. *Let $x \in V_{-1}(\Gamma)$. Then*

$$(3.23) \quad \sum_{\substack{\langle x,y \rangle \\ y \in V_1(\Gamma)}} b_{x,y} = -2.$$

Proof. W_γ has a Puiseux expansion as in (3.11)

$$(3.24) \quad W_x = \frac{c_x}{z - \lambda} + (\text{higher degree terms}) \quad \text{for } c_x < 0.$$

We compare the term $(z - \lambda)^{-1}$ in both sides of the equations (E) and obtain

$$(3.25) \quad -c_x = \frac{1}{2} \sum_{\substack{\langle x,y \rangle \\ y \in V_1(\Gamma)}} (-c_x + \varepsilon_{x,y} c_x \sqrt{1 + 4a_{x,y}^2 / (c_x c_y)}) \\ + \frac{1}{2} \sum_{-1 \leq \rho < 1} \sum_{\substack{\langle x,y \rangle \\ y \in V_\rho(\Gamma)}} (-c_x + \varepsilon_{x,y} c_x),$$

i.e.,

$$(3.26) \quad -2 = \sum_{\substack{\langle x,y \rangle \\ y \in V_1(\Gamma)}} b_{x,y} + \sum_{-1 \leq \rho < 1} \sum_{\substack{\langle x,y \rangle \\ y \in V_\rho(\Gamma)}} (-1 + \varepsilon_{x,y}).$$

Hence the sum

$$\sum_{\substack{\langle x,y \rangle \\ y \in V_1(\Gamma)}} b_{x,y}$$

must be an even integer at least equal to -2 . Since every $b_{x,y}$ is negative by Lemma 3.2, this sum is just equal to -2 . Lemma 3.4 has been proved.

Proof of Theorem 1. (i) follows from Lemma 3.1. (ii) follows from Lemma 3.3. (iii) is an immediate consequence of (3.22) and (3.23). Theorem 1 has thus been proved.

Proof of Theorem 2. We have only to show that there never occurs $N_1 - (N'_1 - \frac{1}{2}L_1) - N_{-1} > 0$ for any $\lambda \in \mathbb{R}$. Suppose λ is an eigenvalue of A . Let Γ_1 be defined as at the beginning of the preceding section. For $x \in V_1(\Gamma)$, let $\text{deg}(x)$ be the number of edges of Γ_1 incident with x . Assume that m (≥ 2) edges emanate from each vertex in T . Let

K be the number of edges between Γ_1 and Γ_{-1} . Then $K \leq mN_{-1}$. On the other hand, according to Lemma 3.3,

$$(3.27) \quad K = \sum_{x \in V_1(\Gamma)} (m - \deg(x)) = mN_1 - (2N'_1 - L_1).$$

Hence, $mN_{-1} \geq mN_1 - 2(N'_1 - \frac{1}{2}L_1) \geq mN_1 - m(N'_1 - \frac{1}{2}L_1)$, and $N_1 - (N'_1 - \frac{1}{2}L_1) - N_{-1} \leq 0$. Theorem 2 has thus been proved.

REMARK 3. In order to show that A is point spectrum free, it is not sufficient that Γ is finite. The following example is very illuminating as a counter example.

Let Γ be a complete bipartite graph consisting of $(p+q)$ points $\{1, 2, \dots, p+q\}$, $p > q \geq 2$, such that each vertex $\{j\}$, $1 \leq j \leq p$, is adjacent to the points $\{p+k\}$, $1 \leq k \leq q$. We assume that $a_{j,p+k} = a_{p+k,j} = 1$, and other $a_{x,y}$ all vanish. The group of automorphisms G is isomorphic to the free group of $(p-1)(q-1)$ generators. Then the equations (\mathcal{E}) reduce to $W_1 = \dots = W_p$, $W_{p+1} = \dots = W_{p+q}$ and

$$(3.28) \quad z - W_1 = \frac{q}{2}(-W_1 + \sqrt{W_1^2 + 4W_1/W_{p+1}}),$$

$$(3.29) \quad z - W_{p+1} = \frac{p}{2}(-W_{p+1} + \sqrt{W_{p+1}^2 + 4W_{p+1}/W_1}),$$

i.e.,

$$(3.28') \quad \frac{z}{W_1} - 1 = \frac{q}{2}(-1 + \sqrt{1 + 4/(W_1 W_{p+1})}),$$

$$(3.29') \quad \frac{z}{W_{p+1}} - 1 = \frac{p}{2}(-1 + \sqrt{1 + 4/(W_1 W_{p+1})}).$$

Hence

$$(3.30) \quad \frac{1}{q}G(1, 1|z) - \frac{1}{p}G(p+1, p+1|z) = \left(\frac{1}{q} - \frac{1}{p}\right) \frac{1}{z} \neq 0.$$

This shows that $G(1, 1|z)$ or $G(p+1, p+1|z)$ has a pole at $z = 0$. 0 is an eigenvalue. Indeed in Theorem 1 we have $V_1(\Gamma) = \{1, 2, \dots, p\}$ and $V_{-1}(\Gamma) = \{p+1, p+2, \dots, p+q\}$ so that $N_1 - N'_1 - N_{-1} = p - q > 0$. Hence

$$\sum_{j=1}^p \frac{1}{c_j} = \frac{p}{c_1} = p - q, \quad \text{i.e.} \quad 1/c_1 = 1 - \frac{q}{p} > 0.$$

The spectrum of A^2 in this case coincides with a part of the one of an operator obtained from a random walk on a barycentric subdivision of polygonal graphs investigated by G. Kuhn-P. M. Soardi, J. Farault-M. A. Picardello or J. M. Cohen-A. R. Trenholme (see [Ku], [Fa] or [Co]). By using this result one can also compute the point spectrum of A as above. The author is indebted to the referee for having informed it to us. See also [B].

Question 2. Assume that Γ is fixed. It seems to be an interesting question to ask whether the existence of point spectrum really depends or not on the data $\{a_{x,y}\}_{x,y \in V(\Gamma)}$ under the conditions (E1) and (E2).

REMARK 4. A modified version of Theorems 1 and 2 is probably true even if the action of G is not necessarily free, provided the quotient $G \backslash T$ is finite. But the author does not know any answer.

REMARK 5. Theorem 1 (iii) remains valid when T is a finite connected tree and A is a linear operator on $l^2(T)$ defined as in (1.1). In this case $W_x(z)$, $x \in T$, are rational in z . Hence there occur only $V_1(T)$ and $V_{-1}(T)$. Theorem 1 can then be modified as follows:

THEOREM 1a. For an eigenvalue λ of A , we denote by N_1 and N_{-1} the numbers of $V_1(T)$ and $V_{-1}(T)$ respectively, and by N'_1 the number of edges in T connecting vertices in $V_1(T)$. Then

$$(3.31) \quad \sum_{x \in V(T)} \operatorname{Res}_{z=\lambda} G(x, x|z) = N_1 - N'_1 - N_{-1}.$$

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