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The universal covering group of SL(2, R) is right orderable, but is not locally indicable; in fact, it contains nontrivial finitely generated perfect subgroups.

Introduction. A group G is called *right orderable* if it admits a total ordering $\leq$ such that $a \leq b \Rightarrow ac \leq bc$ $(a, b, c \in G)$. It is known that a group has such an ordering if and only if it is isomorphic to a group of order-preserving permutations of a totally ordered set [**4**, Theorem 7.1.2].

A group is called *locally indicable* if each of its finitely generated nontrivial subgroups admits a nontrivial homomorphism to $\mathbf{Z}$. Every locally indicable group is right orderable (see [**4**, Theorem 7.3.11]); it was an open question among workers in the area whether the converse was true (equivalent to [**9**, Problem 1]). This note gives a counterexample, and a modified example showing that a finitely generated right orderable group can in fact be a perfect group.

Related to the characterization of right orderable groups in terms of actions on totally ordered sets is the result that the fundamental group of a manifold M is right orderable if and only if the universal covering space of M can be embedded over M in $M \times \mathbf{R}$. After distributing a preprint of this note, I was informed by W. Thurston and P. Kropholler that examples with the same properties were already known among topologists from this point of view (see §6 below. For another topological use of right ordered groups, in this case locally indicable ones, see [**11**].) However, as the present examples are easily established and self-contained, they seem worth presenting.

Since the classes of locally indicable groups and of right orderable groups are distinct, it will now be of interest to investigate whether various results that have been proved for the former also hold for the latter; cf. [**8**], [**2**, §9], [**3**, §4].

2. The group G. Let G denote the universal covering group of $\mathrm{SL}(2, \mathbf{R})$. Elements of G may be thought of as linear transformations of the plane, in which the angle through which each ray is moved is specified, not merely modulo 2π, but as a real number (in a continuous fashion).

Note that $\mathrm{SL}(2, \mathbf{R})$ acts faithfully on the set of rays through 0 in the plane. Similarly, G acts faithfully on the set of rays through 0 in the infinite-sheeted branched covering X of the plane with branch-point 0. This set of rays, "the circle unwound", can be identified with the real line $\mathbf{R}$, and G clearly acts on this set in a way preserving the usual ordering of the line; hence G is right orderable. Members of G, thought of as order-preserving maps from $\mathbf{R}$ to itself, have the diagonal periodicity property

$$f(x+\pi) = f(x) + \pi \qquad (x \in \mathbf{R}). \tag{1}$$

Note that if $h \in \mathrm{SL}(2, \mathbf{R})$ has a positive eigenvalue λ, then exactly one of h's infinitely many liftings to G fixes the rays in X that map into the eigenspace of λ (i.e., moves each of these rays through the angle 0, rather than through 2π, 4π, etc.). If an element of $\mathrm{SL}(2, \mathbf{R})$ has two positive eigenvalues, then the lifting to G that fixes the rays that map into one eigenspace is also the lifting that fixes the rays mapping into the other. For it is easy to see that an element of $\mathrm{SL}(2, \mathbf{R})$ with at least one positive eigenvalue moves every ray through an angle of magnitude strictly less than π, hence it cannot be lifted to a map on X that moves one ray by 0 and another by a nonzero multiple of 2π.

Consider now the following three one-parameter families of elements of $\mathrm{SL}(2, \mathbf{R})$, the first two parametrized by a formal real exponent r, the third by a positive real subscript s:

$$a^r = \begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix}, \qquad b^r = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}, \qquad c_s = \begin{pmatrix} s^{1/2} & 0 \\ 0 & s^{-1/2} \end{pmatrix}.$$

Clearly,

$$a^r a^{r'} = a^{r+r'}, \qquad b^r b^{r'} = b^{r+r'}, \qquad c_s c_{s'} = c_{ss'}. \tag{2}$$

Moreover, we note that

$$c_s a^r c_s^{-1} = a^{sr}, \qquad c_s^{-1} b^r c_s = b^{sr}, \tag{3}$$

and if we define

$$d = aba = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{4}$$

then we have

$$(5) \qquad d^{-1}c_s d = c_{s^{-1}},$$

$$(6) \qquad d^{-1}a^r d = b^r, \qquad d^{-1}b^r d = a^r.$$

For all real numbers r and positive real numbers s, let us define A^r, B^r, C_s to be the unique liftings to G of a^r, b^r, d_s which fix the rays in X lying over the eigenspaces of those elements. We claim that (2) and (3) imply the corresponding equations in G:

$$(2') \qquad A^r A^{r'} = A^{r+r'}, \qquad B^r B^{r'} = B^{r+r'}, \qquad C_s C_{s'} = C_{ss'},$$

$$(3') \qquad C_s A^r C_s^{-1} = A^{sr}, \qquad C_s^{-1} B^r C_s = B^{sr}.$$

Indeed, by (2) and (3), in each of these five formulas the two sides map to the same element of $\mathrm{SL}(2, \mathbf{R})$, hence it suffices to show that both sides of each equation have fixed rays. The right-hand side of each equation has fixed rays by definition. The left-hand side of the first formula has fixed rays because all real powers of a have a *common* fixed ray. Hence the same is true of their chosen liftings; corresponding considerations apply to the remaining two formulas of $(2')$. The left-hand sides of the two equations of $(3')$ have fixed rays because they are conjugates of elements having fixed rays; this completes the verification of these formulas.

Defining D by the analog of (4):

$$(4') \qquad D = ABA,$$

we similarly deduce the analogs of (5) and (6):

$$(5') \qquad D^{-1}C_s D = C_{s^{-1}},$$

$$(6') \qquad D^{-1}A^r D = B^r, \qquad D^{-1}B^r D = A^r.$$

3. The first example. Consider now for any integer $n > 1$ the finitely generated subgroup $\langle A, B, C_n \rangle \subseteq G$. Equations $(3')$ and $(5')$ show that each of A, B, C_n is conjugate in this group to a distinct power of itself, hence any homomorphism $\langle A, B, C_n \rangle \to \mathbf{Z}$ annihilates these three generators, hence is trivial.

Thus G is not locally indicable.

4. Motivation for the construction. Before finding this example, I wondered whether it might be possible to prove all right orderable groups locally indicable by using the fact that one can write a finitely generated right orderable group G as a group of order-automorphisms f of the real line with $|f(x) - x|$ bounded, and mapping such a group into $\mathbf{R}$ by some measure of the "long-term forward flow" induced by each element. Some more subtle measure would then have to be applied to groups whose elements all had zero flow, in this sense. But it turned out that such a flow function, even if nonzero, would not in general be a homomorphism: two order-preserving functions f and g each with "blocked" flow can have composite with nontrivial flow. E.g., this can happen if f is a function which fixes the even integers but pushes all other real numbers a bit forward, and g a function which fixes the odd integers and likewise pushes other values forward.

If such functions have a periodicity property like (1), their behavior can be examined with the help of the induced functions on the circle. Experimentation with convenient cases led to the example described.

5. Further questions, and a modified example. Our above example is equivalent to an affirmative answer to the question, "Is there a nontrivial finitely generated right orderable group H such that the abelianization H/H' is a torsion group?" Question 20 of [1] asks whether a nontrivial finitely generated right orderable group can in fact be simple, and Question 20′ asks the intermediate question of whether such an H can be perfect (have trivial abelianization).

The group $\langle A, B, C_n\rangle$ of §3 is certainly not simple, since it has the central subgroup $\langle D^2\rangle$. It is also quite possibly never perfect: (3′) and (5′) merely tell us that in the abelianization, the images of A and B have exponent $n-1$, and that of C_n has exponent 2. (If the integer n is not a square, the possible 2-torsion of the image of C_n is indeed realized, for we can map $\langle a, b, c_n\rangle$ to Z_2 by sending elements with rational entries to 0, and elements of the form $n^{1/2}$ times a rational matrix to 1.) However, we shall now describe a finitely generated perfect subgroup of G.

Note that by (4′),

$$D \in \langle A, B\rangle.$$

Conjugating by any element C_s, and simplifying the left-hand side with the help of (5′) and the right-hand side with the help of (3′), we get

$$DC_{s^2} \in \langle A^{s^{-1}}, B^s\rangle.$$

Taking for s any positive integer n, we see that by the two preceding displays, the group $\langle A^{n^{-1}}, B\rangle$ will contain both D and DC_{n^2}, hence

$$C_{n^2} \in \langle A^{n^{-1}}, B\rangle. \tag{7}$$

Now consider the group $\langle A^{1/6}, B\rangle$. Since this contains $\langle A^{1/2}, B\rangle$ and $\langle A^{1/3}, B\rangle$, it contains C_4 and C_9, hence by (3′), the generators $A^{1/6}$ and B are each conjugate in this group both to their 4th and to their 9th powers. But $4-1=3$ and $9-1=8$ are relatively prime, so each of these generators vanishes in the abelianization, proving this group perfect.

We remark that by (4′) and (6′), the above group can be variously described as

$$\langle A^{1/6}, B\rangle = \langle A^{1/6}, B^{1/6}\rangle = \langle A^{1/6}, D\rangle = \langle A^{1/2}, B^{1/3}\rangle.$$

The same argument shows that $\langle A^{1/mn}, B\rangle$ is perfect for any integers m and n such that m^2-1 and n^2-1 are relatively prime. But for this relative primality to hold, the product mn must be divisible by 6, so our example is the simplest of this sort.

In contrast to the above example, it is known that for all real numbers $s \geq 4$, the group $\langle a^s, b\rangle$ is free on the indicated generators [**5**]. (What is actually shown in [**5**] is that $\langle a^s, b^s\rangle$ is free if $s \geq 2$; but (3) shows that the isomorphism class of $\langle a^s, b^t\rangle$ depends only on st. See [**10**, p. 168] for further references.) Hence the same is true of $\langle A^s, B\rangle$.

6. Still more examples and questions. Bill Thurston and Peter Kropholler have pointed out to me an example of a finitely generated perfect right orderable group known to topologists:

$$\langle x, y, z | x^2 = y^3 = z^7 = xyz\rangle. \tag{8}$$

That this is perfect can be verified from the presentation. It is right orderable because it has an embedding in the same group G we have been considering, given by

$$x = ABA = D, \qquad y = A^rBA^{1-r}, \qquad z = y^{-1}x, \tag{9}$$

where r is a root of $r^2 - r + 2 = 2\cos(\pi/7)$. Here I shall merely sketch a proof that the elements (9) satisfy the relations of (8), and thus generate a nontrivial perfect subgroup of G.

In G, y is conjugate to AB, which satisfies $(AB)^3 = (ABA)(BAB) = DD^D = D^2$ (cf. (6′)); so as D^2 is central in G, we also have $y^3 = D^2$. Hence $y^3 = x^2 = xyz$. Note that D^2 represents rotation through an angle of $+\pi$ in the branched covering X of the plane. Now calculation shows that the image of z in $\mathrm{SL}(2, \mathbf{R})$ has eigenvalues $e^{\pm\pi i/7}$, hence has 7th power $-I$, and that it turns rays counterclockwise. Thus after an appropriate change of coordinates in the plane, this element represents a counterclockwise rotation by $\pi/7$. We claim that, in the same coordinates, the preimage $z = y^{-1}x \in G$ of this element represents a rotation by $\pi/7$ rather than, say, $15\pi/7$. Indeed, this follows from the observation that y^{-1} and x move rays in opposite directions, and each through angles $< \pi$ (since they are roots of $D^{\pm 2}$). From this description of z we can see that $z^7 \in G$ is a rotation through exactly π i.e., is D^2, as desired.

Thurston and Kropholler note that another perfect example is a group of periodic piecewise linear transformations of the line, related to work of R. J. Thompson, and called $\tilde{G}$ in [7]; cf. [**6**, pp. 54–55].

The question of whether a nontrivial finitely generated right orderable group can be *simple* remains open. Another open question, [**1**, Question 21′], is whether a nontrivial *solvable* right orderable group must be locally indicable. I also do not know any examples of nontrivial finitely generated right orderable groups having both finite abelianization and trivial center.

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