

# Pacific Journal of Mathematics

**TWO APPLICATIONS OF THE UNIT NORMAL BUNDLE OF A  
MINIMAL SURFACE IN  $R^N$**

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## TWO APPLICATIONS OF THE UNIT NORMAL BUNDLE OF A MINIMAL SURFACE IN $R^N$

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*Dedicated to Professor Shingo Murakami on his sixtieth birthday*

**A Gauss parametrization of a minimal surface in  $R^3$  is well known.  
 We prove a generalization.**

**THEOREM A.** *Let  $U$  be an open set of  $S^N(1)$  and  $f$  a function on  $U$  such that*

$$\Delta_{S^N(1)} f = -Nf$$

*and 0 is an eigenvalue of  $\text{Hess } f + f\langle \cdot, \cdot \rangle$  of multiplicity  $N - 2$ , where  $\langle \cdot, \cdot \rangle$  is the metric of  $S^N(1)$  and  $\Delta_{S^N(1)}$  is the Laplacian of  $S^N(1)$ . Then the map of  $U$  into  $R^{N+1}$  defined by*

$$(*) \quad f\eta + \text{grad } f$$

*is of rank 2 and gives a minimal surface, where  $\eta$  is the identity map on  $S^N(1)$ . Conversely, for a minimal surface  $M$  in  $R^{N+1}$ , a neighborhood of each point of  $M$  without geodesic points has this representation.*

If  $M$  is a complete orientable minimal surface of finite total curvature, then there is a global representation  $(*)$  of  $M$ . Using this idea, we obtain the following.

**THEOREM B.** *Let  $M$  be a complete orientable minimal surface of finite total curvature in  $R^{N+1}$ . Then there exist a positive real number  $c(N)$  depending on  $N$  such that*

$$\text{index}(M) \leq c(N) \int (-K) * 1_M,$$

*where  $K$  is the Gauss curvature of  $M$  and  $*1_M$  is the area form of  $M$ .*

Theorem B gives an answer for an open question posed by Cheng and Tysk in [CY1]. After this paper was submitted, the author learned that Cheng and Tysk in [CT2] obtained a similar result as Theorem B by using another Gauss map (generalized Gauss map).

Finally we consider a generalization of minimal herissons [RT].

I'd like to thank Professors J. Eells, J. Rawnsley, M. Micallef and K. Ohshika for their hospitality while visiting at the University of Warwick.

**2. Second variation formula.** Let  $M$  be a minimal surface in  $R^{N+1}$  and  $\chi$  the immersion. Let  $U(M)$  be the unit normal bundle of the normal bundle  $N(M)$ . Then we define a Gauss map  $G$  of  $U(M)$  into the  $N$ -dimensional unit sphere  $S^N(1)$  by  $G(x, \eta) = \eta$  for  $(x, \eta) \in U(M)$ .  $G$  induces a degenerate Riemannian metric of constant curvature 1 on  $U(M)$ . Let  $\xi$  be a section of  $N(M)$  with compact support. Then a function  $F_\xi$  on  $U(M)$  is defined by

$$F_\xi(x, \eta) = \langle \xi, \eta \rangle,$$

where  $(x, \eta) \in U(M)$ . Let  $I(\xi, \xi)$  be the second variation of the area functional in the direction of  $\xi$ . Then we get

PROPOSITION 2.1.

$$I(\xi, \xi) = ((N - 1)/\omega) \int (|\nabla F_\xi|^2 - NF_\xi^2) * 1_{U(M)},$$

where  $\omega$  is the volume of  $S^{N-2}(1)$  and  $*1_{U(M)}$  is the volume form of  $U(M)$ .

This is well known in the case of  $N = 2$ .

*Proof.* Let  $x$  be a point of  $M$  and  $e_\alpha$  for  $\alpha = 3, \dots, N + 1$  be a local orthonormal framing of  $N(M)$  such that

$$\nabla_X^\perp e_\alpha = 0 \quad \text{for all tangent vectors } X \text{ at } x,$$

where  $\nabla^\perp$  is the normal connection of  $N(M)$ . Furthermore we may consider that the second fundamental form  $A_\eta$  in the direction of  $\eta$  is diagonal and given by

$$\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}.$$

Then we get  $G_*(\tilde{e}_1) = -\lambda e_1$ ,  $G_*(\tilde{e}_2) = \lambda e_2$  and  $G_*(\zeta) = \zeta$ , where  $\tilde{e}_1, \tilde{e}_2$  are horizontal lifts of principal vectors  $e_1, e_2$  at  $x$  to the tangent space of  $U(M)$  at  $(x, \eta)$  and  $\zeta$  is a normal vector with  $\langle \eta, \zeta \rangle = 0$ . Thus the induced metric is given by

$$\begin{pmatrix} \lambda^2 & & & \\ & \lambda^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

and the volume form is  $\lambda^2 * 1_M * 1_{S^{N-1}}$ . Note that  $\lambda^2$  is  $(1/2)|A_\eta|^2$ . Since

$$\tilde{e}_1 F_\xi = \langle \nabla_{e_i}^\perp \xi, \eta \rangle \quad \text{and} \quad \zeta F_\xi = \langle \xi, \zeta \rangle,$$

we have

$$|\nabla F_\xi|^2 = \left( (1/\lambda)^2 \sum \langle \nabla_{e_i}^\perp \xi, \eta \rangle^2 \right) + |\xi|^2 - F_\xi^2,$$

which implies

$$\int |\nabla F_\xi|^2 * 1_{U(M)} = \int (1/2) |\nabla F_\xi|^2 |A_\eta|^2 * 1_M * 1_{S^{N-2}(1)}.$$

Now we have the integral over the fibre at  $x$  as follows:

$$\begin{aligned} & \int (1/2) |\nabla F_\xi|^2 |A_\eta|^2 * 1_{S^{N-2}(1)} \\ &= \int \left\{ \sum \langle \nabla_{e_i}^\perp \xi, \eta \rangle^2 + (1/2) |A_\eta|^2 |\xi|^2 - (1/2) |A_\eta|^2 F_\xi^2 \right\} * 1_{S^{N-2}(1)}. \end{aligned}$$

When we put  $\eta = \sum y^\alpha e_\alpha$ , we have

$$\begin{aligned} & \int \left\{ \sum \langle \nabla_{e_i}^\perp \xi, \eta \rangle^2 \right\} * 1_{S^{N-2}(1)} \\ &= \int \left\{ \sum y^\alpha y^\beta \langle \nabla_{e_i}^\perp \xi, e_\alpha \rangle \langle \nabla_{e_i}^\perp \xi, e_\beta \rangle \right\} * 1_{S^{N-2}(1)}. \end{aligned}$$

It follows from

$$\int y^\alpha y^\beta * 1_{S^{N-2}(1)} = (\omega/(N-1)) \delta_{\alpha\beta}$$

that we obtain

$$\int \left\{ \sum \langle \nabla_{e_i}^\perp \xi, \eta \rangle^2 \right\} * 1_{S^{N-2}(1)} = (\omega/(N-1)) |\nabla^\perp \xi|^2$$

and

$$\begin{aligned} & \int (1/2) |A_\eta|^2 |\xi|^2 * 1_{S^{N-2}(1)} \\ &= (1/2) \int \left\{ \sum h_{ij}^\alpha h_{ij}^\beta y^\alpha y^\beta |\xi|^2 \right\} * 1_{S^{N-2}(1)} \\ &= (\omega/2(N-1)) |\sigma|^2 |\xi|^2, \end{aligned}$$

where  $h_{ij}^\alpha = \langle A_{e_i} e_j, e_j \rangle$  and  $|\sigma|^2 = \sum h_{ij}^\alpha h_{ij}^\alpha$ . On the other hand, since

$$\begin{aligned} & \int (1/2) |A_\eta|^2 F_\xi^2 * 1_{S^{N-2}(1)} \\ &= (1/2) \int \left\{ \sum h_{ij}^\alpha h_{ij}^\beta y^\alpha y^\beta y^\gamma y^\delta \langle e_\gamma, \xi \rangle \langle e_\delta, \xi \rangle \right\} * 1_{S^{N-2}(1)} \end{aligned}$$

holds and we may consider  $e_3 = \xi/|\xi|$ , by

$$\int y^\alpha y^\beta (y^3)^2 * 1_{S^{N-2}(1)} = (\omega/(N+1)(N-1))(\delta_{\alpha\beta} + 2\delta_{3\alpha}\delta_{3\beta}),$$

we obtain

$$\begin{aligned} (1/2) \int |A_\eta|^2 F_\xi^2 * 1_{S^{N-2}(1)} &= (\omega/2(N+1)(N-1))|\sigma|^2|\xi|^2 \\ &+ (\omega(N+1)(N-1))\left\{ \sum \langle \xi, \sigma_{ij} \rangle \langle \xi, \sigma_{ij} \rangle \right\}, \end{aligned}$$

where  $\sigma_{ij} = \sum h_{ij}^\alpha e_\alpha$ . Thus we have

$$\begin{aligned} \int (|\nabla F_\xi|^2 - NF_\xi^2) * 1_{U(M)} &= (\omega/(N-1)) \int \left( |\nabla^\perp \xi|^2 - \sum \langle \xi, \sigma_{ij} \rangle \langle \xi, \sigma_{ij} \rangle \right) * 1_M. \quad \square \end{aligned}$$

**PROPOSITION 2.2.** *Let  $\xi$  be a normal vector field of  $N(M)$ . Then  $\xi$  is a Jacobi field if and only if*

$$\Delta_{U(M)} F_\xi = -NF_\xi.$$

*Proof.* We fix a point  $(x, \eta)$  of  $U(M)$ . Let  $\gamma(s)$  be a geodesic with arc length parameter  $s$  such that  $\gamma(0) = x$ . We denote by  $X$  the tangent of  $\gamma(s)$  at  $x$ . Let  $e_1$  and  $e_2$  be the principal vectors of  $A_\eta$  such that  $A_\eta e_1 = \lambda e_1$  and  $A_\eta e_2 = -\lambda e_2$  and  $e_1(s)$  and  $e_2(s)$  the parallel vector fields along  $\gamma(s)$  with respect to the connection of  $T(M)$  such that  $e_1(0) = e_1$  and  $e_2(0) = e_2$ . Let  $e_\alpha, \alpha = 3, \dots, e_{N+1}$  be an orthonormal basis of  $N_x(M)$  and  $e_\alpha(s)$  the parallel vector fields along  $\gamma(s)$  with respect to  $\nabla^\perp$  such that  $e_\alpha(0) = e_\alpha$ . We may set  $e_3(0) = \eta$ . Then  $(\gamma(s), e_3(s))$  is the horizontal lift of  $\gamma(s)$  through  $(x, \eta)$  in  $U(M)$ . By the definition of  $G$ , we obtain

$$G_*(\tilde{\gamma}_*(s)) = -A_{e_3(s)}\gamma_*(s).$$

Let  $\tilde{\nabla}$  be the covariant differentiation with respect to the degenerate metric induced by  $G$ . Then we have

$$\begin{aligned} G_*(\tilde{\nabla}_{\tilde{\gamma}_*(0)}\tilde{\gamma}_*(s)) &= \text{the component of } [-dA_{e_3(s)}\gamma_*(s)/ds]_{s=0} \\ &\text{orthogonal to } \eta. \end{aligned}$$

It follows that

$$\begin{aligned}\tilde{\nabla}_{\tilde{\gamma}_*(0)}\tilde{\gamma}_*(s) &= (\langle \eta, (\nabla_X \sigma)(X, e_1) \rangle / \lambda) \tilde{e}_1 \\ &\quad - (\langle \eta, (\nabla_X \sigma)(X, e_2) \rangle / \lambda) \tilde{e}_2 \\ &\quad - \sum_{\alpha=4}^{N-1} \sum_{k=1}^2 \langle \eta, \sigma(X, e_k) \rangle \langle e_\alpha, \sigma(X, e_k) \rangle_\alpha.\end{aligned}$$

It is easy to extend  $e_\alpha$  for  $\alpha = 4, \dots, N-1$  to the vertical vector fields  $\tilde{e}_\alpha$  on  $U(M)$  such that

$$\tilde{\nabla}_{\tilde{e}_\alpha} \tilde{e}_\alpha = 0 \quad \text{at } (x, \eta).$$

Furthermore, for the horizontal lift  $\tilde{Y}$  of a vector field  $Y$  defined on a neighborhood at  $x$ , we have

$$\tilde{\nabla}_{\tilde{e}_\alpha} \tilde{Y} = (\langle A_{e_\alpha} Y, e_1 \rangle / \lambda) \tilde{e}_1 - (\langle A_{e_\alpha} Y, e_2 \rangle / \lambda) \tilde{e}_2 \quad \text{at } (x, \eta).$$

Using these vector fields, we obtain the following for each point  $(x, \eta) \in U(M)$ .

$$\begin{aligned}\text{Hess } F_\xi(X, X) &= \left\langle \eta, \text{Hess } \xi(X, X) + \sum \langle \sigma(X, e_k), \xi \rangle \sigma(X, e_k) \right\rangle \\ &\quad - \langle \eta, (\nabla_X \sigma)(X, e_1) \rangle \langle \eta \cdot \nabla_{e_1}^\perp \xi \rangle / \lambda \\ &\quad + \langle \eta, (\nabla_X \sigma)(X, e_2) \rangle \langle \eta \cdot \nabla_{e_2}^\perp \xi \rangle / \lambda \\ &\quad - \sum \langle \eta, \sigma(X, e_k) \rangle^2 F_\xi,\end{aligned}$$

$$\text{Hess } F_\xi(e_\alpha, e_\alpha) = -F_\xi \quad \text{for } \alpha = 4, \dots, N-1$$

$$\begin{aligned}\text{Hess } F_\xi(X, e_\alpha) &= \langle e_\alpha, \nabla_X^\perp \xi - (\langle \eta, \nabla_{e_1}^\perp \xi \rangle / \lambda) \sigma(X, e_1) \\ &\quad + (\langle \eta, \nabla_{e_2}^\perp \xi \rangle / \lambda) \sigma(X, e_2) \rangle.\end{aligned}$$

Thus we have

$$\Delta_{U(M)} F_\xi = -(1/\lambda)^2 \langle \eta, J(\xi) \rangle - N F_\xi,$$

where  $J$  is the Jacobi operator of  $N(M)$ . □

We know that  $\chi^\perp = \sum \langle \chi, e_\alpha \rangle e_\alpha$  is a Jacobi field, where  $\chi$  is the position vector of  $M$ . By the calculation as in Proposition 2.2, we obtain

**LEMMA 2.1.** *Hess  $F_\chi^\perp + F_\chi^\perp \langle \cdot, \cdot \rangle$  has an eigenvalue 0 of multiplicity  $N-2$  at  $(x, \eta) \in U(M)$  such that  $\det A_\eta \neq 0$ .*

Now we may consider that  $F_\chi^\perp$  is locally a function on an open set  $U$  of  $S^N(1)$ . Then we define a map of  $U$  into  $R^{N+1}$  such that

$$F_\chi^{\perp\eta} + \text{grad } F_\chi^\perp.$$

By a simple calculation, it is just  $\chi$ . Conversely let  $f$  be an eigenfunction of eigenvalue  $N$  on an open set  $U$  in  $S^N(1)$  such that the eigenvalue of the Hess  $f + f\langle \cdot, \cdot \rangle$  has 0 of multiplicity  $N - 2$ . Then

$$f\eta + \text{grad } f$$

is a map of rank 2 and hence gives a minimal surface. Thus we obtain a Gauss parametrization of a minimal surface in  $R^{N+1}$ .

As a generalization of Theorem A, we easily obtain the following.

**PROPOSITION 2.3.** *Let  $U$  be an open set of  $S^N(1)$  and  $f$  a function on  $U$  such that Hess  $f + f\langle \cdot, \cdot \rangle$  has an eigenvalue 0 of multiplicity  $N - m$ . Then  $f\eta + \text{grad } f$  is a map of  $U$  into  $R^{N+1}$  of rank  $m$  and furthermore gives an  $m$ -dimensional submanifold such that the  $(m - 1)$ st mean curvature vector vanishes. We call the representation the Gauss parametrization by an eigenfunction. Conversely let  $M$  be an  $m$ -dimensional submanifold in  $R^{N+1}$  such that the  $(m - 1)$ st mean curvature vector vanishes, then a neighborhood of each point such that  $\det A_\eta \neq 0$  for some normal vector  $\eta$  the Gauss parametrization by an eigenfunction.*

**REMARK.** In [DG], similar constructions are presented.

**COROLLARY 2.1.** *Let  $M$  be a complex  $m$ -dimensional Kaehler submanifold in  $C^{N+1}$ . Then a neighborhood of each point such that  $\det A_\eta \neq 0$  for some normal vector  $\eta$  admits the Gauss parametrization by an eigenfunction.*

*Proof.* It is well known that the  $(2m - 1)$ st mean curvature vector vanishes on  $M$ .

Let  $M$  be a minimal surface in  $R^{N+1}$  and  $\xi$  a Jacobi field. Then Proposition 2.2 implies that  $F_\xi$  is an eigenfunction of eigenvalue  $N$ . We define the rank  $\gamma_\xi$  of Jacobi field by  $N - \mu$ , where  $\mu$  is the multiplicity of eigenvalue 0 of

$$\text{Hess } F_\xi + F_\xi\langle \cdot, \cdot \rangle.$$

By Proposition 2.3, we have a  $\gamma_\xi$ -dimensional submanifold with zero  $(\gamma_\xi - 1)$ st mean curvature vector. For example, let  $M$  be a minimal

surface in  $R^3$ . Then  $\gamma_\xi = 0$  or  $2$  holds for a Jacobi field  $\xi$  and if  $\gamma_\xi = 2$  holds, then we obtain a minimal surface

$$\xi - \sum (A_\eta^{-1})^{ij} \langle \nabla_{e_i}^\perp \xi, \eta \rangle e_j,$$

which gives a minimal deformation of  $M - \{\text{geodesic points}\}$  whose normal variation vector field is  $\xi$ . In fact

$$\chi + s \left\{ \xi \sum (A_\eta^{-1})^{ij} \langle \nabla_{e_i}^\perp \xi, \eta \rangle e_j \right\}$$

is a one parameter family of minimal surfaces, where  $\chi$  is the immersion of  $M$  into  $R^3$ .

Next let  $M$  be a minimal surface in  $R^4$  and  $\xi$  a Jacobi field. Then  $\gamma_\xi$  is  $0, 2$  or  $3$ . In the case of  $\gamma_\xi = 3$ , we have a hypersurface of zero second mean curvature in  $R^4$ , which implies zero scalar curvature. Thus the first given minimal surface is a limit of deformation of hypersurfaces of zero scalar curvature in  $R^4$ .

**3. The index of minimal surfaces.** Let  $M$  be a complete orientable minimal surface of finite total curvature in  $R^{N+1}$ . Then there exists a compact orientable Riemann surface  $\overline{M}$  and finite points  $p_1, \dots, p_q \in \overline{M}$  such that  $M$  is conformally equivalent to  $\overline{M} - \{p_1, \dots, p_q\}$  and the generalized Gauss map of  $M$  into  $G_2(R^{N+1})$  is extendable over  $\overline{M}$ . Let  $L$  be the tautological vector bundle over  $G_2(R^{N+1})$  with rank  $N - 1$ . Then the restriction of the induced bundle over  $\overline{M}$  to  $M$  is the normal bundle  $N(M)$ . So the unit sphere bundle  $U(\overline{M})$  over  $\overline{M}$  gives a compactification of  $U(M)$  such that the ends are fibres at  $p_i$ . It is clear that the map  $G$  is extendable on  $U(\overline{M})$  and we denote by  $\overline{G}$  the map. Note that  $\overline{G}$  is real analytic.

**LEMMA 3.1.** *The degenerate set  $S$  for  $\overline{G}$  is an analytic set of codimension  $\geq 2$  if  $M$  is not in some  $R^3$ .*

*Proof.* It is clear that  $S$  is an analytic set. Assume that  $S$  has an open set of  $U(M)$ . Then as analytic function  $|A_\eta|^2$  on  $U(M)$  is zero on some open set, which implies that  $M$  is plane. Assume that  $S$  has codimension  $1$ . Then we note that the rank of  $\theta|_S$  is  $1$  or  $2$ , where  $\theta$  is the projection of  $U(M)$  onto  $M$ . If the rank is  $2$ , there is an open set  $U$  of  $M$  such that each fibre at  $x \in U$  has an  $(N - 3)$ -dimensional submanifold where  $|A_\eta^2| \equiv 0$ . For each  $x \in U$ , we have an orthonormal basis  $e_3, \dots, e_{N+1}$  such that, for all  $\alpha \geq 5$ ,

$$A_{e_3} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}, \quad A_{e_\alpha} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and hence, for any unit normal vector  $\eta = ae_3 + be_4 + \text{others}$ ,  $\det A_\eta = 0$  holds if and only if  $a^2\lambda^2 + b^2\mu^2 = 0$ , which implies that if  $\lambda \neq 0$  and  $\mu \neq 0$ , then the set where  $\det A_\eta = 0$  is an  $(N - 4)$ -dimensional sphere. It is a contradiction and hence  $\lambda$  or  $\mu = 0$ , which implies the first normal space on  $U$  is at most 1-dimensional. It is easy to see that  $M$  is in some  $R^3$ . Next assume that the rank of  $\theta|_S$  is 1. Then, on the image  $\theta(S)$ , the second fundamental form of  $M$  vanishes. On the other hand, it is well known that totally geodesic points are isolated. It is contradiction.  $\square$

By the result in [H], we can have a stratification of  $S$  such that if a stratum  $T$  satisfies  $T \cap S \neq \emptyset$ , then  $S \supset T$ . So  $\overline{G}(S)$  has a stratification and  $\overline{G}^{-1}(\overline{G}(S))$  is a sum of finite strata of codimension  $\geq 2$ . By a simple argument, we get

LEMMA 3.2.

$$\overline{G}: U(\overline{M}) \setminus \overline{G}^{-1}(\overline{G}(S)) \rightarrow S^N(1) \setminus \overline{G}(S)$$

is a  $k$ -sheeted covering map, where  $k$  is the total curvature of  $M/2\pi$ .

From Proposition 2.1, we obtain the following:

$$\text{index}(M) \leq \text{the number of eigenvalues of } \Delta_{U(\overline{M})} \text{ that are strictly less than } N.$$

Let  $\{\lambda_i\}_{i=0}^\infty$  and  $\{\mu_i\}_{i=0}^\infty$  be eigenvalues of  $\Delta_S N(1)$  and  $\Delta_{U(\overline{M})}$ , respectively. A theorem in [5], together with Lemma 3.2 implies

$$\sum e^{-\mu_i t} \leq k \left( \sum e^{-\lambda_i t} \right).$$

Thus we conclude that

$$(\text{index}(M))e^{-Nt} \leq \sum_{\mu_i < N} e^{-\mu_i t} \leq \sum e^{-\mu_i t} \leq k \left( \sum e^{-\lambda_i t} \right).$$

Hence

$$\text{index}(M) \leq e^{Nt} \left( \sum e^{-\lambda_i t} \right) k.$$

Note that if  $M$  is not in some  $R^3$ , then  $c(N)$  is given by

$$2\pi \inf_{t>0} \left\{ e^{Nt} \left( \sum e^{-\lambda_i t} \right) \right\}.$$

**4. A generalization of minimal herissons.** Recently Rosenberg and Toubiana [RT] give some results on complete minimal finite branched

surfaces in  $R^3$  of finite total curvature  $4\pi$ , which are called minimal herissons and parametrized by their Gauss image.

Let  $M$  be an  $m$ -dimensional submanifold of zero  $(m - 1)$ st mean curvature vector in  $R^{N+1}$ . We consider the following condition (\*\*).

- (\*\*) *There exist finite stratum  $S$  of  $U(M)$  and  $S'$  of  $S^N(1)$  such that codimensions of elements of  $S$  and  $S' \geq 2$  and*  

$$G: U(M) \setminus S \rightarrow S^N(1) \setminus S'$$
*is a  $k$ -sheeted covering.*

Let  $\mathfrak{M}$  denote the space of  $m$  ( $2 \leq m \leq N$ )-dimensional submanifolds of zero  $(m - 1)$ st mean curvature vector in  $R^{N+1}$  which satisfy (\*\*). Following as in [RT], we can define a sum operation in  $\mathfrak{M}$ :

$$M_1 + M_2 = \left\{ \sum \theta(x_i) + \sum \theta(y_i): G_1^{-1}(z) = \{x_i\}, \right. \\ \left. G_2^{-1}(z) = \{y_i\}, \text{ where } z \in S^N(1) \setminus S'_1 \cup S'_2 \right\},$$

where  $G_1$  and  $G_2$  are the Gauss map of  $M_1$  and  $M_2$ , respectively and  $S'_1$  and  $S'_2$  satisfy (\*\*) for  $G_1$  and  $G_2$ . Note that the equality of dimensions of  $M_1$  and  $M_2$  is not necessary. This operation may be considered as follows: for  $z \in S^N(1) \setminus S'_1 \cup S'_2$ , we define a function  $f$  by

$$f(z) = \sum F_{\chi_1}^\perp(x_i) + \sum F_{\chi_2}^\perp(y_i),$$

where  $\chi_1$  and  $\chi_2$  are immersions of  $M_1$  and  $M_2$  into  $R^{N+1}$ , respectively. It is clear that

$$\Delta_S N_{(1)} f = -Nf$$

on  $U = S^N(1) \setminus S'_1 \cup S'_2$  and hence  $f$  is analytic on  $U$ . By the analyticity of  $f$  on  $U$ , the multiplicity of the eigenvalue 0 of  $\text{Hess } f + f \langle \cdot, \cdot \rangle$  is constant  $N - m$  on some open dense set of  $U$ . Thus we get an  $m$ -dimensional submanifold of zero  $(m - 1)$ st mean curvature vector in  $R^{N+1}$  which gives  $M_1 + M_2$ .

**PROPOSITION 4.1.** *Assume that  $M_1 + M_2$  is of dimension  $m$ . Then  $M_1 + M_2$  is of zero  $(m - 1)$ st mean curvature vector and parametrized by Gauss image. In particular, the total absolute curvature is the volume of  $S^N(1)$ .*

**REMARK.** The study of  $f$  which satisfies  $\Delta_S N_{(1)} f = -Nf$  has a relation to  $N$ -dimensional space-like minimal submanifolds of constant curvature 1 in an  $(N + 2)$ -dimensional deSitter space time [K].

In [N], Nayatani proves that, if  $M$  be a complete orientable minimal surface of finite total curvature, then  $M$  has a finite index. But it does not imply the existence of  $c(N)$ .

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Received January 6, 1989, the author was partially supported by a grant of the Japanese Education Ministry.

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